A proof of the Holmes–Holroyd–Ramírez cyclic-product conjecture

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Abstract

Given positive numbers $z_0, \dots, z_{m-1}$ and a fixed $1 \le r \le m$, we show that, among all permutations of the given numbers, the circularly symmetric greedy arrangement is a maximizer of

$$\sum_{k=0}^{m-1} \prod_{j=0}^{r-1} z_{k+j},$$

where indices are read modulo m. This resolves the cyclic-product conjecture of Holmes, Holroyd and Ramírez.

1 The conjecture

Let $m \ge 2$ and let $a_0 \ge a_1 \ge \cdots \ge a_{m-1} > 0$. For a circular arrangement $z = (z_0, \dots, z_{m-1})$ of these numbers, write

$$P_r(z) = \sum_{k=0}^{m-1} \prod_{j=0}^{r-1} z_{k+j}, \qquad r \in \{1, \dots, m\},$$

with indices read modulo m. The *greedy* (circular-symmetric) arrangement a^G is defined by

$$a_i^G = a_{2i}, \qquad a_{m-1-i}^G = a_{2i+1}, \qquad i = 0, \dots, \lfloor m/2 \rfloor - 1, \tag{1}$$

with $a_{\lfloor m/2 \rfloor}^G = a_{m-1}$ when m is odd.

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Theorem 1.1. *For every $r \in \{1, \dots, m\}$ and every circular arrangement z of $(a_0, \dots, a_{m-1})$,*

$$P_r(a^G) \ge P_r(z).$$

The cases $r \in \{1, m\}$ are tautological. Holmes–Holroyd–Ramírez formulated this statement as Conjecture 5 and proved it for $r \le 3$ and $r \ge m-3$ (hence for all r when $m \le 7$) [2, Conjecture 5 and Theorem 6].

Application to periodic random walks. A principal motivation for Conjecture 5 was the following "greedy is least speedy" problem, formulated as Conjecture 8 in [2]. Fix

$$p_0 \ge p_1 \ge \cdots \ge p_{m-1}, \qquad p_i \in (0,1),$$

and consider a particle on $\mathbb{Z}$ moving in the corresponding m-periodic environment: when the particle is at x, it moves to $x+1$ with probability $p_{x \bmod m}$ and to $x-1$ with probability $1 - p_{x \bmod m}$. More generally, for any circular arrangement $q = (q_0, \dots, q_{m-1})$ of these probabilities, let $v(q)$ denote the almost-sure limiting velocity

$$v(q) = \lim_{n \to \infty} \frac{X_n}{n}.$$

This limit exists and is deterministic [2, Proposition 1]. Set

$$\rho_i(q) = \frac{1-q_i}{q_i}, \qquad \gamma = \prod_{i=0}^{m-1} \rho_i(q) = \frac{\prod_{i=0}^{m-1}(1-p_i)}{\prod_{i=0}^{m-1} p_i};$$

the value of γ is independent of the arrangement. We restrict to the right-transient regime $\gamma < 1$, equivalently

$$\prod_{i=0}^{m-1}(1-p_i) < \prod_{i=0}^{m-1} p_i,$$

in which every arrangement has positive velocity [2].

The question is then how to arrange the fixed probabilities so that the particle escapes to $+\infty$ as slowly as possible. Holmes–Holroyd–Ramírez proved the velocity formula

$$v(q) = \frac{1-\gamma}{1-\gamma+\dfrac{2}{m}\displaystyle\sum_{r=1}^{m} P_r\big(\rho(q)\big)} \tag{2}$$

[2, Proposition 7]. Let p^G be the pendulum, or greedy, arrangement obtained from (1). The map $p \mapsto (1-p)/p$ reverses the order of the entries, but the greedy arrangements of the p_i and of the corresponding ρ_i agree up to a rotation and a reversal; these operations leave every cyclic sum P_r unchanged. Theorem 1.1 therefore maximises the denominator in (2) at p^G, and hence

$$v(p^G) \le v(q)$$

for every circular arrangement q. Thus Theorem 1.1 proves the "greedy is least speedy" conjecture [2, Conjecture 8]: among all rearrangements with positive velocity, the greedy arrangement makes the walk escape to $+\infty$ as slowly as possible.

Continuous rearrangement viewpoint. The stronger majorisation statement used below in Section 2 has a close continuous analogue. For a nonnegative measurable function u on $\mathbb{R}$, let $u^* : [0,\infty) \to [0,\infty]$ denote its decreasing rearrangement, and say that u majorises v if

$$\int_0^s u^*(t)\,dt \ge \int_0^s v^*(t)\,dt \quad \text{for every } s \ge 0, \qquad \int_{\mathbb{R}} u = \int_{\mathbb{R}} v.$$

Writing $f^\#$ for the symmetric decreasing rearrangement of f on $\mathbb{R}$, the Riesz rearrangement inequality, or more generally the Brascamp–Lieb–Luttinger inequality [1], implies for nonnegative integrable f, g that

$$f^\# * g^\# \ \text{ majorises } \ f * g.$$

Indeed, one applies Riesz rearrangement to f, g, and the indicator of a set of prescribed measure. Continuous Karamata then gives

$$\int_{\mathbb{R}} \Phi(f^\# * g^\#) \ge \int_{\mathbb{R}} \Phi(f * g)$$

for every convex Φ with $\Phi(0) = 0$ for which the integrals are finite. Theorem 2.1 below is discrete $\mathbb{Z}/m\mathbb{Z}$ version of this phenomenon: one convolution factor is the indicator of an interval (an arc in $\mathbb{Z}/m\mathbb{Z}$), while the symmetric decreasing rearrangement of the other factor is precisely the pendulum arrangement. Finally, taking logarithms converts consecutive products into consecutive sums, and choosing the convex function $\varphi(t) = e^t$ recovers the original cyclic-product problem.

2 A more general majorisation

For $u \in \mathbb{R}^m$, write $u^{\downarrow}$ for the decreasing rearrangement of u. For $u, v \in \mathbb{R}^m$, we say that u *majorises* v, and write $u \succ v$, if

$$\sum_{i=1}^{k} u_i^{\downarrow} \geq \sum_{i=1}^{k} v_i^{\downarrow} \quad \text{for } k = 1, \ldots, m-1, \qquad \sum_{i=1}^{m} u_i = \sum_{i=1}^{m} v_i.$$

Set $x_i = \log a_i$, so $x_0 \geq \cdots \geq x_{m-1}$, and let $x_i^G = \log a_i^G$. For a circular arrangement $b = (b_0, \ldots, b_{m-1})$ of $(x_0, \ldots, x_{m-1})$, write

$$S_k(b) := \sum_{j=0}^{r-1} b_{k+j}, \qquad S(b) := \big(S_0(b), \ldots, S_{m-1}(b)\big), \qquad S^{\downarrow}(b) := S(b)^{\downarrow}.$$

If $z_i = e^{b_i}$, then

$$P_r(z) = \sum_k e^{S_k(b)}, \qquad P_r(a^G) = \sum_k e^{S_k(x^G)}.$$

Thus Theorem 1.1 is the case $\varphi(u) = e^u$ of the following statement.

Theorem 2.1. *For every $r \in \{1, \ldots, m\}$ and every circular arrangement b of $(x_0, \ldots, x_{m-1})$,*

$$S^{\downarrow}(x^G) \succ S^{\downarrow}(b).$$

Consequently, by Karamata's inequality,

$$\sum_k \varphi\big(S_k(x^G)\big) \geq \sum_k \varphi\big(S_k(b)\big)$$

for every convex $\varphi : \mathbb{R} \to \mathbb{R}$.

The rest of the note proves Theorem 2.1.

3 Ky Fan sums and coverage

For $s \in \{0, \ldots, m\}$ write

$$\Phi_{r,s}(b) = \text{sum of the } s \text{ largest coordinates of } S(b).$$

Theorem 2.1 is equivalent to $\Phi_{r,s}(x^G) \geq \Phi_{r,s}(b)$ for all s, together with $\sum_k S_k = r \sum_i x_i$.

For functions $f, h : \mathbb{Z}/m\mathbb{Z} \to \mathbb{R}$, their circular convolution is

$$(f * h)(t) = \sum_{u \in \mathbb{Z}/m\mathbb{Z}} f(u)h(t-u).$$

Let $B \subset \mathbb{Z}/m\mathbb{Z}$ with $|B| = s$, and put $V_t = \{t, t-1, \ldots, t-r+1\}$. Then

$$\sum_{k \in B} S_k(b) = \sum_{t=0}^{m-1} c_B(t) b_t,$$

where the *coverage* $c_B : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}$ is the circular convolution

$$c_B(t) = |B \cap V_t| = \big(\mathbf{1}_B * \mathbf{1}_{\{0,\ldots,r-1\}}\big)(t). \tag{3}$$

Thus

$$\Phi_{r,s}(b) = \max_{|B|=s} \langle c_B, b\rangle \le \max_{|B|=s} \langle c_B^{\downarrow}, x^{\downarrow}\rangle, \tag{4}$$

the last step being the classical rearrangement inequality.

Lemma 3.1. *For every $B \subset \mathbb{Z}/m\mathbb{Z}$ with $|B| = s$,*

$$|c_B(t+1) - c_B(t)| \le 1, \qquad \mu \le c_B(t) \le M, \qquad \sum_t c_B(t) = sr,$$

where $\mu = \max(0, s+r-m)$ and $M = \min(s, r)$.

Proof. The identity $c_B(t+1) - c_B(t) = \mathbf{1}_B(t+1) - \mathbf{1}_B(t-r+1)$ gives the Lipschitz bound. The upper bound is immediate, while

$$|B \cap V_t| \ge |B| + |V_t| - m$$

gives the lower bound. Finally, each element of B lies in exactly r of the windows V_t. □

For $s \in \{0, m\}$, the corresponding Ky Fan inequalities are equalities, while Theorem 2.1 is immediate for $r \in \{1, m\}$. We therefore assume below that

$$0 < s < m, \qquad 1 < r < m.$$

We now identify the candidate extremal coverage vector. By an s-arc we mean a set of s consecutive residues in $\mathbb{Z}/m\mathbb{Z}$. Let $I \subset \mathbb{Z}/m\mathbb{Z}$ be an s-arc. Its coverage c_I has the steepest profile permitted by the bounds and the Lipschitz condition in Lemma 3.1: it is a circular trapezoid. Indeed, $c_I(t) = |I \cap V_t|$ is the overlap of a fixed s-arc with a moving r-arc. As the latter moves around the circle, the overlap increases by one at each step until one arc is contained in the other, remains maximal, then decreases by one at each step until the minimum overlap is reached. Thus c_I increases with slope $+1$ from μ to M, has a plateau of length

$$H = |s - r| + 1$$

at M, decreases with slope -1 to μ, and has a plateau of length

$$L = |m - s - r| + 1$$

at μ. The upper plateau counts the positions in which the smaller arc lies inside the larger, while the lower plateau counts the positions for which the minimum overlap persists. The two ramps have length $M - \mu - 1$ each, since

$$M - \mu = \min(s, r, m - s, m - r).$$

In particular c_I attains both μ and M. Writing $\Delta = M - \mu$, the plateau lengths and the sum satisfy

$$H + L + 2(\Delta - 1) = m, \qquad sr = m\mu + \Delta(H + \Delta - 1).$$

Thus c_I is exactly the abstract steep $[\mu, M]$-trapezoid of sum sr introduced below.

Theorem 3.2. *If I is an s-arc and B is an arbitrary s-set, then $c_I \succ c_B$.*

We first derive Theorem 2.1 from Theorem 3.2 and Lemma 4.2. Sections 5 and 6 then prove Theorem 3.2 through a more general Lipschitz-isoperimetric statement.

Lemma 3.3. *If $u \succ v$ in $\mathbb{R}^m$ and $y \in \mathbb{R}^m$ is decreasing, then*

$$\langle u^{\downarrow}, y \rangle \geq \langle v^{\downarrow}, y \rangle.$$

Proof. Summation by parts gives

$$\sum_{k=1}^{m} y_k u_k^{\downarrow} = y_m \sum_{k=1}^{m} u_k^{\downarrow} + \sum_{j=1}^{m-1} (y_j - y_{j+1}) \sum_{k=1}^{j} u_k^{\downarrow}.$$

The total sums of u and v agree, each coefficient $y_j - y_{j+1}$ is nonnegative, and the partial sums of $u^{\downarrow}$ dominate those of $v^{\downarrow}$. $\square$

Choose I as in Lemma 4.2. Granting Theorem 3.2 and Lemmas 3.3 and 4.2,

$$\begin{aligned}
\Phi_{r,s}(b) &= \max_{|B|=s} \langle c_B, b\rangle \\
&\le \max_{|B|=s} \langle c_B^{\downarrow}, x^{\downarrow}\rangle \\
&\le \langle c_I^{\downarrow}, x^{\downarrow}\rangle && \text{by Theorem 3.2 and Lemma 3.3} \\
&= \langle c_I, x^G\rangle && \text{by Lemma 4.2} \\
&\le \Phi_{r,s}(x^G).
\end{aligned}$$

This proves Theorem 2.1.

4 Comonotonicity

For $c \in \frac{1}{2}\mathbb{Z}/m\mathbb{Z}$ and $t \in \mathbb{Z}/m\mathbb{Z}$, let

$$d_c(t) = \min_{k\in\mathbb{Z}} |t - c + km|$$

be the circular distance from t to c, using any representatives of t and c. A function $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{R}$ is *circularly symmetrically decreasing about the centre* c if $f(t)$ depends only on $d_c(t)$ and is nonincreasing as $d_c(t)$ increases. If c is a half-integer, the centre is the seam between two adjacent residues.

Lemma 4.1 (Convolution preserves circular symmetry)**.** *If $f, h : \mathbb{Z}/m\mathbb{Z} \to [0, \infty)$ are circularly symmetrically decreasing about centres c_f and c_h, respectively, then $f * h$ is circularly symmetrically decreasing about $c_f + c_h$.*

Proof. Such a function is a nonnegative linear combination of indicators of arcs with the same centre, with coefficients given by the successive drops on the distance-rings. If A and C are arcs centred at c_A and c_C, respectively, then

$$(\mathbf{1}_A * \mathbf{1}_C)(t) = |A \cap (t - C)|.$$

As t moves away from $c_A + c_C$, this overlap is nonincreasing and changes by at most one at each step. Its profile is therefore a possibly degenerate circular trapezoid, circularly symmetrically decreasing about $c_A + c_C$. The claim follows by bilinearity. □

Lemma 4.2 (Comonotonicity)**.** *Place x^G according to* (1)*. Let*

$$\sigma = -\left\lfloor \frac{r+s-1}{2} \right\rfloor \pmod{m}, \qquad I = \{\sigma, \sigma+1, \ldots, \sigma+s-1\},$$

where the elements of I are read modulo m. Then x^G and c_I are comonotone:

$$(c_I(p) - c_I(q))(x^G_p - x^G_q) \geq 0 \qquad \textit{for all } p, q.$$

Consequently

$$\langle c_I, x^G \rangle = \langle c_I^{\downarrow}, x^{\downarrow} \rangle.$$

Proof. The indicator $\mathbf{1}_I$ is circularly symmetrically decreasing about $\sigma + (s - 1)/2$, and $\mathbf{1}_{\{0,\ldots,r-1\}}$ is circularly symmetrically decreasing about $(r-1)/2$. Lemma 4.1 puts the centre of c_I at

$$\sigma + \frac{s+r-2}{2} = \begin{cases} -1/2, & r+s \text{ odd}, \\ 0, & r+s \text{ even}. \end{cases}$$

In the first case

$$c_I(0) = c_I(m-1) \geq c_I(1) = c_I(m-2) \geq \cdots,$$

while in the second case $c_I(d) = c_I(-d)$ and c_I is nonincreasing in the distance from 0. Thus c_I is nonincreasing along

$$0,\ m-1,\ 1,\ m-2,\ 2,\ \ldots,$$

the order in which (1) places the decreasing sequence x. Hence x^G and c_I are comonotone. □

The next two sections complete the proof of Theorem 3.2.

5 Superlevel sizes

A *steep circular* $[\mu, M]$*-trapezoid* with plateau lengths $H, L \geq 1$ is a circular sequence which, up to rotation and reversal, consists of a plateau of H copies of M, a unit-slope descent through $M-1, \ldots, \mu+1$, a plateau of L copies of μ, and a unit-slope ascent through $\mu+1, \ldots, M-1$.

For the remainder of the proof, $\mu < M$ are integers, $\Delta = M - \mu$, and S is such that the steep $[\mu, M]$-trapezoid of sum S exists. Writing $Q = S - m\mu$, this means

$$H = \frac{Q}{\Delta} - \Delta + 1, \qquad L = m - H - 2(\Delta - 1)$$

are integers with $H, L \geq 1$. Its decreasing rearrangement is

$$\tau^{\downarrow} = \big(\underbrace{M, \ldots, M}_{H}, \underbrace{M-1, M-1}_{2}, \ldots, \underbrace{\mu+1, \mu+1}_{2}, \underbrace{\mu, \ldots, \mu}_{L}\big),$$

and its superlevel sizes are

$$n_j(\tau) = |\{t : \tau(t) \geq \mu + j\}| = H + 2(\Delta - j), \qquad j = 1, \dots, \Delta. \tag{5}$$

For an integer sequence $v : \mathbb{Z}/m\mathbb{Z} \to [\mu, M]$, set

$$n_j(v) = |\{t : v(t) \geq \mu + j\}|, \qquad j = 1, \dots, \Delta,$$

and write

$$n(v) = \big(n_1(v), \dots, n_\Delta(v)\big).$$

For $k \in \{0, \dots, m\}$, the layer-cake formula gives

$$\sum_{i=1}^{k} v_i^{\downarrow} = k\mu + \sum_{j=1}^{\Delta} \min(k, n_j(v)). \tag{6}$$

Thus $\tau \succ v$ precisely when the right-hand side for τ dominates that for v for every k, together with equality of the total sums.

Lemma 5.1. *Let $x, y \in \mathbb{R}^\Delta$ be decreasing and satisfy $x \succ y$. Then, for every $k \geq 0$,*

$$\sum_{i=1}^{\Delta} \min(k, x_i) \leq \sum_{i=1}^{\Delta} \min(k, y_i).$$

Proof. The function $u \mapsto \min\{k, u\}$ is concave. Since $x \succ y$, Karamata's inequality gives

$$\sum_{i=1}^{\Delta} \min\{k, x_i\} \leq \sum_{i=1}^{\Delta} \min\{k, y_i\}.$$

$\square$

Lemma 5.2. *Let v be a circular Lipschitz-1 integer sequence with values in $[\mu, M]$, attaining both endpoints, and satisfying $\sum v = S$. Then*

$$n(v) \succ n(\tau), \qquad \textit{and consequently} \qquad \tau \succ v.$$

Proof. Choose p, q with $v(p) = M$ and $v(q) = \mu$. The two arcs joining p to q are internally disjoint. Along each arc, integer-valuedness and the Lipschitz-1 condition force every intermediate value. Hence

$$n_j(v) - n_{j+1}(v) = |\{v = \mu + j\}| \geq 2, \qquad j = 1, \dots, \Delta - 1.$$

The sequence (5) has consecutive differences exactly 2, so

$$d_j = n_j(v) - n_j(\tau)$$

is nonincreasing. Moreover,

$$\sum_{j=1}^{\Delta} n_j(v) = \sum_t \big(v(t) - \mu\big) = S - m\mu = Q = \sum_{j=1}^{\Delta} n_j(\tau),$$

so $\sum_{j=1}^{\Delta} d_j = 0$. Hence the partial sums of (d_j) are nonnegative, and therefore $n(v) \succ n(\tau)$. Lemma 5.1 and (6) give $\tau \succ v$. $\square$

6 Smaller boxes

Lemma 6.1 (Padded comparison)**.** *Let α, β be nonnegative integers and $\Delta' = \Delta - \alpha - \beta \geq 1$. Write*

$$Q' = Q - m\alpha, \qquad H' = \frac{Q'}{\Delta'} - \Delta' + 1,$$

and define

$$n^\sharp = \big(\underbrace{m, \dots, m}_{\alpha}, \big(H' + 2(\Delta' - j)\big)_{j=1}^{\Delta'}, \underbrace{0, \dots, 0}_{\beta}\big) \in \mathbb{R}^\Delta.$$

If $H' \geq 0$ and $H' + 2(\Delta' - 1) \leq m$, then $n^\sharp \succ n(\tau)$.

Proof. Both vectors are decreasing and sum to Q. Write $P^\sharp(p)$ and $P^\tau(p)$ for the sums of their first p coordinates. If $p \leq \alpha$, then

$$P^\sharp(p) - P^\tau(p) = p(L + p - 1) \geq 0.$$

If $p \geq \alpha + \Delta'$, then $P^\sharp(p) = Q \geq P^\tau(p)$; this also handles $\Delta' = 1$.

Suppose $\Delta' \geq 2$ and $\alpha < p < \alpha + \Delta'$. Put $t = p - \alpha$. Then

$$P^\sharp(p) = \alpha m + \frac{t(Q - m\alpha)}{\Delta'} + t\Delta' - t^2,$$
$$P^\tau(p) = (\alpha + t)\left(\frac{Q}{\Delta} + \Delta - \alpha - t\right).$$

Using $Q = \Delta(H + \Delta - 1)$ and $m = H + L + 2\Delta - 2$, a direct expansion gives

$$\Delta'\big(P^\sharp(p) - P^\tau(p)\big) = Ht\beta + L\alpha(\Delta' - t) + t\beta(\beta - 1) + \alpha(1 - \alpha)(t - \Delta'). \quad (7)$$

Every term on the right is nonnegative: the first two are immediate, $\beta(\beta - 1) \geq 0$ for integer $\beta \geq 0$, and the last term vanishes for $\alpha \in \{0, 1\}$ and is nonnegative for $\alpha \geq 2$. Hence $P^\sharp(p) \geq P^\tau(p)$. $\square$

Lemma 6.2 (Smaller box)**.** *Let τ be the steep $[\mu, M]$-trapezoid of sum S, and let v be a circular Lipschitz-1 integer sequence with the same sum. Set*

$$\mu' = \min_t v(t), \qquad M' = \max_t v(t).$$

If $[\mu', M'] \subsetneq [\mu, M]$, then $\tau \succ v$.

Proof. Let $\Delta' = M' - \mu'$, $\alpha = \mu' - \mu$, $\beta = M - M'$, and $Q' = S - m\mu'$. If $\Delta' = 0$, then v is constant, and every vector with the same sum majorises v.

Assume $\Delta' \geq 1$ and set

$$n_j^{\mathrm{loc}}(v) = |\{t : v(t) \geq \mu' + j\}|, \qquad j = 1, \ldots, \Delta'.$$

The two-arc argument in Lemma 5.2 gives

$$n_j^{\mathrm{loc}}(v) - n_{j+1}^{\mathrm{loc}}(v) \geq 2, \qquad j = 1, \ldots, \Delta' - 1.$$

Since v attains μ' and M', we have $n_{\Delta'}^{\mathrm{loc}}(v) \geq 1$ and $n_1^{\mathrm{loc}}(v) \leq m-1$. Therefore, for $j = 1, \ldots, \Delta'$,

$$1 + 2(\Delta' - j) \leq n_j^{\mathrm{loc}}(v) \leq m - 1 - 2(j-1).$$

Summing these inequalities over j gives

$$\Delta'^2 \leq Q' = \sum_{j=1}^{\Delta'} n_j^{\mathrm{loc}}(v) \leq \Delta'(m - \Delta').$$

Thus, for

$$H' = \frac{Q'}{\Delta'} - \Delta' + 1,$$

we have $H' \geq 1$ and $H' + 2(\Delta' - 1) \leq m - 1$.

Let

$$n_{\mathrm{loc}}^{\sharp} = (H' + 2(\Delta' - 1), \ldots, H')$$

and set

$$d_j = n_j^{\mathrm{loc}}(v) - \big(H' + 2(\Delta' - j)\big), \qquad j = 1, \ldots, \Delta'.$$

Then

$$d_j - d_{j+1} = n_j^{\mathrm{loc}}(v) - n_{j+1}^{\mathrm{loc}}(v) - 2 \geq 0, \qquad j = 1, \ldots, \Delta' - 1,$$

so $d_1 \geq \cdots \geq d_{\Delta'}$. Moreover,

$$\sum_{j=1}^{\Delta'} d_j = Q' - \big(\Delta' H' + \Delta'(\Delta' - 1)\big) = 0$$

by the definition of H'. A decreasing sequence with zero sum has nonnegative partial sums, and hence

$$n^{\mathrm{loc}}(v) \succ n^{\sharp}_{\mathrm{loc}}.$$

Define

$$\widetilde{n}(v) = \big(\underbrace{m, \dots, m}_{\alpha}, n^{\mathrm{loc}}_1(v), \dots, n^{\mathrm{loc}}_{\Delta'}(v), \underbrace{0, \dots, 0}_{\beta}\big).$$

Since both local vectors are decreasing and take values in $[0, m]$, adjoining the same initial block of m's and terminal block of 0's preserves majorisation. Therefore Lemmas 5.1 and 6.1 give, for every k,

$$\sum_{j=1}^{\Delta} \min(k, \widetilde{n}_j(v)) \leq \sum_{j=1}^{\Delta} \min(k, n^{\sharp}_j) \leq \sum_{j=1}^{\Delta} \min(k, n_j(\tau)).$$

The vector $\widetilde{n}(v)$ is the superlevel vector of v measured from the baseline μ. By (6), this is $\tau \succ v$. $\square$

Theorem 6.3 (Lipschitz isoperimetry)**.** *Let $\mu < M$ be integers, and let τ be the steep circular trapezoid with values in $[\mu, M]$, plateaus only at μ and M, and sum S. Then $\tau \succ v$ for every circular Lipschitz-1 integer sequence v with values in $[\mu, M]$ and sum S.*

Proof. If v attains both endpoints, apply Lemma 5.2. Otherwise its actual range $[\min v, \max v]$ is a strict subbox of $[\mu, M]$, so Lemma 6.2 applies. $\square$

Proof of Theorem 3.2. Lemma 3.1 shows that c_B is a circular Lipschitz-1 integer sequence with values in $[\mu, M]$ and sum sr. The identities displayed after the description of c_I show that c_I is exactly the steep circular $[\mu, M]$-trapezoid of sum sr. Hence Theorem 6.3 gives $c_I \succ c_B$. $\square$

This completes the proof of Theorem 3.2, hence of Theorem 2.1, and therefore of Theorem 1.1.

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complete transcript of the conversation with Grok is available at `https://grok.com/share/c2hhcmQtNA_73c57e1f-2ff7-467b-a325-250dcd6b9e96`.

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