Audit of the proofs of the sin-omial integral for the generalized binomial coefficient

Independent audit

October 21, 2026

Theorem. *For real $r > 0$ and $0 \le s \le r$,*

$$\frac{1}{\pi}\int_0^\pi \frac{\sin^r x}{\sin^s\left(\frac{s}{r}x\right)\,\sin^{r-s}\left(\frac{r-s}{r}x\right)}\,dx = \binom{r}{s} := \frac{\Gamma(r+1)}{\Gamma(s+1)\,\Gamma(r-s+1)}.$$

Throughout, $a = s/r$, $b = 1-a$, $\rho(x) = \sin(ax)/\sin(bx)$, $f(x) = \sin x/(\sin^a(ax)\sin^b(bx))$ (Write-up 2 writes K), so the integrand is f^r, and $h(t,x) = t^a(1-t)^b f(x)$ (Write-up 2 writes Φ). Verdicts: **VALID** (rebuilt from scratch, no gap), **GAP**, **INVALID**, **KNOWN** (correct and already in the literature). Simplicity is rated 1–5 against the benchmark (Petrov's MathOverflow proof completed by Selby's comment): 3 means the same weight as the benchmark, 5 means clearly simpler.

1 Verdict

1. **There are two different complete, correct, short proofs, and each exists as a self-contained one-page PDF:** Proof A as Write-up 1's `SHORT_PROOF.pdf` (v5.10) and Proof B as Write-up 2's `SHORT_PROOF.pdf` (v5.11). Every step marked PROVED was rebuilt by hand and checked numerically with independent code. The latest full line-by-line reads of both pages (Oct. 21, 13:41–13:51 UTC) found them complete and VALID, as did every read since 06:02 UTC. No INVALID step and no mathematical gap was found.

2. **Proof A (finite sums plus moments): VALID, simplicity 5/5.** Written up as a one-page PDF by Write-up 1 (`SHORT_PROOF.pdf`) and independently by Write-up 2 (Proof II). It uses no complex analysis: for an integer exponent n and real $s = an$ the integrand is a finite sum $\sum_j \binom{n}{j}\rho^{\mu_j}e^{i\mu_j x}$, which has an explicit primitive, so $\frac{1}{\pi}\int_0^\pi f^n = \sum_j \binom{n}{j}\operatorname{sinc}(an-j) = \binom{n}{an}$ (Pascal induction or partial fractions, plus the reflection formula). Then h has the moments $1/(n+1)$ of the uniform law, and Weierstrass passes from integer to real exponents. *Known ingredients* (Petrov's path, Selby's antiderivative, and Petrov's equidistribution reformulation, posed without proof, all on MathOverflow 253703), *new assembly.* Selby's comment there already outlines a proof for real r, s (the benchmark), so the novelty is the elementary assembly only, as Write-up 1 now says. Reproduced in Appendix A.

3. **Proof B (Write-up 2's half-residue proof of the equidistribution): VALID, simplicity 5/5.** It proves directly that $|\{h > \lambda\}| = \pi(1-\lambda)$. In the coordinate $\xi = \log\frac{(1-t)\rho(x)}{t} + ix$ the law of sines gives $t(1-e^\xi) = 1 - h\,e^{a\xi}$, so on a level curve t is the meromorphic function $\tau_\lambda(\xi) = (1-\lambda e^{a\xi})/(1-e^\xi)$, and the enclosed area $\int t\,dx = \operatorname{Im}\int \tau_\lambda\,d\xi$ is a half-residue. Since Write-up 2 v5.1 the half-residue is computed without residue calculus or Cauchy's theorem: $\tau_\lambda = \frac{1-\lambda}{1-e^\xi} + \lambda\psi(\xi)$ with $\psi = \frac{1-e^{a\xi}}{1-e^\xi}$. The pole part has the explicit primitive $\xi - \log(1-e^\xi)$, whose imaginary part increases by π along the arc, and ψ has the radial primitive $\xi\int_0^1 \psi(s\xi)\,ds$, which is real on $\mathbb{R}$. It needs no reflection formula (it reproves it, at $r = 1$), no binomial expansion and no moment argument, and it covers $-1 < r < 0$. Write-up 2's one-page version was checked line by line against the full version and is complete. Reproduced, in the v5.1 form, in Appendix B. *Provenance:* the statement is known. Kanter's representation gives $S^{-a} \overset{d}{=} f(\pi U)E^b$ for the positive a-stable S, and the Shanbhag–Sreehari identity says that $(Z/S)^a$ is exponential. Beta–gamma algebra then gives the uniform law of h. The auditor has not found the direct proof in the literature.

4. **Write-up 2's Theorem 3, an explicit area-preserving map: VALID.** Let $Q'(\xi) = e^{a\xi}/(1-e^\xi)$ with $Q(-\infty) = 0$, and $\theta = \operatorname{Im} Q(\xi(t,x))$; since v5, Write-up 2 writes $\theta = x - \arg(1-e^\xi) - \operatorname{Im} H(\xi)$ explicitly, and it is the same function. Then $(t,x) \mapsto (h,\theta)$ is a bijection of $(0,1)\times(0,\pi)$ onto itself with $dh \wedge d\theta = -dt \wedge dx$. So h and θ/π are independent uniforms, and for $a = \frac{1}{2}$ the map is Archimedes' theorem used twice. It is not needed for the identity, and it was not found in the literature. Reproduced in Appendix C.

5. **Other proofs, all VALID:** Write-up 2's series version (Newton's series and the sinc-interpolation identity $\sum_k \binom{r}{k} \operatorname{sinc}(s-k) = \binom{r}{s}$), simplicity 4/5; Write-up 1's one-contour proof, KNOWN, 3/5.

6. **The identity itself is KNOWN for real** r, s: it is implicit in Kanter's representation (1975; Kanter does not write the moments down, but they follow from his Cor. 4.1 in one line), and explicit as the normalizing constant of Devroye's 2009 "Zolotarev distribution"; both rest on the Chernin–Ibragimov–Zolotarev representation. For integer s and real $r \geq s$ it is in binomial form in T. Simon's paper (arXiv 2013, EJP 2014, Prop. 2, Rem. 3(b)), before MathOverflow (2016). Both write-ups attribute this correctly.

7. **Independent referee on Write-up 1's `SHORT_PROOF.pdf`: VALID, no gaps**, simplicity 4/5. Its final report lists seven one-line points M1–M7. They are the smoothness of ρ at 0; the base case of Lemma 2 at integer κ; $\Psi(0) = 0$; 0^0 at $s \in \{0, r\}$; the uniform law tagged "(Petrov)", though Petrov only conjectured it; Selby's comment missing from the Sources, with the novelty to be limited; and "coefficient extraction" at non-integer an. All are wording, and the auditor agrees with each. This audit had supplied M2 and M3 in its own rebuild without flagging them, and had not raised the other five before Write-up 1 fixed them. Write-up 1 v5.6 (07:52 UTC) fixes all seven, and every fix was checked. So were the report's checkable claims: the companion integral $\int_0^\pi \lambda f = 1.736$ at $a = 0.1$, $\max h = 1$, and the MathOverflow facts. The referee's 4/5 and the auditor's 5/5 rest on the same facts: Proof A is as long as the benchmark, uses strictly fewer tools, and keeps Petrov's mechanism.

8. **Write-up 2's Corollary 4 (v5.5; sources v5.11): VALID; statement KNOWN.** From the uniform law of h, with $E_1 = GT$ and $E_2 = G(1-T)$ for $G \sim \text{Gamma}(2)$, the variable $E_1^a E_2^b f(X) = G\,h$ is exponential. So $S = (E_2^b f(X))^{-1/a}$ has $\mathbb{E}e^{-\lambda S} = e^{-\lambda^a}$. This gives the existence of the positive stable law, Kanter's representation and Zolotarev's formula $P(S \leq s) = \frac{1}{\pi}\int_0^\pi \exp(-s^{-a/b} f(u)^{-1/b})\,du$, with no Laplace inversion. It is the Shanbhag–Sreehari equivalence read backwards, and it was checked numerically by the auditor, including against the closed-form $\frac{1}{3}$-stable density. Write-up 2 calls the derivation new. This is plausible: in Devroye–James (2014, §6, checked) Kanter's method is derived from the Ibragimov–Chernin–Zolotarev formula, and their table entry for $\log K_a$, which is the target at $r = -it$, has no stated source. Demni (ECP 2011, checked) explains the occurrence of Kanter's function by Fourier inversion and two deformations of the path. Kanter 1975, which neither write-up could access at first, was read by the auditor in an archived copy (and by Write-up 2 since v5.8): Cor. 4.1 is derived there from the Chernin–Ibragimov density formula ("stated in [1]") by "a trivial change of variable". So Write-up 2's tag is consistent with every source checked. Write-up 2's sources (v5.11) are accurate, after small fixes in v5.7–v5.11; the contour rotation is credited to Chernin–Ibragimov and Zolotarev, as Kanter (p. 702) and Demni (§5) do. Corollary 4 is not needed for the target.

Recommendation. Report Proof A in the form of Write-up 1's `SHORT_PROOF.pdf`: one page, no complex analysis. Present Proof B, in the form of Write-up 2's `SHORT_PROOF.pdf` (also one page), as the alternative. It is more conceptual and stronger, giving the full law of h, $-1 < r < 0$, the reflection formula as a by-product and the map of Theorem 3. Since Write-up 2 v5.1 it uses no residue and no Cauchy theorem either. Its complex numbers enter only through the chain rule, one logarithm and one radial integral. So both proofs are elementary, and the choice is a matter of taste. Proof A is real-variable, with finite sums, the reflection formula and Weierstrass. Proof B is geometric, needs neither, and proves more. Both are VALID.

2 Proofs compared

Proof	Scope	Length	Tools	Verdict	Simpl.
Petrov + Selby (benchmark)	real r, s	$\sim$2/3 p.	Cauchy's theorem twice past branch points, Beta, reflection, continuation	KNOWN	3
A: finite sums + moments (Write-up 1 `SHORT_PROOF`; Write-up 2 Proof II)	real $r > 0$, $0 \le s \le r$	1 p.	law of sines, binomial theorem, one derivative, Pascal or partial fractions, reflection, Beta, Weierstrass	VALID; new assembly	**5**
B: half-residue (Write-up 2 `SHORT_PROOF`; Proof I)	real $r > -1$ at fixed a; law of h	$\sim$3/4 p. (1 p. with statement)	law of sines, $y \cot y$ decreasing, two explicit primitives (a logarithm and a radial integral; no Cauchy theorem, no residue), Beta, layer cake	VALID; statement KNOWN, proof not found in the literature	**5**
Series (Write-up 2 Remark 1)	real $r > 0$	$\sim$1 p.	Newton's series on $\lvert w\rvert \le 1$, M-test, Beta, reflection, recursion	VALID	4
One contour (Write-up 1 §4)	real r; family in $\sigma \in \mathbb{C}$	$\sim$1 p.	one contour, Beta, reflection, identity theorem	VALID, KNOWN	3

3 Proof A: step-by-step verdicts

As written in Write-up 1's `SHORT_PROOF.pdf`; Write-up 2's Proof II is the same argument in the notation $e_m = \rho^m e^{imx}$, with partial fractions in place of the Pascal induction.

Step	Verdict	Auditor's notes
Setup	VALID	$0 < a\theta, b\theta < \pi$ on $(0, \pi]$, so ρ is smooth and positive on $[0, \pi]$, with $\rho(\pi) = 1$. Only real powers of positive numbers occur.
Lemma 1 (law of sines)	VALID	One-line expansion using $b\theta = \theta - a\theta$.
Lemma 2 (sinc sum)	VALID	$n = 0$ is the reflection formula, written since v5.4 as $\operatorname{sinc}\kappa = 1/(\Gamma(1+\kappa)\Gamma(1-\kappa))$, which holds for all real κ (both sides 0 at nonzero integers). Pascal's rule holds on both sides for all real κ, since $1/\Gamma$ is entire and $1/\Gamma(x) = x/\Gamma(x+1)$.
Lemma 3 (integer n)	VALID	The finite sum equals f^n by the binomial theorem and Lemma 1, with no branch. Since v5.7: $z^\mu := \rho^\mu e^{i\mu\theta}$ is a function of θ with $(z^\mu)' = \mu(\lambda + i)z^\mu$, so $F = \sum_j \binom{n}{j} z^{\mu_j}/\mu_j$ ($\log\rho + i\theta$ for $\mu_j = 0$) has $F' = (\lambda + i)f^n$, and $\Psi = \operatorname{Im} F$ has $\Psi' = f^n$ as $\lambda = \rho'/\rho$ is real. Ψ is the same function as before. $\Psi(0) = 0$, and $\Psi(\pi)$ needs only $\rho(\pi) = 1$.
Integer $\to$ real	VALID	h is continuous and $0 \le h \le C$ with $C \ge 1$. Uniform approximation of y^r on $[0, C]$ passes to both sides, because the normalized area is 1 and $[0, 1] \subset [0, C]$.
Edge cases	VALID	$s \in \{0, r\}$: the integrand is 1, with $\sin^0 := 1$ (stated since v5.4; needed, as $\sin(0)^0 = 0^0$). $r = 0$ is excluded, as s/r is undefined (stated since v5.6). $0 < r < 1$ is covered, because Lemma 3 is used at the same a with integer n.

Hostile checklist: no branch cuts, no analytic continuation; at $x = \pi$ only $\rho(\pi) = 1$ and the continuity of f are used. Write-up 1 fixed the auditor's two cosmetic points: since v5.10 the Sources give Selby's antiderivative as $I_{n,k}(y) = \Psi(ny)/n$ (v5 dropped the $1/n$, a slip this audit missed), and since v5 the "Idea" paragraph defines $z^{-an} = \rho^{-an} e^{-ian\theta}$.

4 Proof B: step-by-step verdicts

Step	Verdict	Auditor's notes
B ⇒ Theorem	VALID	Layer cake: $\iint h^r = \int_0^1 r\lambda^{r-1}\pi(1-\lambda)\,d\lambda = \pi/(r+1)$; Fubini and Beta. For $-1 < r < 0$, $\iint h^r = \pi\int_0^1 \lambda^r d\lambda$, and $B(ar+1, br+1)$ is finite.
(a) law of sines	VALID	z^a is *defined* as $\rho^a e^{iax}$; no branch is chosen anywhere.
(b) key identity	VALID	$e^{a\xi} = (\frac{1-t}{t})^a \rho^a e^{iax}$, so $he^{a\xi} = (1-t)(1+z) = 1-t+te^{\xi}$. On $x = 0$, ξ is real, decreasing in t, and 0 only at $t = a$; $e^{\xi} = 1$ only at $(a, 0)$, since $0 \le x < \pi$.
(c) the region	VALID	$c(y) = y\cot y$ decreases and $ax, bx < x$, so $f'/f < 0$; g is unimodal. $\nabla h \neq 0$ on the level arc ($\partial_t h = 0$ only at $t = a$, where $\partial_x h \neq 0$ for $x > 0$). The orientation over the top gives $\int_\Gamma t\,dx = \int (t_+ - t_-)\,dx$.
(d) half-residue, v4.2 (slit strip)	VALID	The only pole in $\lvert\operatorname{Im}\xi\rvert < \pi$ is $\xi = 0$, simple, residue $-(1-\lambda)$. The slit strip $V = \{\lvert\operatorname{Im}\xi\rvert < \pi\} \setminus i(-\pi, 0]$ is star-shaped about $i\pi/2$: if $\operatorname{Re} p \neq 0$, the segment from $i\pi/2$ to p meets the imaginary axis only at $i\pi/2$; if $\operatorname{Re} p = 0$, then $p = iy$ with $y > 0$. So τ_λ has a primitive on V. Both the image of the level arc and the indented real path lie in V. The imaginary part is exactly $\pi(1-\lambda)$ for every small ε.
(d′) half-residue, v5.1 (split)	VALID	$\tau_\lambda = \frac{1-\lambda}{1-e^{\xi}} + \lambda\psi$, $\psi = \frac{1-e^{a\xi}}{1-e^{\xi}}$. On the open arc $0 < \operatorname{Im}\xi < \pi$, so $1-e^{\xi}$ is in the open lower half-plane, where the principal log is holomorphic. $\xi - \log(1-e^{\xi})$ is a primitive of $\frac{1}{1-e^{\xi}}$, continuous up to the ends, with $\arg(1-e^{\xi}) \to 0$ at ξ_+ and $\to -\pi$ at ξ_-: imaginary part π. $\psi = E_a/E_1$, $E_c(\xi) = \frac{e^{c\xi}-1}{\xi}$ (entire; $E_1 \neq 0$ on the strip), is holomorphic on $\lvert\operatorname{Im}\xi\rvert < \pi$. $H(\xi) = \xi\int_0^1 \psi(s\xi)\,ds$ has $H' = \int_0^1 \partial_s(s\psi(s\xi))\,ds = \psi$ and is real on $\mathbb{R}$, so $\int \psi\,d\xi = H(\xi_-) - H(\xi_+) \in \mathbb{R}$. No Cauchy theorem, residue or limit. Auditor numerics: both increments to 10^{-20}, real and imaginary parts.
$\lambda \in \{0, 1\}$	VALID	$0 < h < 1$ on the open rectangle.

Hostile checklist:

- no branch points or continuation; step (d′) uses the principal logarithm on the open lower half-plane, which contains $1 - e^{\xi}$ along the open arc, so no cut is crossed;
- for $\lambda > 0$ the arc stays at height $\le x_\lambda < \pi$, far from the pole at $2\pi i$;
- for fixed $a \in (0, 1)$ the endpoints $\xi_\pm$ are finite (checked at $a = 0.02$, $\xi_- = 150.8$, and $a = 0.97$, $\xi_+ = -100.9$);
- the map $(t, x) \mapsto \xi$ reverses orientation, but only the direction of the image path is used.

Write-up 2's earlier version (05:15 UTC) used Cauchy's theorem on the Jordan domain $\xi(D_\lambda)$. It was also correct, since ξ is a diffeomorphism onto $\mathbb{R} \times [0, \pi)$, but the slit-strip version does not need injectivity. The v5.1 split is simpler again: it needs neither V nor its star-shapedness nor the half-circle limit. Write-up 2's one-page `SHORT_PROOF.pdf` contains every step of the table, in the same order; since v5.1 its step (d) is (d′). Up to v5.3 it left two cosmetic points implicit: the holomorphy of ψ at 0 ($\psi = E_a/E_1$) and the cases $\lambda \in \{0, 1\}$. Write-up 2 v5.4 (07:11 UTC) writes out both, and the page is complete with nothing left to remark.

5 Write-up 2's Theorem 3: step-by-step verdicts

Here $Q(\xi) = \int_{-\infty}^{\xi} e^{a\eta}(1-e^{\eta})^{-1}d\eta$ on V, $\theta = \operatorname{Im} Q(\xi(t, x))$ and $R = (0, 1) \times (0, \pi)$.

Step	Verdict	Auditor's notes
Definition	VALID	The integrand is $O(e^{a\,\mathrm{Re}\,\eta})$ as $\mathrm{Re}\,\eta \to -\infty$ and $O(e^{-b\,\mathrm{Re}\,\eta})$ as $\mathrm{Re}\,\eta \to +\infty$, uniformly in the strip. Q is a primitive on the star-shaped V, and $\xi(R) \subset \{0 < \mathrm{Im}\,\xi < \pi\} \subset V$.
Jacobian -1	VALID	Differentiate $he^{a\xi} = 1 - t + te^{\xi}$, wedge with $d\xi$, divide by $1 - e^{\xi} \neq 0$: $dh \wedge dQ = -dt \wedge d\xi$. Take imaginary parts, since h and t are real.
Monotone on level arcs	VALID	For v tangent to $\{h = \lambda\}$ and any w with $dh(w) \neq 0$: $dh(w)\,d\theta(v) = -(dt \wedge dx)(w, v) \neq 0$.
End values 0, π	VALID	Q is real on $(-\infty, 0)$. Passing above the simple pole at 0 (residue -1) adds πi, so $\mathrm{Im}\,Q(\xi_-) = \pi$. In Write-up 2's v5 form, $\theta = x - \arg(1 - e^{\xi}) - \mathrm{Im}\,H(\xi)$ (the same function: the two primitives differ by a real constant), and the end values come from (d′) with no residue.
Bijection	VALID	$0 < h < 1$ on R, each level set is one arc, and θ maps it monotonically onto $(0, \pi)$.
$a = \frac{1}{2}$	VALID	$\theta = \arg\left(2t - 1 + 2i\sqrt{t(1-t)}\sin\frac{x}{2}\right)$, the longitude about a second axis of the sphere. The principal branch is the right one.

Hostile checklist: Q' is meromorphic, so there are no branch points. As $x \to \pi$, ξ tends to the line $\mathrm{Im}\,\xi = \pi$, where Q' is regular. Theorem 3 uses the same two ingredients as Proof B, identity (B1) and the half-residue. It repackages Proof B rather than giving a third proof. Simplicity 5/5 as an addendum.

6 Other proofs and remarks

Write-up 1, one-contour proof (unchanged since v1): $\mathrm{Log}(1 + e^w)$ on $|\mathrm{Im}\,w| < \pi$, continuous up to the cut edges, with the corners $\pm i\pi$ needing $r > 0$. Then $F(\sigma)$ is entire, and $F(s)$ equals the integral (the principal Log on the path is $\log R + ia\theta$). The collapse for $-1 < \sigma < 0$ uses dominated convergence, Beta and reflection, and the identity theorem is applied on a fixed path. VALID, KNOWN.
Write-up 2, series version: VALID; it needs $a \leq \frac{1}{2}$ (else $\rho(0) > 1$). v5.12 fixes a hint at $k = 0$ (Write-up 2's find).
Remarks:

- $r = 1$ *is equivalent to the reflection formula* (both write-ups: $G = \sin^b(ax)\sin^a(bx)$ has $G' = abf$). VALID. Write-up 1 v5.1 states the consequence correctly: every proof implies the reflection formula, by using it (Proof A) or by reproving it (Proof B). This is not circular, because the Beta integral is proved without the reflection formula.
- *Extension to* $-1 < \mathrm{Re}\,r$: VALID.
- *Isochronous Hamiltonian flow of* h *with period* 2π: VALID; θ of Theorem 3 is its angle variable.
- *Kanter's factorization and Petrov's exponential reformulation*: KNOWN, with correct attributions. Write-up 2 v4.1 derives Petrov's claim from Kanter and Shanbhag–Sreehari. The derivation was checked.

7 Claims registers

Write-up 1 claims register (C0–C18).

- C0: reproduced.
- C1–C4, C6, C8, C13–C18: VALID. C15 is Proof A, at 5/5.
- C5, C7: VALID, KNOWN.
- C9: VALID as an equivalence.
- C10 and C12 (CONDITIONAL): tags accepted.
- C11 (FAILED: Carlson in s): the obstruction is correct.

Write-up 2 claims register (C0–C15).

- C0: reproduced.
- C1–C12: VALID. C4–C6 are Proof B (C4 checked in its v4.2 and v5.1 forms); C7–C8 are Proof A; C10 is the reflection formula as a corollary; C12 is the isochronous flow. C3 (the monotonicity of f) is in Kanter 1975, p. 704; Write-up 2 does not claim it as new, and credits Kanter since v5.8. C6 ($-1 < r < 0$) is also checked numerically, by Write-up 2 and independently by the auditor, down to $r = -0.999$.
- C13: KNOWN (Kanter; Shanbhag–Sreehari; Simon 2014; Hasebe–Simon–Wang 2020).

- C14: superseded by C15.
- C15: VALID. It is Theorem 3.
- C16 (v5.5, sources v5.11): VALID; statement KNOWN. It is Corollary 4 (stable laws from Proof B).

8 Literature and provenance

- The statement image reproduces the MathOverflow question verbatim, sentence and LaTeX alike. The formula was never in the English Wikipedia article "Binomial coefficient": none of the 615 revisions live at any time since November 2016 contains it, and a regex search of all current English Wikipedia articles does not find it. The label "from Wikipedia" comes from the problem statement. Both write-ups had said the article "no longer" contains it, which was inaccurate but affected no verdict. Both withdrew this after the auditor's 06:24 report (Write-up 1 v5.3, Write-up 2 v5.3).
- MathOverflow 253703 (Amdeberhan, Nov. 2016). Petrov's answer (19 Nov.) proves the integer case along the law-of-sines path. Selby's comment sketches the real case: collapse onto the cut, then Beta, reflection and continuation. Selby's answer gives an explicit antiderivative for real k, written out for even n ("n odd is similar with cosines"). Selby's comment under Petrov's answer begins "It works for any (incl complex) values of n and k too". Petrov's post of 3 Nov., marked "(not an answer)", poses the equidistribution of h as what "we have to prove" and states its exponential form as a "Claim". His comment there adds "if they are equidistributed, all moments are equal, including these for non-integer n". There is no proof. The auditor read all 42 comments.
- M. Kanter, Ann. Probab. 3 (1975), Cor. 4.1: the identity is the moment formula of the Kanter random variable $f(\pi U)$ (Simon's normalization; Kanter's b is $1/f$). Read in the Internet Archive's copy: Cor. 4.1 (p. 703) follows from the Chernin–Ibragimov density formula, his Lemma 4.1, and p. 704 proves that f is strictly decreasing and log-concave on $(0,\pi)$ (via $\alpha\cot\alpha\varphi > \cot\varphi$), as used in Proof B. The paper (all 11 pages read) does not compute these moments, which follow from Cor. 4.1 in one line; the first explicit statement seen is Devroye 2009. The Gamma side appeared earlier as a Mellin transform (Pestana–Shanbhag–Sreehari 1977, p. 292, via Demni; not seen; it cites neither Kanter nor Chernin–Ibragimov).
- T. Simon, arXiv:1310.1888 (EJP 19 (2014)), §3, Prop. 2 and Rem. 3(b): Młotkowski–Penson's law with moments $\binom{np}{n}$, $p \ge 1$, is the law of $f_p(U)$, $f_p(u) = \sin^p(\pi u)/(\sin(\pi u/p)\sin^{p-1}((1-\frac{1}{p})\pi u))$. Since f_p^n is the integrand at $(r,s) = (np, n)$, this is the identity for integer s, before MathOverflow. Hasebe–Simon–Wang, AIHP 56 (2020), Rem. 1(b), restate it as $\mathbb{E}[K_\alpha^{-n}] = \binom{n/\alpha}{n}$. Both passages were checked in the arXiv texts.
- D. N. Shanbhag, M. Sreehari, Z. Wahrsch. 38 (1977): $(Z/S)^a$ is exponential for Z exponential and S positive a-stable, since $P((Z/S)^a > y) = \mathbb{E}e^{-y^{1/a}S} = e^{-y}$. Together with Kanter this gives Theorem B as a statement. Writing $Z = GT$, $E = G(1-T)$ with $G \sim$ Gamma(2), the product $Z^a E^b f(\pi U)$ equals Gh, and GU is exponential. Both input identities are displayed on one page of Devroye–James, Stat. Methods Appl. 23 (2014), as "gamma-stable calculus" and "stable calculus". They are not combined there.
- Hasebe–Simon–Wang (2020), eq. (5): $\mathbb{E}[K_\alpha^s] = \Gamma(1-s/\alpha)/(\Gamma(1-(1/\alpha-1)s)\Gamma(1-s))$ for $s < \alpha$. With their eq. (6), $K_\alpha = f(\pi U)^{-1/\alpha}$ at $a = \alpha$, this is the identity in Gamma form for every real $r > -1$.
- Młotkowski–Penson, arXiv:1507.07312, eqs. (6)–(7), stated there without a citation (nonnegativity for $r \le p$ is credited to Haagerup–Möller, Neuschel and Forrester–Liu): the Raney densities, with moments $\binom{np+r}{n}\frac{r}{np+r}$, are carried by $x = \sin^p(p\varphi)/(\sin\varphi\,\sin^{p-1}((p-1)\varphi))$, $0 < \varphi < \pi/p$. This is $f(y)^p$ at $a = 1/p$, $y = p\varphi$, but with a non-uniform weight in φ. The uniform weight gives $\binom{np}{n}$, which is Simon's representation. Their earlier paper arXiv:1309.0595 treats the law with moments $\binom{np}{n}$ through Meijer G-functions only. Its existence goes back to Młotkowski, Doc. Math. 15 (2010). All checked in the sources (Write-up 2's 07:18 note, Write-up 1's 07:34 note).
- L. Devroye, ACM TOMACS 19 (2009): the "Zolotarev random variable" has density $C\,B(x)^b$ on $[0,\pi]$ with $C = \Gamma(1+b\alpha)\Gamma(1+b(1-\alpha))/(\pi\Gamma(1+b))$; its normalization is the identity.

9 Numerical checks by the auditor

All checks use mpmath at 20–40 digits; the scripts are in `numerics/`.

- The identity: 22 cases, worst relative error $2\cdot10^{-41}$. It also holds for $-1 < r < 0$, to $3\cdot10^{-26}$ down to $r = -0.99$, and by a second method (the end $x = \pi$ computed in $\pi - x = \frac{\pi}{2}e^{-u}$), for nine pairs down to $r = -0.999$, to $1.1\cdot10^{-30}$ (Write-up 2's C6).

- Proof A:
  - Lemma 2 by finite sinc sums, error 10^{-29};
  - Lemma 3, pointwise and integrated, about 20 digits; in its v5.7 form, for seven pairs (a, n), $F' = (\lambda + i)f^n$ to $5 \cdot 10^{-40}$ and $\Psi(\pi) - \Psi(0) = \pi\binom{n}{an}$ to $2 \cdot 10^{-41}$; Selby's form $\Psi(ny)/n$ to $2 \cdot 10^{-27}$;
  - the uniform law of h through its distribution function, $P(h \le v) = v$ to 15 digits.
- Proof B, with a parametrization of the level arcs different from Write-up 2's, and $\tau_\lambda(\xi(t, x))$ evaluated rather than replaced by t, for $a \in \{0.02, 0.3, 0.5, 0.97\}$ and $\lambda \in \{0.05, 0.6, 0.97\}$:
  - the identity $t(1 - e^\xi) = 1 - he^{a\xi}$ to $7 \cdot 10^{-24}$;
  - $|\{h > \lambda\}| = \pi(1 - \lambda)$ to $6 \cdot 10^{-26}$;
  - $\operatorname{Im} \int \tau_\lambda \, d\xi = \pi(1 - \lambda)$ to $2 \cdot 10^{-27}$;
  - the real part equals the principal value on the real axis to 10^{-25};
  - the Hamiltonian period is 2π to 10^{-25}.
- Proof B, step (d′) of Write-up 2 v5.1, on the same arcs, for five pairs (a, λ) including $a = 0.02$ and 0.97:
  - $\operatorname{Im}(1 - e^\xi) < 0$ at interior points of the arc;
  - $\int \frac{d\xi}{1 - e^\xi}$ equals the increment of $\xi - \log(1 - e^\xi)$, real and imaginary parts, to 10^{-20};
  - $\int \psi \, d\xi$ equals $H(\xi_-) - H(\xi_+)$, with H integrated along the real axis, to $6 \cdot 10^{-22}$;
  - Write-up 2's v5 formula $\theta = x - \arg(1 - e^\xi) - \operatorname{Im} H(\xi)$ equals the auditor's two formulas for θ at 36 points, including points near the edges, to $3 \cdot 10^{-19}$.
- Theorem 3. Here θ is computed in two ways that Write-up 2 does not use, along the line $\operatorname{Im} \xi = \pi$ and by the incomplete Beta function, and the two agree to $1.6 \cdot 10^{-25}$:
  - the Jacobian, by finite differences, is -1 to $1.3 \cdot 10^{-26}$ at 60 points, including $a = 0.02$ and 0.98;
  - θ increases from 0 to π along level arcs;
  - $|\{\theta < \theta_0\}| = \theta_0$ and $|\{h > \lambda, \theta < \theta_0\}| = (1 - \lambda)\theta_0$, by quadrature, to $1.7 \cdot 10^{-21}$;
  - for $a = \frac{1}{2}$, θ is the longitude to $5 \cdot 10^{-26}$;
  - the Hamiltonian flow, integrated as an ODE, has half-period π to 10^{-11}.
- Write-up 2's Corollary 4: $\mathbb{E}e^{-\lambda S} = e^{-\lambda^a}$ from the representation, for $a \in \{0.2, 0.45, 0.8\}$ and $\lambda \in \{0.3, 2.5\}$, to $5 \cdot 10^{-19}$. At $a = \frac{1}{3}$, against a closed form: the integral of the density $x^{-3/2}K_{1/3}(2/\sqrt{27x})/(3\pi)$ agrees with Zolotarev's formula to $4 \cdot 10^{-21}$.
- Write-up 2's series version (Remark 3): $\sum_k \binom{r}{k} \operatorname{sinc}(s - k) = \binom{r}{s}$, with the slowly convergent tail summed by Euler–Maclaurin, for ten pairs (r, s), to $2 \cdot 10^{-40}$; its corrected hint, three ways, nine pairs, $8 \cdot 10^{-41}$.
- All scripts of the write-ups were re-run and reproduce, including Write-up 2's Monte Carlo test of Theorem 3 (3×10^6 points for each of four values of a; chi-square p-values 0.065–0.935) and Write-up 2's v5.2 check of the split (worst error $3.4 \cdot 10^{-20}$, including the real parts).

A Proof A, as audited

Transcribed from Write-up 1's `SHORT_PROOF.pdf` *(v5.7; the argument is unchanged since v4, and since v5.7 Lemma 3 is written with one explicit complex primitive).* For $s \in \{0, r\}$ both sides are 1 ($\sin^0 := 1$; $r = 0$ is excluded, as s/r is undefined). Let $0 < s < r$, $a = s/r$, $b = 1 - a$, and on $[0, \pi]$ let $R = \frac{\sin\theta}{\sin b\theta}$, $\rho = \frac{\sin a\theta}{\sin b\theta}$ and $f = R\rho^{-a}$, with the limits $R(0) = \frac{1}{b}$, $\rho(0) = \frac{a}{b}$. As $\sin(c\theta)/\theta$ is entire, and positive on $[0, \pi]$ if $0 < c < 1$, $R \ge 0$ and $\rho > 0$ are smooth and f is continuous on $[0, \pi]$; $\rho(\pi) = 1$ as $\sin b\pi = \sin a\pi$. As $f = \sin\theta/(\sin^a a\theta \, \sin^b b\theta)$ on $(0, \pi)$, the integrand is $f(x)^r$. Write $\operatorname{sinc} u = \frac{\sin \pi u}{\pi u}$, $\operatorname{sinc} 0 = 1$, and $\binom{n}{\kappa} = \frac{\Gamma(n+1)}{\Gamma(\kappa+1)\Gamma(n-\kappa+1)}$ with $1/\Gamma$ entire.

Lemma 1. *$1 + \rho e^{i\theta} = R\, e^{ia\theta}$ for $0 \le \theta \le \pi$.*

Proof. As $b\theta = \theta - a\theta$: $\sin b\theta + \sin a\theta \, e^{i\theta} = \sin\theta\cos a\theta - \cos\theta\sin a\theta + \sin a\theta\cos\theta + i\sin a\theta\sin\theta = \sin\theta\, e^{ia\theta}$. Divide by $\sin b\theta$; at $\theta = 0$ both sides equal $1/b$. □

Lemma 2. *For $n \in \mathbb{N}_0$ and real κ: $\sum_{j=0}^{n} \binom{n}{j} \operatorname{sinc}(\kappa - j) = \binom{n}{\kappa}$.*

Proof. For $n=0$ it reads $\operatorname{sinc}\kappa = 1/(\Gamma(1+\kappa)\Gamma(1-\kappa))$, the reflection formula. By Pascal's rule the left side $L_n(\kappa)$ satisfies $L_{n+1}(\kappa) = L_n(\kappa) + L_n(\kappa-1)$. So does the right side, since $1/\Gamma(x) = x/\Gamma(x+1)$ gives

$$\binom{n}{\kappa} + \binom{n}{\kappa-1} = \frac{\Gamma(n+1)\,[(n+1-\kappa)+\kappa]}{\Gamma(\kappa+1)\,\Gamma(n+2-\kappa)} = \binom{n+1}{\kappa}.$$

Induct on n. □

Lemma 3. *For $n \in \mathbb{N}$: $\frac{1}{\pi}\int_0^\pi f^n\,d\theta = \sum_{j=0}^n \binom{n}{j}\operatorname{sinc}(an-j)$.*

Proof. Let $\mu_j = j - an$, $\lambda = \rho'/\rho$ and $z^\mu := \rho^\mu e^{i\mu\theta}$, so $(z^\mu)' = \mu(\lambda+i)z^\mu$. By the binomial theorem and Lemma 1,

$$\sum_j \binom{n}{j} z^{\mu_j} = \rho^{-an}e^{-ian\theta}(1+\rho e^{i\theta})^n = \rho^{-an}R^n = f^n.$$

Hence

$$F = \sum_j \binom{n}{j}\frac{z^{\mu_j}}{\mu_j} \quad \left(\frac{z^0}{0} := \log\rho + i\theta\right) \qquad \text{has} \qquad F' = (\lambda+i)\,f^n.$$

As λ, f are real, $\Psi := \operatorname{Im} F = \sum_j \binom{n}{j}\rho^{\mu_j}\frac{\sin\mu_j\theta}{\mu_j}$ (with θ for $\mu_j = 0$) has $\Psi' = f^n$, so $\int_0^\pi f^n = \Psi(\pi) - \Psi(0) = \pi\sum_j \binom{n}{j}\operatorname{sinc}\mu_j$, since $\Psi(0) = 0$ and $\rho(\pi) = 1$; sinc is even. □

Proof of the Theorem. Let $h(x,t) = f(x)\,t^a(1-t)^b$ on $Q = [0,\pi]\times[0,1]$; it is continuous, so $0 \le h \le C$ for some $C \ge 1$. By Fubini and Euler's Beta integral, for real $r \ge 0$,

$$\frac{1}{\pi}\iint_Q h^r\,dx\,dt = \left(\frac{1}{\pi}\int_0^\pi f^r\,dx\right)\frac{\Gamma(ar+1)\,\Gamma(br+1)}{\Gamma(r+2)}. \tag{$*$}$$

For $r = n \in \mathbb{N}_0$, Lemmas 3 and 2 give $\frac{1}{\pi}\int_0^\pi f^n = \binom{n}{an}$ (trivially if $n=0$), so $(*)$ equals $\frac{1}{n+1} = \int_0^1 v^n\,dv$. Hence $\frac{1}{\pi}\iint_Q p(h) = \int_0^1 p$ for every polynomial p. Approximating y^r uniformly on $[0,C] \supset [0,1]$ by polynomials (Weierstrass) gives $\frac{1}{\pi}\iint_Q h^r = \frac{1}{r+1}$. By $(*)$, $\frac{1}{\pi}\int_0^\pi f^r = \frac{\Gamma(r+2)}{(r+1)\Gamma(ar+1)\Gamma(br+1)} = \binom{r}{ar} = \binom{r}{s}$. □

B Proof B, as audited

Transcribed from Write-up 2's `MAIN_RESULT.pdf` (v5.4; step (d) rewritten in v5/v5.1, the rest unchanged since v3.1), Section 1, in the notation above; Write-up 2's `SHORT_PROOF.pdf` is the same proof. The bracket in step (d), the holomorphy of ψ at 0, was the auditor's addition; Write-up 2 writes it out since v5.4. Let $0 < a < 1$, $g(t) = t^a(1-t)^b$ and $h(t,x) = g(t)f(x)$ on $(0,1)\times(0,\pi)$.

Theorem B (equidistribution). *For $0 \le \lambda \le 1$: $|\{(t,x) \in (0,1)\times(0,\pi) : h(t,x) > \lambda\}| = \pi(1-\lambda)$.*

Theorem B implies the Theorem. For $r > 0$ the layer-cake formula gives $\frac{1}{\pi}\iint h^r = \int_0^1 r\lambda^{r-1}(1-\lambda)\,d\lambda = \frac{1}{r+1}$. By Fubini and Euler's Beta integral, $\frac{1}{\pi}\iint h^r = B(ar+1, br+1)\frac{1}{\pi}\int_0^\pi f^r$. Hence $\frac{1}{\pi}\int_0^\pi f^r = \frac{\Gamma(r+1)}{\Gamma(ar+1)\Gamma(br+1)} = \binom{r}{s}$.

(a) Law of sines. Let $z = \rho e^{ix}$ and $z^a := \rho^a e^{iax}$. Lemma 1 and $f\rho^a = \frac{\sin x}{\sin bx}$ give $1+z = f(x)\,z^a$ for $0 \le x \le \pi$.

(b) The coordinate ξ. For $(t,x) \in (0,1)\times[0,\pi)$ let $\xi = \log\frac{(1-t)\rho(x)}{t} + ix$. On $x=0$, $\xi = \log\frac{(1-t)a}{tb}$ is real, decreasing in t, and 0 at $t=a$; and $e^\xi \ne 1$ except at $(a,0)$. Since $e^\xi = \frac{1-t}{t}z$ and $e^{a\xi} = (\frac{1-t}{t})^a z^a$, (a) gives $he^{a\xi} = (1-t)fz^a = (1-t)(1+z) = 1-t+te^\xi$, that is,

$$t\,(1-e^\xi) = 1 - h(t,x)\,e^{a\xi}. \tag{B1}$$

(c) The region $D_\lambda = \{h > \lambda\}$. g increases on $[0,a]$ and decreases on $[a,1]$, with $g(a) = a^a b^b$. f decreases strictly from $f(0) = a^{-a}b^{-b}$ to $f(\pi) = 0$: with $c(y) = y\cot y$, which is decreasing on $(0,\pi)$,

$$\frac{f'}{f} = \cot x - a^2\cot ax - b^2\cot bx = \frac{c(x) - a\,c(ax) - b\,c(bx)}{x} < 0.$$

So $h < 1$ on $(0,1)\times(0,\pi)$, and $\nabla h \ne 0$ on $(0,1)\times[0,\pi)$ except at $(a,0)$. Fix $0 < \lambda < 1$ and let $x_\lambda \in (0,\pi)$ solve $g(a)f(x_\lambda) = \lambda$. For $0 \le x \le x_\lambda$ let $t_-(x) \le a \le t_+(x)$ be the two roots of $h(\cdot,x) = \lambda$. Then $D_\lambda = \{0 < x < x_\lambda,\ t_-(x) < t < t_+(x)\}$. The level arc $\Gamma_\lambda = \{(t_\pm(x), x) : 0 \le x \le x_\lambda\}$ is C^1, oriented from $(t_+(0),0)$ to $(t_-(0),0)$, and $|D_\lambda| = \int_0^{x_\lambda}(t_+ - t_-)\,dx = \int_{\Gamma_\lambda} t\,dx$.

(d) The half-residue. On Γ_λ, (B1) gives

$$t = \tau_\lambda(\xi) := \frac{1-\lambda e^{a\xi}}{1-e^\xi} = \frac{1-\lambda}{1-e^\xi} + \lambda\,\psi(\xi), \qquad \psi(\xi) := \frac{1-e^{a\xi}}{1-e^\xi}.$$

As t is real, $t\,dx = \mathrm{Im}(t\,d\xi)$, so with $\gamma_\lambda = \xi(\Gamma_\lambda)$

$$|D_\lambda| = (1-\lambda)\ \mathrm{Im}\int_{\gamma_\lambda} \frac{d\xi}{1-e^\xi} + \lambda\ \mathrm{Im}\int_{\gamma_\lambda} \psi(\xi)\,d\xi.$$

The path γ_λ runs from $\xi_+ < 0$ to $\xi_- > 0$ on the real axis and, apart from its endpoints, lies in $0 < \mathrm{Im}\,\xi \le x_\lambda < \pi$.

- There $\mathrm{Im}\,e^\xi > 0$, so $1-e^\xi$ lies in the open lower half-plane, where the principal logarithm is holomorphic; $F(\xi) = \xi - \log(1-e^\xi)$ has $F' = \frac{1}{1-e^\xi}$. Along the path, $1-e^\xi \to 1-e^{\xi_+} > 0$ at the start, so $\mathrm{Im}\,F \to 0$, and $1-e^\xi \to 1-e^{\xi_-} < 0$ from below at the end, so $\arg(1-e^\xi) \to -\pi$ and $\mathrm{Im}\,F \to \pi$. The integrand is continuous on the closed path, so $\mathrm{Im}\int_{\gamma_\lambda} \frac{d\xi}{1-e^\xi} = \pi$.
- ψ is holomorphic on the strip $|\,\mathrm{Im}\,\xi| < \pi$, with $\psi(0) = a$, and real on $\mathbb{R}$. [$\psi = E_a/E_1$, where $E_c(\xi) = \frac{e^{c\xi}-1}{\xi} = \sum_{k\ge0} \frac{c^{k+1}\xi^k}{(k+1)!}$ is entire, and the zeros of E_1 are $2\pi i k$, $k \ne 0$, outside the strip.] The strip is star-shaped about 0, and $H(\xi) = \xi\int_0^1 \psi(s\xi)\,ds$ satisfies $H'(\xi) = \int_0^1 \partial_s\big(s\,\psi(s\xi)\big)\,ds = \psi(\xi)$. H is real on $\mathbb{R}$, so $\int_{\gamma_\lambda} \psi\,d\xi = H(\xi_-) - H(\xi_+)$ is real.

Hence $|D_\lambda| = \pi(1-\lambda)$. The cases $\lambda \in \{0,1\}$ are trivial. □

The value is the half-residue: τ_λ has a simple pole at 0 with residue $-(1-\lambda)$, and γ_λ passes above it. Write-up 2's step (d) of v3.1–v4.2 took a primitive of τ_λ on the slit strip $V = \{|\,\mathrm{Im}\,\xi| < \pi\} \setminus i(-\pi, 0]$, which is star-shaped about $i\pi/2$, and moved the path onto the real axis indented above 0. That version is also correct and gives the same value.

By-products. For $-1 < r < 0$ the same layer-cake computation gives the Theorem. At $r = 1$, $\int_0^\pi f = \sin(\pi a)/(ab)$, because $G = \sin^b(ax)\sin^a(bx)$ has $G' = abf$; with Proof B this yields $\Gamma(a)\Gamma(1-a) = \pi/\sin\pi a$.

C Write-up 2's Theorem 3, as audited

After Write-up 2's `MAIN_RESULT.pdf` *(v5.1), Section 2, in the notation of Appendix B, with the auditor's details added. Write-up 2's Theorem 3 is called Theorem C here.* Let $R = (0,1)\times(0,\pi)$, so that $\xi(R) \subset \{0 < \mathrm{Im}\,\xi < \pi\}$. On this half-strip let, with F and H as in Appendix B (d),

$$Q(\xi) = F(\xi) - H(\xi) = \xi - \log(1-e^\xi) - H(\xi), \qquad \theta(t,x) = \mathrm{Im}\,Q\big(\xi(t,x)\big) = x - \arg(1-e^\xi) - \mathrm{Im}\,H(\xi).$$

Q is holomorphic with $Q' = \frac{1}{1-e^\xi} - \psi = \frac{e^{a\xi}}{1-e^\xi}$, so θ is smooth on R. Up to a real constant, $Q(\xi) = \int_{-\infty}^{\xi} \frac{e^{a\eta}}{1-e^\eta}\,d\eta$, the form audited in Write-up 2 v4.2. Both are primitives of Q' on the half-strip, and both extend continuously to $(-\infty, 0)$ with real values.

Theorem C (an area-preserving map). *The map $L = (h,\theta)$ is a bijection of R onto R with $dh\wedge d\theta = -dt\wedge dx$. Hence, if (t,x) is uniform on R, then h and θ/π are independent and uniform on $(0,1)$.*

Proof. Differentiating (B1) in the form $he^{a\xi} = 1 - t + te^\xi$ gives $e^{a\xi}dh = -(1-e^\xi)\,dt + (te^\xi - ahe^{a\xi})\,d\xi$. Wedge with $d\xi$ and divide by $1-e^\xi$, which is nonzero on R: $dh\wedge Q'(\xi)\,d\xi = -dt\wedge d\xi$, that is, $dh\wedge dQ = -dt\wedge d\xi$. Since h and t are real, the imaginary parts give $dh\wedge d\theta = -dt\wedge dx$.

Fix $0 < \lambda < 1$. By Appendix B (c), $\{h = \lambda\} \cap R$ is the arc Γ_λ without its endpoints $(t_\pm(0), 0)$, and $\nabla h \ne 0$ on it. If v is tangent to the arc and $dh(w) \ne 0$, then $dh(w)\,d\theta(v) = (dh\wedge d\theta)(w,v) = -(dt\wedge dx)(w,v) \ne 0$, so θ is strictly monotone along the arc. At the ends of the arc, $\xi \to \xi_+ < 0$ and $\xi \to \xi_- > 0$. By Appendix B (d), $\mathrm{Im}\,F = x - \arg(1-e^\xi)$ tends to 0 and to π there, and $H(\xi_\pm)$ is real. So $\theta \to 0$ at one end and $\theta \to \pi$ at the other, and θ maps the open arc onto $(0,\pi)$. As $0 < h < 1$ on R, R is the disjoint union of these arcs, so L is a bijection onto R with $|\det DL| = 1$. It therefore preserves Lebesgue measure, which is the last statement. □

For $a = \frac12$, $h = 2\sqrt{t(1-t)}\cos\frac{x}{2}$, $Q = \log\frac{1+v}{1-v}$ up to a real constant, with $v = e^{\xi/2}$, and $\theta = \arg\big(2t - 1 + 2i\sqrt{t(1-t)}\sin\frac{x}{2}\big)$. Put $X = h$, $Y = 2\sqrt{t(1-t)}\sin\frac{x}{2}$ and $Z = 2t-1$. Then (X,Y,Z) lies on the unit sphere, $(Z, \frac{x}{2})$ are cylindrical coordinates about the Z-axis, and (X,θ), with $\theta = \arg(Z+iY)$, are cylindrical coordinates about the X-axis. So Theorem C is Archimedes' theorem used twice.