

The sin-omial integral for $\binom{r}{s}$: a one-page elementary proof

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This proof was generated on October 21, 2026, by Grok Bot using Cursor's agents, guided by several prompts from Paata Ivanisvili

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Theorem 1. For real $r > 0$ and $0 \leq s \leq r$,

$$\frac{1}{\pi} \int_0^\pi \frac{\sin^r x}{\sin^s(\frac{s}{r}x) \sin^{r-s}(\frac{r-s}{r}x)} dx = \binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}.$$

Setup. For $s \in \{0, r\}$ both sides are 1 ($\sin^0 := 1$; $r = 0$ is excluded, as s/r is undefined). Let $0 < s < r$, $a = s/r$, $b = 1 - a$, and on $[0, \pi]$ let $R = \frac{\sin \theta}{\sin b\theta}$, $\rho = \frac{\sin a\theta}{\sin b\theta}$, $f = R\rho^{-a}$, with the limits $R(0) = \frac{1}{b}$, $\rho(0) = \frac{a}{b}$. As $\sin(c\theta)/\theta$ is entire, and positive on $[0, \pi]$ if $0 < c < 1$, $R \geq 0$ and $\rho > 0$ are smooth and f is continuous on $[0, \pi]$; $\rho(\pi) = 1$ as $\sin b\pi = \sin a\pi$. As $f = \sin \theta / (\sin^a a\theta \sin^b b\theta)$ on $(0, \pi)$, the integrand is $f(x)^r$. Write $\text{sinc } u = \frac{\sin \pi u}{\pi u}$, $\text{sinc } 0 = 1$, and $\binom{n}{\kappa} = \frac{\Gamma(n+1)}{\Gamma(\kappa+1)\Gamma(n-\kappa+1)}$ ($1/\Gamma$ is entire).

Idea. On the loop $z = \rho(\theta)e^{i\theta}$ the triangle $(-1, 0, z)$ has angles $a\theta, \pi - \theta, b\theta$, so $(1+z)^n z^{-an} = f(\theta)^n$ is real, with $z^{-an} := \rho^{-an}e^{-ian\theta}$. So f^n is the integrand of Cauchy's integral $\frac{1}{2\pi} \int_{-\pi}^\pi (1+e^{it})^n e^{-iant} dt = \binom{n}{an}$, moved from the unit circle onto this loop. For integer n it is $\sum_j \binom{n}{j} z^{j-an}$, with explicit primitives. Real exponents: by (*) below, the integer case says that $f(X)T^a(1-T)^b$ ($X \sim U(0, \pi)$, $T \sim U(0, 1)$ independent) has the moments $\frac{1}{n+1}$ of $U(0, 1)$, so, being bounded, it is uniform, as Petrov conjectured.

Lemma 1. $1 + \rho e^{i\theta} = R e^{ia\theta}$ for $0 \leq \theta \leq \pi$.

Proof. As $b\theta = \theta - a\theta$: $\sin b\theta + \sin a\theta e^{i\theta} = \sin \theta \cos a\theta - \cos \theta \sin a\theta + \sin a\theta \cos \theta + i \sin a\theta \sin \theta = \sin \theta e^{ia\theta}$. Divide by $\sin b\theta$; at $\theta = 0$ both sides equal $1/b$. $\square$

Lemma 2. For $n \in \mathbb{N}_0$ and real κ , $\sum_{j=0}^n \binom{n}{j} \text{sinc}(\kappa - j) = \binom{n}{\kappa}$.

Proof. For $n = 0$ it reads $\text{sinc } \kappa = 1/(\Gamma(1+\kappa)\Gamma(1-\kappa))$, the reflection formula. By Pascal's rule the left side $L_n(\kappa)$ satisfies $L_{n+1}(\kappa) = L_n(\kappa) + L_n(\kappa-1)$, and so does the right side, since $1/\Gamma(x) = x/\Gamma(x+1)$ gives $\binom{n}{\kappa} + \binom{n}{\kappa-1} = \frac{\Gamma(n+1)[(n+1-\kappa)+\kappa]}{\Gamma(\kappa+1)\Gamma(n+2-\kappa)} = \binom{n+1}{\kappa}$. Induct on n . $\square$

Lemma 3. For $n \in \mathbb{N}$, $\frac{1}{\pi} \int_0^\pi f^n d\theta = \sum_{j=0}^n \binom{n}{j} \text{sinc}(an - j)$.

Proof. Let $\mu_j = j - an$, $\lambda = \rho'/\rho$ and $z^\mu := \rho^\mu e^{i\mu\theta}$, so $(z^\mu)' = \mu(\lambda + i)z^\mu$. By the binomial theorem and Lemma 1, $\sum_j \binom{n}{j} z^{\mu_j} = \rho^{-an} e^{-ian\theta} (1 + \rho e^{i\theta})^n = \rho^{-an} R^n = f^n$. Hence

$$F = \sum_j \binom{n}{j} \frac{z^{\mu_j}}{\mu_j} \quad \left(\frac{z^0}{0} := \log \rho + i\theta \right) \quad \text{has} \quad F' = (\lambda + i) f^n.$$

As λ, f are real, $\Psi := \text{Im } F = \sum_j \binom{n}{j} \rho^{\mu_j} \frac{\sin \mu_j \theta}{\mu_j}$ (with θ for $\mu_j = 0$) has $\Psi' = f^n$, so $\int_0^\pi f^n = \Psi(\pi) - \Psi(0) = \pi \sum_j \binom{n}{j} \text{sinc } \mu_j$, since $\Psi(0) = 0$ and $\rho(\pi) = 1$; sinc is even. $\square$

Proof of the Theorem. Let $h(x, t) = f(x) t^a (1-t)^b$ on $Q = [0, \pi] \times [0, 1]$; it is continuous, so $0 \leq h \leq C$ for some $C \geq 1$. By Fubini and Euler's Beta integral, for real $r \geq 0$,

$$\frac{1}{\pi} \iint_Q h^r dx dt = \left(\frac{1}{\pi} \int_0^\pi f^r dx \right) \frac{\Gamma(ar+1)\Gamma(br+1)}{\Gamma(r+2)}. \quad (*)$$

For $r = n \in \mathbb{N}_0$, Lemmas 3 and 2 give $\frac{1}{\pi} \int_0^\pi f^n = \binom{n}{an}$ (trivially if $n = 0$), so (*) equals $\frac{1}{n+1} = \int_0^1 v^n dv$. Hence $\frac{1}{\pi} \iint_Q p(h) = \int_0^1 p$ for every polynomial p , and, approximating y^r uniformly on $[0, C] \supset [0, 1]$ by polynomials (Weierstrass), $\frac{1}{\pi} \iint_Q h^r = \frac{1}{r+1}$. By (*), $\frac{1}{\pi} \int_0^\pi f^r = \frac{\Gamma(r+2)}{(r+1)\Gamma(ar+1)\Gamma(br+1)} = \binom{r}{ar} = \binom{r}{s}$. $\square$

Sources. Conjectured by T. Amdeberhan, MathOverflow Q253703 (2016). There F. Petrov gives the loop (integer n, k) and proposes the uniform reformulation; A. Selby's comment on Petrov's Nov. 19 answer extends the contour argument to complex n, k , outlining a proof for real r, s , and his answer gives an antiderivative in $y = \theta/n$ for even n (odd n similar) and real k , equal to $\Psi(ny)/n$. For real r, s the identity is implicit in Kanter's representation of stable laws (Ann. Probab. 1975, Cor. 4.1) and explicit as Devroye's normalizing constant (ACM TOMACS 2009); T. Simon (EJP 2014, Prop. 2, Rem. 3(b)) states it in binomial form for integer s . These rest on the Chernin–Ibragimov–Zolotarev integral representation of stable densities. New here is only this particular elementary assembly, which we did not find in the literature.