

# The sharp $L^1$ Poincaré inequality on the cube

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This explanation is based on Tomar's recent preprint [2], which uses the semigroup identity of Ivanisvili–van Handel–Volberg [1]. We state only the three ingredients and the final deduction. For each lemma, one can ask AI to explore alternative or simpler proofs.

For the uniform measure on  $\{-1, 1\}^n$ , put  $D_j f(x) = (f(x) - f(x^{(j)}))/2$  and  $|\nabla f| = (\sum_j (D_j f)^2)^{1/2}$ , where  $x^{(j)}$  flips the  $j$ th coordinate of  $x$ . The sharp dimension-independent inequality is

$$\mathbb{E}|f - \mathbb{E}f| \leq \sqrt{\frac{\pi}{2}} \mathbb{E}|\nabla f|.$$

**Lemma 1 (semigroup reduction).** For every  $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E}|f - \mathbb{E}f| \leq \mathbb{E}J(\nabla f), \quad J(a) = \frac{1}{2} \int_0^1 \frac{\mathbb{E}|\sum_{j=1}^n a_j (B_j(q) - q)|}{q(1-q)} dq,$$

where, for each  $q$ , the variables  $B_j(q)$  are independent Bernoulli variables of parameter  $q$ .

*Comment.* This follows from the semigroup identity in [1, Lemma 2.1 and Remark 2.2] and is the main input from that paper, which also resolved Enflo's conjecture that Rademacher type and Enflo type coincide for Banach spaces.

**Lemma 2 (removing signs; Tomar).** For every  $a \in \mathbb{R}^n$ ,

$$J(a) \leq J(|a|), \quad |a| = (|a_1|, \dots, |a_n|).$$

*Comment.* This comparison was first proved in Tomar's preprint [2, Theorem 4.1]. Its proof is technical and uses Fourier analysis.

**Lemma 3 (positive coefficients; Tomar).** If  $b_j \geq 0$  for every  $j$ , then

$$J(b) = \int_0^\infty \left[ 1 - \prod_{j=1}^n (1 + b_j t) e^{-b_j t} \right] \frac{dt}{t^2}.$$

*Comment.* This identity also appears in Tomar's preprint [2, Theorem 3.6]. It admits different proofs, including a short application of the residue theorem.

**Deduction of the inequality.** For  $x \geq 0$ ,  $\log(1+x) \geq x - x^2/2$ . Thus Lemma 3 gives, for  $b \geq 0$ ,

$$J(b) \leq \int_0^\infty \frac{1 - e^{-t^2 \|b\|_2^2/2}}{t^2} dt = \sqrt{\frac{\pi}{2}} \|b\|_2,$$

where the equality follows by integration by parts and the Gaussian integral. By Lemma 2, the same bound holds for every  $a \in \mathbb{R}^n$ . Lemma 1 therefore gives

$$\mathbb{E}|f - \mathbb{E}f| \leq \mathbb{E}J(\nabla f) \leq \sqrt{\frac{\pi}{2}} \mathbb{E}|\nabla f|.$$

## References

- [1] P. Ivanisvili, R. van Handel and A. Volberg, *Rademacher type and Enflo type coincide*, Ann. of Math. (2) **192** (2020), 665–678. [doi:10.4007/annals.2020.192.2.8](https://doi.org/10.4007/annals.2020.192.2.8).
- [2] T. S. Tomar, *The  $L^1$  Poincaré constant of the Hamming cube*, preprint. [Online PDF](#).

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