A claim by claim proof of a polynomial AA bound

Theorem. *For an absolute constant $C > 0$, every $f : \{-1,1\}^n \to [-1,1]$ of multilinear degree at most $d \geq 1$ satisfies*

$$\max_j \mathrm{Inf}_j(f) \geq \frac{\mathrm{Var}(f)^4}{Cd^8} \geq \frac{1}{C}\left(\frac{\mathrm{Var}(f)}{d}\right)^8, \qquad \mathrm{Inf}_j(f) = \tfrac{1}{4}\mathbb{E}(f(X) - f(X^j))^2.$$

Here X, Y are independent uniform vertices, x^S flips the coordinates in S, and $\Delta(x,y) = \{j : x_j \neq y_j\}$. A *row* for h at x is a vector $a(x) \geq 0$ such that

$$|h(x) - h(y)| \leq \sum_{j \in \Delta(x,y)} a_j(x) \quad \text{for all } y. \tag{1}$$

Write $L_j(h) = \mathbb{E}|h(X) - h(X^j)|$.

Claim 1 (Rows of bounded cost)**.** *The function f has rows $a(x)$ satisfying $\sum_j a_j(x) \leq K = Cd^2$ at every x.*

Proof. This is the fractional-certificate theorem of Agarwal and Ben-David [1, Theorem 1.7]. For $F : \{0,1\}^n \to [0,1]$ of degree at most d, it gives nonnegative $c_i(z)$ with

$$|F(z) - F(w)| \leq \sum_{i : z_i \neq w_i} c_i(z), \qquad \sum_i c_i(z) \leq \frac{\pi^2}{4} d^2.$$

Apply it to $F(z) = (1 + f(2z - \mathbf{1}))/2$ and put $a_i(x) = 2c_i((x + \mathbf{1})/2)$. Thus one may take $K = (\pi^2/2)d^2$. □

Claim 2 (The antipodal inequality)**.** *Let $g : \{-1,1\}^n \to \{0,1\}$ have rows $\lambda(x)$, and let $\ell_j > 0$. Put $\bar{x} = -x$, $O = \{g = 1\}$, $Z = \{u \in O : g(\bar{u}) = 0\}$, and $b_j = \#\{x \in O : g(x^j) = 0\}$. Then*

$$\sum_{u \in Z} \left(\sum_j \frac{\lambda_j(u)\lambda_j(\bar{u})}{\ell_j} \right)^{-1} \leq \sum_j \ell_j b_j. \tag{2}$$

Proof. This is the nonstandard structural input from the supplied manuscript [2, Theorem 4.3]; its determinant argument is included here. First, rows satisfy

$$|g(p) - g(q)| \leq \sum_{j \in \Delta(p,q)} \min\{\lambda_j(p), \lambda_j(q)\}. \tag{3}$$

Indeed, take the intermediate vertex obtained from p by flipping precisely those differing coordinates with $\lambda_j(p) \leq \lambda_j(q)$, and use the two endpoint rows and the triangle inequality. Hence all denominators in (2) are positive; write their reciprocals as ω_u. Set

$$A_u = \{j : \ell_j < \omega_u \lambda_j(\bar{u})\} \quad (u \in Z), \qquad A_x = \varnothing \quad (x \in O \setminus Z).$$

Then $\sum_{j \in A_u} \lambda_j(u) < 1$: if the sum is positive, multiply the defining strict inequalities by $\lambda_j(u)/\ell_j$ and sum; otherwise this is immediate. By the row property, $x^S \in O$ whenever $x \in O$ and $S \subseteq A_x$.

Let $N = |O|$, assume $O \neq \varnothing$, and introduce indeterminates z_j. Define the $O \times O$ matrix

$$M_{xy} = \prod_{j \in \mathrm{ag}(x,y)} z_j, \qquad \mathrm{ag}(x,y) = [n] \setminus \Delta(x,y), \qquad P = \det M.$$

The monomial $\prod_j z_j^N$ arises only from the identity permutation, so $P \neq 0$ and $\deg_{z_j} P = N$. If e_j counts internal undirected j-edges of O, subtract z_j times one endpoint row from the other on each such edge. The edges are disjoint, and each changed row has a factor $1 - z_j^2$. Therefore

$$P = (1 - z_j^2)^{e_j} R_j, \qquad \deg_{z_j} R_j = N - 2e_j = b_j.$$

A multiplier with constant term 1 preserves the lowest nonzero homogeneous part. Thus every monomial z^k in that part of P has $k_j \leq b_j$ for every j.

Now replace row x by

$$\sum_{S \subseteq A_x} (-1)^{|S|} \left(\prod_{j \in S} z_j \right) M_{x^S,\cdot}.$$

All these rows exist. Coordinatewise expansion shows that the new entry at (x, y) equals

$$\begin{cases} \left(\prod_{j \in A_x} (1 - z_j^2) \right) M_{xy}, & \operatorname{ag}(x, y) \cap A_x = \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Let Q be M with the latter entries replaced by zero. The row transformation matrix has constant term I, and the extracted row factors have constant term 1. Consequently $\det Q$ and P have the same lowest nonzero homogeneous part. Choose a monomial z^k in this part and a permutation σ contributing to its nonzero coefficient in $\det Q$. For $u \in Z$, put $G = \operatorname{ag}(u, \sigma(u))$. The entry survives, so $G \cap A_u = \varnothing$. Since $\bar{u}^G = \sigma(u) \in O$, the row at $\bar{u}$ gives $\sum_{j \in G} \lambda_j(\bar{u}) \geq 1$. Hence

$$\sum_{j \in G} \ell_j \geq \omega_u \sum_{j \in G} \lambda_j(\bar{u}) \geq \omega_u.$$

Summing first over $u \in Z$ and then over all permutation rows yields $\sum_{u \in Z} \omega_u \leq \sum_j \ell_j k_j \leq \sum_j \ell_j b_j$. □

Claim 3 (A weighted inequality for real functions)**.** *Let $h : \{-1, 1\}^n \to \mathbb{R}$ have rows $a(x)$, and let $w_j > 0$. Set*

$$D = |h(X) - h(Y)|, \qquad s = \sum_{j \in \Delta(X,Y)} w_j a_j(X) a_j(Y).$$

Then $s > 0$ whenever $D > 0$, and, with a zero quotient when $D = 0$,

$$\mathbb{E} \frac{D^3}{s} \leq 3 \sum_j \frac{L_j(h)}{w_j}. \tag{4}$$

Proof. Apply Claim 2 to a Boolean g on every face, restricting the rows and taking $\ell_j = 1/w_j$ [2, Proposition 5.2]. Each ordered $1, 0$ pair is antipodal in exactly one face. Each boundary j-edge lies in 2^{n-1} faces, and there are $2^{n-1} L_j(g)$ such edges. Summing, including both orientations, and dividing by 4^n gives

$$\mathbb{E} \frac{\mathbf{1}_{\{g(X) \neq g(Y)\}}}{\sum_{j \in \Delta(X,Y)} w_j \lambda_j(X) \lambda_j(Y)} \leq \frac{1}{2} \sum_j \frac{L_j(g)}{w_j}. \tag{5}$$

For t not a value of h, $g_t = \mathbf{1}_{\{h > t\}}$ has rows $\lambda_j(x) = a_j(x)/|h(x) - t|$. Integrate (5) in t, using the standard layer-cake identity and, for endpoint values $u > v$,

$$\int_v^u (u - t)(t - v)\, dt = \frac{(u - v)^3}{6}, \qquad \int_{\mathbb{R}} L_j(g_t)\, dt = L_j(h).$$

This gives (4); positivity follows from (3) at an intermediate level. Equal-value summands are zero throughout. □

Full proof of the theorem. Take the rows of Claim 1, and put $\alpha_j = \mathbb{E}a_j(X)$ and $M = \max_j L_j(f)$. Use Claim 3 with $h = f$ and $w_j = 1/\alpha_j$ when $\alpha_j > 0$. If $\alpha_j = 0$, then a_j and $L_j(f)$ vanish, so choose any positive w_j and omit that coordinate from the displayed sums. We obtain

$$\mathbb{E}\frac{D^3}{s} \le 3\sum_j \alpha_j L_j(f) \le 3KM, \qquad s = \sum_{j\in\Delta(X,Y)} \frac{a_j(X)a_j(Y)}{\alpha_j}.$$

Writing $\alpha_{j,\pm} = \mathbb{E}[a_j(X)\mathbf{1}_{\{X_j=\pm 1\}}]$, independence gives

$$\mathbb{E}s = \sum_j \frac{2\alpha_{j,+}\alpha_{j,-}}{\alpha_j} \le \frac{1}{2}\sum_j \alpha_j \le \frac{K}{2}.$$

Let $V = \mathrm{Var}(f)$. If $V = 0$ the conclusion is immediate. Otherwise, since $D \le 2$ and $\mathbb{E}D^2 = 2V$, the Cauchy–Schwarz inequality gives

$$4V^2 = (\mathbb{E}D^2)^2 \le \mathbb{E}\frac{D^3}{s}\,\mathbb{E}[Ds] \le (3KM)(2\mathbb{E}s) \le 3K^2M.$$

A second application of Cauchy–Schwarz gives $\mathrm{Inf}_j(f) \ge L_j(f)^2/4$. Therefore

$$\max_j \mathrm{Inf}_j(f) \ge \frac{M^2}{4} \ge \frac{4V^4}{9K^4} = \frac{V^4}{Cd^8}.$$

Finally $V \le 1$, so $V^4 \ge V^8$. □

References

[1] A. Agarwal and S. Ben-David, *Oracle separations for the quantum-classical polynomial hierarchy.* arXiv:2410.19062v2 (3 December 2025), Theorem 1.7. Published in ITCS 2026, LIPIcs 362, Article 2 (Theorem 7).

[2] D. Turturean, *A proof of the Aaronson–Ambainis conjecture*, supplied manuscript, November 2026. Theorem 3.1 and its determinant proof, pp. 18–23; Theorem 4.3, p. 26; Proposition 5.2, pp. 27–29.