

# The sin-omial integral for $\binom{r}{s}$ : the area of a level set is a half-residue

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**Theorem 1.** For real  $r > 0$  and  $0 \leq s \leq r$ ,

$$\frac{1}{\pi} \int_0^\pi \frac{\sin^r x}{\sin^s\left(\frac{sx}{r}\right) \sin^{r-s}\left(\frac{(r-s)x}{r}\right)} dx = \binom{r}{s} := \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}.$$

If  $s \in \{0, r\}$  the integrand is  $\equiv 1$  ( $\sin^0 := 1$ ) and both sides are 1. Otherwise let  $a = s/r \in (0, 1)$ ,  $b = 1 - a$ ,  $K(x) = \frac{\sin x}{\sin^a(ax) \sin^b(bx)}$  (so the integrand is  $K^r$ ),  $\rho(x) = \frac{\sin ax}{\sin bx} = \frac{aS(ax)}{bS(bx)}$  with  $S(u) = \frac{\sin u}{u}$  (so  $\rho$  is real-analytic and positive on  $[0, \pi]$ , and  $\rho(0) = \frac{a}{b}$ ),  $g(t) = t^a(1-t)^b$ , and  $\Phi(t, x) = g(t)K(x)$  on  $R = (0, 1) \times (0, \pi)$ .

**Theorem 2.**  $|\{\Phi > \lambda\}| = \pi(1 - \lambda)$  for  $0 \leq \lambda \leq 1$ ; that is,  $\Phi$  is uniform on  $(0, 1)$  for the measure  $dt dx/\pi$  on  $R$ . (For  $a = \frac{1}{2}$  this is Archimedes' hat-box theorem.)

*Theorem 2  $\Rightarrow$  Theorem 1.* By the layer-cake formula,  $\frac{1}{\pi} \iint_R \Phi^r = \int_0^1 r\lambda^{r-1}(1-\lambda) d\lambda = \frac{1}{r+1}$ . By Fubini and Euler's Beta integral,  $\frac{1}{\pi} \iint_R \Phi^r = B(ar+1, br+1) \cdot \frac{1}{\pi} \int_0^\pi K^r$ . Hence  $\frac{1}{\pi} \int_0^\pi K^r = \frac{\Gamma(r+2)}{(r+1)\Gamma(ar+1)\Gamma(br+1)} = \binom{r}{s}$ .  $\square$

*Idea of the proof of Theorem 2.* On the level curve  $\Phi = \lambda$ , the law of sines makes  $t$  a meromorphic function of the coordinate  $\xi$  of (b), real on  $\mathbb{R}$ . Its only pole in the strip  $|\operatorname{Im} \xi| < \pi$  is simple, with residue  $-(1-\lambda)$ , at the image of the maximum point  $(a, 0)$ . The curve joins two real points, halfway around the pole, so the area is  $\pi(1-\lambda)$ .

*Proof of Theorem 2. (a) Law of sines.* Let  $z = \rho e^{ix}$  and  $z^a := \rho^a e^{iax}$ . The identity  $\sin bx + \sin ax e^{ix} = \sin x e^{iax}$  (its real part is  $\sin bx = \sin(x - ax)$ ) and  $K\rho^a = \frac{\sin x}{\sin bx}$  give  $1 + z = K(x) z^a$  on  $[0, \pi]$ .

**(b) Key identity.** For  $(t, x) \in (0, 1) \times [0, \pi]$  let  $\xi = \log \frac{(1-t)\rho(x)}{t} + ix$ . Then  $e^\xi = \frac{1-t}{t} z$  and  $e^{a\xi} = (\frac{1-t}{t})^a z^a$ , so (a) gives  $\Phi e^{a\xi} = (1-t)(1+z) = 1 - t + te^\xi$ , that is,

$$t(1 - e^\xi) = 1 - \Phi(t, x) e^{a\xi}. \quad (*)$$

On  $x = 0$ ,  $\xi = \log \frac{(1-t)a}{tb}$  is real, decreasing in  $t$ , and 0 at  $t = a$ . Elsewhere  $e^\xi \neq 1$ .

**(c) The region**  $D_\lambda = \{\Phi > \lambda\}$ ,  $0 < \lambda < 1$ .  $g$  increases on  $[0, a]$  and decreases on  $[a, 1]$ .  $K$  decreases from  $K(0) = a^{-a}b^{-b}$  to  $K(\pi) = 0$  (as in Kanter 1975): with  $c(y) = y \cot y$ , which decreases because  $c'(y) = \frac{\sin 2y - 2y}{2 \sin^2 y} < 0$ , we have  $x(\log K)' = c(x) - ac(ax) - bc(bx) < 0$ . So  $\Phi < g(a)K(0) = 1$  on  $R$ ,  $\nabla \Phi \neq 0$  on  $(0, 1) \times [0, \pi]$  except at  $(a, 0)$ , and  $D_\lambda = \{0 < x < x_\lambda, t_-(x) < t < t_+(x)\}$ . Here  $g(a)K(x_\lambda) = \lambda$ , and  $t_-(x) \leq a \leq t_+(x)$  are the roots of  $\Phi(\cdot, x) = \lambda$ . The level arc  $\Gamma_\lambda = \{(t_\pm(x), x) : 0 \leq x \leq x_\lambda\}$  is  $C^1$ ; orient it from  $(t_+(0), 0)$  to  $(t_-(0), 0)$ . By Fubini,  $|D_\lambda| = \int_0^{x_\lambda} (t_+ - t_-) dx = \int_{\Gamma_\lambda} t dx$ .

**(d) Half-residue.** On  $\Gamma_\lambda$ , (\*) gives

$$t = \tau_\lambda(\xi) := \frac{1 - \lambda e^{a\xi}}{1 - e^\xi} = \frac{1 - \lambda}{1 - e^\xi} + \lambda \psi(\xi), \quad \psi(\xi) := \frac{1 - e^{a\xi}}{1 - e^\xi} = \frac{E_a(\xi)}{E_1(\xi)}, \quad E_c(\xi) := \frac{e^{c\xi} - 1}{\xi} = \sum_{k \geq 0} \frac{c^{k+1} \xi^k}{(k+1)!}.$$

As  $t$  is real,  $t dx = \operatorname{Im}(t d\xi)$ , so  $|D_\lambda| = \operatorname{Im} \int_{\gamma_\lambda} \tau_\lambda d\xi$  with  $\gamma_\lambda = \xi(\Gamma_\lambda)$ . By (b),  $\gamma_\lambda$  is a  $C^1$  path from  $\xi_+ < 0$  to  $\xi_- > 0$  (both real) and otherwise lies in  $0 < \operatorname{Im} \xi < \pi$ . There  $1 - e^\xi$  lies in the lower half-plane, so  $\xi - \log(1 - e^\xi)$ , with  $\arg(1 - e^\xi) \in (-\pi, 0)$ , is a primitive of  $\frac{1}{1 - e^\xi}$ . Its imaginary part  $x - \arg(1 - e^\xi)$  goes from 0 at  $\xi_+$  (where  $1 - e^\xi > 0$ ) to  $\pi$  at  $\xi_-$  (where  $1 - e^\xi < 0$ ). And  $\psi$  is holomorphic on the convex strip  $|\operatorname{Im} \xi| < \pi$ , since  $E_a, E_1$  are entire and  $E_1 \neq 0$  there (its zeros are  $2\pi ik$ ,  $k \neq 0$ );  $\psi(0) = a$  and  $\psi$  is real on  $\mathbb{R}$ . So its primitive  $H(\xi) = \xi \int_0^1 \psi(s\xi) ds$  (indeed  $H' = \int_0^1 \partial_s(s\psi(s\xi)) ds = \psi$ ) is real on  $\mathbb{R}$ , and  $\int_{\gamma_\lambda} \psi d\xi = H(\xi_-) - H(\xi_+)$  is real. Hence  $|D_\lambda| = \pi(1 - \lambda)$ . This is the half-residue of  $\tau_\lambda$  at its pole  $\xi = 0$  (residue  $-(1 - \lambda)$ ), which  $\gamma_\lambda$  passes above. For  $\lambda \in \{0, 1\}$ , use  $0 < \Phi < 1$ .  $\square$

**Remarks.** (1) *Tools:* one trigonometric identity,  $y \cot y$  decreasing, two explicit primitives (no Cauchy theorem, residue or branch cut), Euler's Beta integral. The reflection formula is not used. At  $r = 1$  the integral is elementary,  $\int_0^\pi K = \frac{\sin \pi a}{ab}$  because  $(\sin^b ax \sin^a bx)' = abK$ , so the proof yields  $\Gamma(a)\Gamma(1-a) = \pi/\sin \pi a$ . It also gives Theorem 1 for  $-1 < r < 0$ . (2) *An explicit Archimedes map.* Let  $\theta = x - \arg(1 - e^\xi) - \operatorname{Im} H(\xi) = \operatorname{Im} Q(\xi)$ , where  $Q = \xi - \log(1 - e^\xi) - H$  has  $Q' = \frac{1}{1 - e^\xi} - \psi = \frac{e^{a\xi}}{1 - e^\xi}$ . Differentiating (\*) gives  $Q'(\xi) d\Phi \wedge d\xi = -dt \wedge d\xi$ , hence  $d\Phi \wedge d\theta = -dt \wedge dx$ . As  $\theta$  runs from 0 to  $\pi$  along each level arc (as in (d)),  $(\Phi, \theta)$  is an area-preserving bijection of  $R$  onto itself. For  $a = \frac{1}{2}$  it is the passage between cylindrical coordinates about two perpendicular axes of the sphere. (3) *Sources.* Conjectured by T. Amdeberhan, MathOverflow 253703 (2016); F. Petrov proved the integer case there by a contour shift and noted the equivalence with Theorem 2. In probability the identity is the moment formula of Kanter's variable (implicit in Kanter 1975, explicit in Devroye 2009; in binomial form for integer  $s$ , T. Simon 2014), proved via the Chernin–Ibragimov–Zolotarev formula; Theorem 2 then follows by the Shanbhag–Sreehari identity. The direct proof of Theorem 2 and the map in (2) are new to our knowledge. Figures, a second (finite-sum) proof and numerics: `MAIN_RESULT.pdf`.