The Sánchez-Ruiz entropy bound for all harmonic-oscillator eigenstates: an audited proof of his Eqs. (30) and (37)

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Abstract

Let $\rho_n = H_n^2 e^{-x^2}/(2^n n!\sqrt{\pi})$ be the position density of the n-th eigenstate of the one-dimensional harmonic oscillator, and let $S_n = -\int \rho_n \ln \rho_n$. J. Sánchez-Ruiz [J. Math. Phys. 38 (1997) 5031] conjectured the upper bound (37) for S_n. He noted that it follows from his conjecture (30), which compares the logarithmic potential of H_n at its zeros with its value at the origin. We prove (30) in a strong form: the potential is strictly monotone along the non-negative zeros. This proves (37) for every n, with equality exactly for $n \in \{0, 1\}$, and also proves his (36), (39), (42) and (45). The Hermite-specific input is an integral formula for the amplitude $y_1^2 + \frac{4}{\pi} y_2^2$ of two solutions of the Weber equation, which shows that this amplitude increases with $|t|$.

1 Statement

H_n are the physicists' Hermite polynomials, $\int_{\mathbb{R}} H_n^2 e^{-x^2}\,\mathrm{d}x = 2^n n!\sqrt{\pi}$, and $\psi = \Gamma'/\Gamma$. As in the paper, $m = \lfloor n/2 \rfloor$ and $\varepsilon = n - 2m \in \{0, 1\}$; note $m + \varepsilon + \frac{1}{2} = \frac{n+\varepsilon+1}{2}$. Put

$$\rho_n(x) = \frac{H_n(x)^2 e^{-x^2}}{2^n n!\sqrt{\pi}}, \qquad S_n = -\int_{\mathbb{R}} \rho_n \ln \rho_n \,\mathrm{d}x, \qquad U_n(t) = \int_{\mathbb{R}} \rho_n(x) \ln|x - t| \,\mathrm{d}x.$$

The paper's logarithmic potential (11) is $V_n(t) = -\int H_n^2 \ln|x - t|\, e^{-x^2}\,\mathrm{d}x = -2^n n!\sqrt{\pi}\, U_n(t)$, and its S_n (Eq. (4), with $\alpha = 1$) is the S_n above. The zeros of H_n are $x_{n,1}, \dots, x_{n,n}$, and $0 < \xi_1 < \cdots < \xi_m$ are the positive ones. (From its Eq. (12) on, the paper writes $x_{n,i}$, $i \le m$, for the positive zeros. Since V_n is even this does not affect (30).) Let

$$R_n := \ln\Big(\frac{n!\sqrt{\pi}}{2^n}\Big) + n + \frac{1}{2} - n\,\psi\Big(\frac{n+\varepsilon+1}{2}\Big) = \ln(2^n n!\sqrt{\pi}) + \frac{1}{2} + n\Big(1 + \gamma - 2\sum_{k=1}^{m+\varepsilon} \frac{1}{2k-1}\Big),$$

the right side of the paper's (37). The two forms agree because $\psi(k + \frac{1}{2}) = -\gamma - 2\ln 2 + 2\sum_{j=1}^{k} \frac{1}{2j-1}$, which is the paper's (31).

Theorem A (Eq. (37))**.** *$S_n \le R_n$ for every integer $n \ge 0$, with equality if and only if $n \in \{0, 1\}$.*

Theorem B (Eq. (30), strong form)**.** *Let $n \ge 1$. Then $U_n(0) < U_n(\xi_1) < U_n(\xi_2) < \cdots < U_n(\xi_m)$. Hence $V_n(x_{n,i}) \le V_n(0)$ for every zero $x_{n,i}$, with equality if and only if $x_{n,i} = 0$.*

*This note was written by an AI system acting as an independent referee. The proof mechanism (the factorization of U_n', the amplitude formula (N) and the half-turn pairing) is that of an AI-generated proof submitted for audit. Every step was re-derived independently and checked numerically; the changes made in the audit are listed in §8. Readers should treat it as unrefereed.

The paper observed both statements of Theorem B numerically for $n \le 5$ (its Fig. 1). Theorem A is deduced from Theorem B in §2, and Theorem B is proved in §§3–5. §6 proves the paper's (39) and (42), and hence (45).

2 Reduction of (37) to the values of U_n at the zeros

Lemma 1 (Eq. (29))**.** $U_n(0) = \int_{\mathbb{R}} \rho_n(x) \ln|x| \,\mathrm{d}x = \frac{1}{2}\psi(m+\varepsilon+\frac{1}{2})$.

Proof. Put $\alpha = \varepsilon - \frac{1}{2}$. By Szegő's (5.6.1), $H_{2m+\varepsilon}(x) = (-1)^m 2^{2m+\varepsilon} m!\, x^\varepsilon L_m^{(\alpha)}(x^2)$, so $\rho_n = c\, x^{2\varepsilon} L_m^{(\alpha)}(x^2)^2 e^{-x^2}$ with a constant $c > 0$. Since ρ_n is even, $y = x^2$ gives

$$\int_{\mathbb{R}} \rho_n \ln|x| \,\mathrm{d}x = \frac{c}{2}\int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y)^2 \ln y \,\mathrm{d}y, \qquad 1 = \int_{\mathbb{R}} \rho_n \,\mathrm{d}x = c\int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y)^2 \,\mathrm{d}y.$$

For $\alpha > -1$ let $N(\alpha) = \int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y)^2 \,\mathrm{d}y = \Gamma(m+\alpha+1)/m!$. $L_m^{(\alpha)}(y)$ is a polynomial in (α, y). On $[\alpha_0, \alpha_1] \subset (-1,\infty)$ the α-derivative of the integrand is bounded by $(y^{\alpha_0} + y^{\alpha_1})(1 + |\ln y|)P(y)e^{-y}$ for a polynomial P, which is integrable. So we may differentiate under the integral sign. The leading coefficient $(-1)^m/m!$ of $L_m^{(\alpha)}$ does not depend on α, so $\partial_\alpha L_m^{(\alpha)}$ has degree $\le m-1$ and is orthogonal to $L_m^{(\alpha)}$. Hence $\int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y)^2 \ln y \,\mathrm{d}y = N'(\alpha) = N(\alpha)\,\psi(m+\alpha+1)$. The ratio of the two displayed integrals is therefore $\frac{1}{2}\psi(m+\alpha+1) = \frac{1}{2}\psi(m+\varepsilon+\frac{1}{2})$. □

Lemma 2. *For every $n \ge 0$, $R_n - S_n = 2\sum_{i=1}^n \big(U_n(x_{n,i}) - U_n(0)\big)$.*

Proof. $H_n = 2^n \prod_i (x - x_{n,i})$, so off the zeros $-\ln\rho_n = \ln(2^n n!\sqrt{\pi}) + x^2 - 2n\ln 2 - 2\sum_i \ln|x - x_{n,i}|$. Each term is integrable against ρ_n: the logarithmic singularities are integrable, and ρ_n decays like a Gaussian. From $xH_n = \frac{1}{2}H_{n+1} + nH_{n-1}$ and orthogonality, $\int x^2 \rho_n = n + \frac{1}{2}$. Hence

$$S_n = \ln(2^n n!\sqrt{\pi}) + n + \tfrac{1}{2} - 2n\ln 2 - 2\sum_{i=1}^n U_n(x_{n,i}). \tag{1}$$

By Lemma 1, $2nU_n(0) = n\psi(\frac{n+\varepsilon+1}{2})$. Subtracting (1) from R_n gives the claim. □

With $E_n(H) = 2^n n!\sqrt{\pi}\,(S_n - \ln(2^n n!\sqrt{\pi}) - n - \frac{1}{2})$, which is the paper's (4), and $V_n = -2^n n!\sqrt{\pi}\,U_n$, (1) is the paper's (10). So (36), which is (10) with (30) and (29) inserted, is equivalent to (37).

Proof of Theorem A, given Theorem B. For $n = 0$ the sum in Lemma 2 is empty. For $n = 1$ the only zero is 0. In both cases $S_n = R_n$. Let $n \ge 2$. U_n is even because ρ_n is even, so by Theorem B every term $U_n(x_{n,i}) - U_n(0)$ is ≥ 0, and the $n - \varepsilon \ge 2$ terms with $x_{n,i} \ne 0$ are > 0. By Lemma 2, $S_n < R_n$. □

3 The derivative of the potential

$\mathcal{S}$ is the Schwartz class. For $f \in \mathcal{S}$ put

$$\mathcal{H}f(t) = \int_0^\infty \frac{f(t-y) - f(t+y)}{y}\,\mathrm{d}y.$$

The integrand is bounded by $2\|f'\|_\infty$ for $y \le 1$ and by $|f(t-y)| + |f(t+y)|$ for $y > 1$, so the integral converges absolutely. Substituting $x = t \mp y$ shows that $\mathcal{H}f(t) = \lim_{\delta\to 0}\int_{|x-t|>\delta} \frac{f(x)}{t-x}\,\mathrm{d}x$. We use

Peetre's inequality $1+|y| \le (1+|t\pm y|)(1+|t|)$ repeatedly: if $|f(x)| \le C(1+|x|)^{-2}$ and $|t| \le T$, then $|f(t\pm y)| \le C(1+T)^2(1+|y|)^{-2}$.

Let $F(t) = e^{-t^2}\int_0^t e^{u^2}\,\mathrm{d}u$ be Dawson's function, and put

$$G_n = (-1)^n F^{(n)}, \qquad y_1(t) = e^{-t^2/2}H_n(t), \qquad y_2(t) = e^{t^2/2}G_n(t), \qquad W = 2^n n!.$$

Lemma 3. *$U_n \in C^1(\mathbb{R})$ and $U_n'(t) = \frac{2}{W}\,H_n(t)G_n(t) = \frac{2}{W}\,y_1(t)y_2(t)$.*

Proof. (a) $U_n \in C^1$ and $U_n' = \mathcal{H}\rho_n$. We have $U_n(t) = \int \rho_n(t+y)\ln|y|\,\mathrm{d}y$. Choose C with $|\rho_n| + |\rho_n'| \le C(1+|x|)^{-2}$. For $|t| \le T$, both $\rho_n(t+y)\ln|y|$ and its t-derivative $\rho_n'(t+y)\ln|y|$ are bounded by $C(1+T)^2(1+|y|)^{-2}|\ln|y|| \in L^1(\mathbb{R})$. So U_n is C^1, with $U_n'(t) = \int \rho_n'(t+y)\ln|y|\,\mathrm{d}y = \int_0^\infty h'(y)\ln y\,\mathrm{d}y$, where $h(y) = \rho_n(t+y) - \rho_n(t-y)$. On $[\delta, R]$, integration by parts gives $\int_\delta^R h'\ln y = h(R)\ln R - h(\delta)\ln\delta - \int_\delta^R h(y)y^{-1}\,\mathrm{d}y$. Here $|h(\delta)| \le 2\delta\|\rho_n'\|_\infty$ and $h(R)\ln R \to 0$, while $h(y)/y$ is the negative of the integrand of $\mathcal{H}\rho_n$, which is absolutely integrable. Letting $\delta \to 0$ and $R \to \infty$ gives $U_n' = \mathcal{H}\rho_n$. Nothing changes at zeros of ρ_n.

(b) $(\mathcal{H}f)' = \mathcal{H}(f')$ for $f \in \mathcal{S}$. The t-derivative of the integrand is $(f'(t-y) - f'(t+y))/y$. For $|t| \le T$ it is bounded by $2\|f''\|_\infty$ if $y \le 1$, and by $2C(1+T)^2(1+y)^{-2}$ if $y > 1$, where $|f'| \le C(1+|x|)^{-2}$. This bound is integrable, so we may differentiate under the integral sign.

(c) $\mathcal{H}[e^{-x^2}] = 2\sqrt{\pi}F$. $\mathcal{H}[e^{-x^2}](t) = e^{-t^2}K(t)$ with $K(t) = \int_0^\infty e^{-y^2}\frac{2\sinh 2ty}{y}\,\mathrm{d}y$ and $K(0) = 0$. The t-derivative of the integrand is $4e^{-y^2}\cosh 2ty \le 4e^{-y^2}\cosh 2Ty$ for $|t| \le T$. So $K'(t) = 4\int_0^\infty e^{-y^2}\cosh(2ty)\,\mathrm{d}y = 2\int_{\mathbb{R}} e^{-y^2+2ty}\,\mathrm{d}y = 2\sqrt{\pi}e^{t^2}$, and $K(t) = 2\sqrt{\pi}\int_0^t e^{u^2}\,\mathrm{d}u$.

(d) $\mathcal{H}[H_n^2 e^{-x^2}](t) = H_n(t)\,\mathcal{H}[H_n e^{-x^2}](t)$. Fix t, and write $H_n(x) - H_n(t) = (x-t)Q_t(x)$, where Q_t is a polynomial of degree $\le n-1$. For $\varphi \in \mathcal{S}$ and $g(x) = (x-t)\varphi(x)$, the definition gives $\mathcal{H}g(t) = -\int_0^\infty[\varphi(t-y) + \varphi(t+y)]\,\mathrm{d}y = -\int_{\mathbb{R}}\varphi$. By linearity, $\mathcal{H}[H_n^2e^{-x^2}](t) - H_n(t)\mathcal{H}[H_ne^{-x^2}](t) = \mathcal{H}[(x-t)Q_tH_ne^{-x^2}](t) = -\int_{\mathbb{R}} Q_tH_ne^{-x^2}\,\mathrm{d}x = 0$, by orthogonality.

Assembly. By Rodrigues' formula, $H_ne^{-x^2} = (-1)^n(e^{-x^2})^{(n)}$. Then (b) and (c) give $\mathcal{H}[H_ne^{-x^2}] = (-1)^n(2\sqrt{\pi}F)^{(n)} = 2\sqrt{\pi}\,G_n$, and (a) and (d) give $U_n' = H_n\cdot 2\sqrt{\pi}G_n/(2^n n!\sqrt{\pi})$. □

Lemma 4. *y_1 and y_2 solve $y'' + (2n+1-t^2)y = 0$, and $y_1y_2' - y_1'y_2 = W = 2^n n!$. Moreover G_n has parity $(-1)^{n+1}$.*

Proof. For y_1 this is the equation of the Hermite functions. From $F' = 1-2tF$ we get $F''+2tF'+2F = 0$. Differentiating n times, $g = F^{(n)}$ satisfies $g'' + 2tg' + 2(n+1)g = 0$, and then $y = e^{t^2/2}g$ satisfies $y'' = e^{t^2/2}(g'' + 2tg' + (1+t^2)g) = (t^2 - 2n - 1)y$. Differentiating $F' = 1 - 2tF$ k times gives

$$G_{k+1} = 2tG_k - 2kG_{k-1}\ (k \ge 1), \qquad G_0 = F, \quad G_1 = 2tF - 1, \qquad G_k' = -G_{k+1},$$

and H_k satisfies the same recurrence, with $H_k' = 2kH_{k-1}$. Using these, $y_1y_2' - y_1'y_2 = H_nG_n' - H_n'G_n + 2tH_nG_n = 2n(H_nG_{n-1} - H_{n-1}G_n)$ for $n \ge 1$. The Casoratian $C_k = H_kG_{k-1} - H_{k-1}G_k$ satisfies $C_{k+1} = 2kC_k$ and $C_1 = 1$, so $W = 2nC_n = 2^n n!$. For $n = 0$, $W = F' + 2tF = 1$. Finally, F is odd, so $F^{(n)}$ has parity $(-1)^{n+1}$. □

4 The amplitude

Let $\kappa = 2/\sqrt{\pi}$, $Y = y_1 + i\kappa y_2$, and $A_n(t)^2 := |Y(t)|^2 = e^{-t^2}H_n(t)^2 + \frac{4}{\pi}e^{t^2}G_n(t)^2$.

Lemma 5 (Nicholson-type formula)**.** *For every real t,*

$$A_n(t)^2 = \frac{2^{n+1}n!}{\pi}\int_0^1 e^{\mu t^2}\,\frac{(1-\mu)^n}{\mu^{1/2}(1+\mu)^{n+1}}\,\mathrm{d}\mu. \tag{N}$$

Consequently A_n is even, and $A_n(t_2) > A_n(t_1)$ whenever $0 \le t_1 < t_2$. Moreover $A_n(0)^2 = 2^n n!\,\Gamma(\frac{n+1}{2})/(\sqrt{\pi}\,\Gamma(\frac{n}{2}+1))$.

Proof. Step 1. $\int_0^\infty e^{-s^2/4}\cos(ts)\,\mathrm{d}s = \sqrt{\pi}e^{-t^2}$. Let $\Phi(t) = \frac{1}{2}\int_0^\infty e^{-s^2/4}\sin(ts)\,\mathrm{d}s$. Differentiating under the integral sign (dominated by $\frac{1}{2}se^{-s^2/4}$) and integrating by parts gives $\Phi' = 1 - 2t\Phi$. Since $\Phi(0) = 0$, uniqueness for this linear equation gives $\Phi = F$. Hence $e^{-t^2} + i\kappa F(t) = \pi^{-1/2}\int_0^\infty e^{-s^2/4}e^{its}\,\mathrm{d}s$. Differentiate n times under the integral sign (dominated by $s^n e^{-s^2/4}$), use $(e^{-t^2})^{(n)} = (-1)^n H_n e^{-t^2}$ and $F^{(n)} = (-1)^n G_n$, and multiply by $(-1)^n e^{t^2/2}$. This gives

$$Y(t) = \frac{(-i)^n}{\sqrt{\pi}}\,e^{t^2/2}J_n(t), \qquad J_n(t) = \int_0^\infty s^n e^{-s^2/4}e^{its}\,\mathrm{d}s, \qquad A_n(t)^2 = \frac{e^{t^2}}{\pi}|J_n(t)|^2.$$

Step 2. $|J_n|^2 = \iint_{(0,\infty)^2}(su)^n e^{-(s^2+u^2)/4}e^{it(s-u)}\,\mathrm{d}s\,\mathrm{d}u$, an absolutely convergent double integral. The map $s = p + q$, $u = p - q$ has Jacobian 2 and image $\{p > |q|\}$, and Fubini gives $|J_n|^2 = 2\int_{\mathbb{R}} e^{2itq}e^{-q^2/2}I_n(q)\,\mathrm{d}q$, where $I_n(q) = \int_{|q|}^\infty (p^2-q^2)^n e^{-p^2/2}\,\mathrm{d}p$. Substitute $p = \sqrt{q^2+v}$, and use $a^{-1/2} = \pi^{-1/2}\int_0^\infty \lambda^{-1/2}e^{-\lambda a}\,\mathrm{d}\lambda$ for $a = q^2 + v$. With Tonelli (the integrand is non-negative) and $\int_0^\infty v^n e^{-v(\lambda+1/2)}\,\mathrm{d}v = n!(\lambda+\frac{1}{2})^{-n-1}$,

$$I_n(q) = \frac{e^{-q^2/2}}{2}\int_0^\infty \frac{v^n e^{-v/2}}{\sqrt{q^2+v}}\,\mathrm{d}v = \frac{n!\,e^{-q^2/2}}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\lambda q^2}\,\mathrm{d}\lambda}{\lambda^{1/2}(\lambda+\frac{1}{2})^{n+1}}.$$

Step 3. Hence $|J_n|^2 = \frac{n!}{\sqrt{\pi}}\int_{\mathbb{R}}\int_0^\infty e^{2itq}D(q,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}q$, where $D = \lambda^{-1/2}(\lambda+\frac{1}{2})^{-n-1}e^{-(1+\lambda)q^2} > 0$. The integral of D over $\mathbb{R}\times(0,\infty)$ is at most $\sqrt{\pi}\int_0^\infty \lambda^{-1/2}(\lambda+\frac{1}{2})^{-n-1}\,\mathrm{d}\lambda$. This is finite for every $n \ge 0$: the integrand is $O(\lambda^{-1/2})$ at 0 and $O(\lambda^{-3/2})$ at ∞. By Fubini and $\int_{\mathbb{R}} e^{2itq-(1+\lambda)q^2}\,\mathrm{d}q = \sqrt{\pi/(1+\lambda)}\,e^{-t^2/(1+\lambda)}$,

$$A_n(t)^2 = \frac{e^{t^2}}{\pi}|J_n(t)|^2 = \frac{n!}{\pi}\int_0^\infty \frac{e^{t^2\lambda/(1+\lambda)}\,\mathrm{d}\lambda}{\lambda^{1/2}(\lambda+\frac{1}{2})^{n+1}(1+\lambda)^{1/2}}.$$

Step 4. Substitute $\mu = \lambda/(1+\lambda) \in (0,1)$, so that $\lambda + \frac{1}{2} = \frac{1+\mu}{2(1-\mu)}$, $1+\lambda = \frac{1}{1-\mu}$ and $\mathrm{d}\lambda = \frac{\mathrm{d}\mu}{(1-\mu)^2}$. This gives (N). The integrand of (N) is positive on $(0,1)$ and strictly increasing in t^2, which gives the monotonicity. At $t = 0$, put $\mu = v^2$ and $v = \tan\frac{\phi}{2}$. Then $(1-v^2)/(1+v^2) = \cos\phi$, and (N) becomes $\frac{2^{n+1}n!}{\pi}\int_0^{\pi/2}\cos^n\phi\,\mathrm{d}\phi$, which is the stated Wallis value. □

Remark 1. (N) can also be checked without Fourier analysis. Every product of two solutions of $y'' + qy = 0$, with $q = 2n+1-t^2$, solves $P''' + 4qP' + 2q'P = 0$. In the variable $z = t^2$ this reads $2zf''' + 3f'' + 2(2n+1-z)f' - f = 0$. The right side of (N), as a function of z, satisfies this equation because of the identity $\frac{\mathrm{d}}{\mathrm{d}\mu}[2\mu(\mu^2-1)\omega] = (3\mu^2 + (4n+2)\mu - 1)\,\omega$ for $\omega = \mu^{-1/2}(1-\mu)^n(1+\mu)^{-n-1}$, after one integration by parts with vanishing boundary terms. Parity rules out y_1y_2. The growth as $t\to\infty$ fixes the coefficient of y_2^2 at $4/\pi$, and the decay along $t = i\tau$ fixes the coefficient of y_1^2 at 1. Also $A_n(t)^2 = \frac{2^{n+1}(n!)^2}{\pi}|D_{-n-1}(i\sqrt{2}\,t)|^2$, where D_ν is Weber's parabolic cylinder function. Formula (N) is the analogue of Nicholson's integral for $J_\nu^2 + Y_\nu^2$. Steps 2–4, applied to $\int_0^\infty s^{c-1}e^{-s^2/4}e^{its}\,\mathrm{d}s$ in place of J_n and combined with the integral representation [DLMF 12.5.1] of D_{-c}, give for every real $c > 0$ and real x

$$|D_{-c}(ix)|^2 = \frac{1}{\Gamma(c)}\int_0^1 e^{x^2\mu/2}\,\frac{(1-\mu)^{c-1}}{\mu^{1/2}(1+\mu)^c}\,\mathrm{d}\mu;$$

the Fubini bound is $\sqrt{\pi}\int_0^\infty \lambda^{-1/2}(\lambda+\frac{1}{2})^{-c}(1+\lambda)^{-1/2}\,\mathrm{d}\lambda < \infty$, where the factor $(1+\lambda)^{-1/2}$ from the q-integral is needed when $c \le \frac{1}{2}$.

5 Phase and pairing; proof of Theorem B

Lemma 6. *There is $\theta \in C^\infty(\mathbb{R})$ with $y_1 = A_n \cos\theta$, $\kappa y_2 = A_n \sin\theta$ and*

$$\theta' = \frac{\kappa W}{A_n^2} > 0, \qquad y_1 y_2 = \frac{A_n^2 \sin 2\theta}{2\kappa}.$$

The zeros of H_n are exactly the points where $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$. If $a < b$ are consecutive zeros of H_n, then $\theta(b) = \theta(a) + \pi$. If n is odd, $\theta(0) \in \frac{\pi}{2} + \pi\mathbb{Z}$. If $n \geq 2$ is even, $\theta(0) \in \pi\mathbb{Z}$ and $\theta(\xi_1) = \theta(0) + \frac{\pi}{2}$.

Proof. By Lemma 4, y_1 and y_2 have no common zero, so $Y \in C^\infty(\mathbb{R}; \mathbb{C} \setminus \{0\})$. Choose ℓ_0 with $e^{\ell_0} = Y(0)$, and put $L = \ell_0 + \int_0^t Y'/Y$. Then $(Ye^{-L})' = 0$, so $Y = e^L$. Set $\theta = \operatorname{Im} L$. Then $Y = A_n e^{i\theta}$, and $\theta' = \operatorname{Im}(Y'\bar{Y})/|Y|^2 = \kappa(y_1 y_2' - y_1' y_2)/A_n^2 = \kappa W/A_n^2$. The product formula follows from $y_1 y_2 = A_n^2 \cos\theta \sin\theta/\kappa$.

Since $A_n > 0$, $H_n(t) = 0$ if and only if $\cos\theta(t) = 0$. Let $a < b$ be consecutive zeros. Since θ is strictly increasing, $\theta(b) \geq \theta(a) + \pi$. If the inequality were strict, the intermediate value theorem would give a zero of H_n in (a, b). If n is odd, then $H_n(0) = 0$. If n is even, then G_n is odd (Lemma 4), so $y_2(0) = 0$ and $\theta(0) \in \pi\mathbb{Z}$. Moreover $H_n \neq 0$ on $[0, \xi_1)$, so the first value in $\frac{\pi}{2} + \pi\mathbb{Z}$ reached after $\theta(0)$, namely $\theta(0) + \frac{\pi}{2}$, is reached at ξ_1. □

Lemma 7 (half-turn pairing)**.** (i) *If $0 \leq a < b$ are consecutive zeros of H_n, then $\int_a^b y_1 y_2 \, \mathrm{d}t > 0$.* (ii) *If $n \geq 2$ is even, then $\int_0^{\xi_1} y_1 y_2 \, \mathrm{d}t > 0$.*

Proof. (i) Let $\theta_a = \theta(a) \in \frac{\pi}{2} + \pi\mathbb{Z}$. By Lemma 6, θ is a C^∞ increasing bijection of $[a, b]$ onto $[\theta_a, \theta_a + \pi]$. Its inverse τ satisfies $\tau'(\varphi) = A_n(\tau(\varphi))^2/(\kappa W)$. Substituting $t = \tau(\varphi)$ and writing $B = A_n^4 \circ \tau$,

$$\int_a^b y_1 y_2 \, \mathrm{d}t = \frac{1}{2\kappa^2 W} \int_{\theta_a}^{\theta_a + \pi} B(\varphi) \sin 2\varphi \, \mathrm{d}\varphi = \frac{1}{2\kappa^2 W} \int_{\theta_a}^{\theta_a + \pi/2} \big[B(\varphi + \tfrac{\pi}{2}) - B(\varphi)\big]\big(-\sin 2\varphi\big) \, \mathrm{d}\varphi.$$

The second form comes from splitting at $\theta_a + \frac{\pi}{2}$ and substituting $\varphi \mapsto \varphi + \frac{\pi}{2}$ in the upper half, using $\sin(2\varphi + \pi) = -\sin 2\varphi$. For $\varphi \in (\theta_a, \theta_a + \frac{\pi}{2})$ we have $2\varphi \in (\pi, 2\pi) + 2\pi\mathbb{Z}$, so $-\sin 2\varphi > 0$. Also $0 \leq a \leq \tau(\varphi) < \tau(\varphi + \frac{\pi}{2}) \leq b$, and A_n is strictly increasing on $[0, \infty)$ by Lemma 5, including at the endpoint 0. So $B(\varphi + \frac{\pi}{2}) > B(\varphi)$. The integrand is therefore continuous on $[\theta_a, \theta_a + \frac{\pi}{2}]$ and positive in the interior. This is the only place where $a \geq 0$ is used. On an interval of negative zeros the inequality reverses, as it must, since U_n is even.

(ii) On $(0, \xi_1)$, Lemma 6 gives $\theta \in (\theta(0), \theta(0) + \frac{\pi}{2})$ with $\theta(0) \in \pi\mathbb{Z}$. So $\sin 2\theta > 0$ and $y_1 y_2 > 0$ there. No monotonicity is needed. □

Proof of Theorem B. Put $\xi_0 = 0$. By Lemma 3, $U_n(\xi_j) - U_n(\xi_{j-1}) = \frac{2}{W} \int_{\xi_{j-1}}^{\xi_j} y_1 y_2 \, \mathrm{d}t$ for $1 \leq j \leq m$. If n is odd, 0 is a zero of H_n. So every $[\xi_{j-1}, \xi_j]$ joins consecutive non-negative zeros, and Lemma 7(i) applies; for $j = 1$ it applies with $a = 0$. If n is even, Lemma 7(ii) applies for $j = 1$ and (i) for $j \geq 2$. All increments are positive. The statement about V_n follows because $V_n = -2^n n! \sqrt{\pi}\, U_n$ and U_n is even. □

This completes the proofs of Theorems B and A. Note that for odd n the first step starts at the zero $t = 0$, where $(A_n^2)'(0) = 0$. There U_n first falls and then rises, and only the exact pairing shows that the rise wins. Numerically, this step gives the smallest increments of the whole chain (about 6.3×10^{-7} at $n = 99$).

6 The series (39), the identity (42) and the inequality (45)

Let $\hat\rho_n(s) = \int_{\mathbb R} \rho_n(x)\cos(sx)\,\mathrm dx$, and, with $\mu_{2r}(n)$ the paper's (41),

$$c_r = \frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}, \qquad F_r = {}_2F_1(-n, r; 1; 2) = \sum_{k=0}^{n} \binom{n}{k} \frac{(r)_k (-2)^k}{k!}, \qquad \mu_{2r}(n) = \frac{1}{n}\sum_{i=1}^{n} x_{n,i}^{2r}.$$

Proposition 8. *For every real t, $U_n(t) - U_n(0) = -\frac12 \sum_{r\ge1} c_r F_r\, t^{2r}$, and the series converges absolutely. With $V_n = -2^n n!\sqrt\pi\, U_n$ and Lemma 1, this is the paper's (39). Consequently the paper's (42) holds,*

$$S_n = R_n + n\sum_{r\ge1} c_r F_r\, \mu_{2r}(n) \qquad (n \ge 1),$$

and so does (45): $\sum_{r\ge1} c_r F_r \mu_{2r}(n) \le 0$ for every $n \ge 1$, with equality if and only if $n = 1$.

Proof. Step 1: $\hat\rho_n(s) = e^{-s^2/4} L_n(s^2/2)$. We have $\sum_{j,k} H_j(x)H_k(x)\frac{a^j b^k}{j!k!} = e^{2x(a+b)-a^2-b^2}$. The coefficients of H_j alternate in sign, so $\sum_j |H_j(x)|\frac{r^j}{j!} \le e^{2|x|r+r^2}$, and the double series may be integrated termwise against e^{-x^2+isx}. Since $e^{-a^2-b^2}\int_{\mathbb R} e^{-x^2+x(2a+2b+is)}\,\mathrm dx = \sqrt\pi e^{-s^2/4} e^{2ab+is(a+b)}$, comparing the coefficients of $a^n b^n$ gives

$$\int_{\mathbb R} H_n^2 e^{-x^2+isx}\,\mathrm dx = \sqrt\pi\, e^{-s^2/4}(n!)^2 \sum_{j=0}^{n} \frac{2^j (is)^{2n-2j}}{j!\,((n-j)!)^2} = 2^n n!\sqrt\pi\, e^{-s^2/4} L_n\!\left(\tfrac{s^2}{2}\right).$$

In particular $|\hat\rho_n(s)| \le C(1+s)^{2n} e^{-s^2/4}$.

Step 2: $U_n'(t) = \int_0^\infty \hat\rho_n(s)\sin(ts)\,\mathrm ds$. Fourier inversion for the even function ρ_n gives

$$\rho_n(t-y) - \rho_n(t+y) = \frac{2}{\pi}\int_0^\infty \hat\rho_n(s)\sin(ts)\sin(ys)\,\mathrm ds.$$

Since $|\sin(ys)/y| \le s$, Fubini on $(0,R)\times(0,\infty)$ gives

$$\int_0^R \frac{\rho_n(t-y)-\rho_n(t+y)}{y}\,\mathrm dy = \frac{2}{\pi}\int_0^\infty \hat\rho_n(s)\sin(ts)\,\mathrm{Si}(Rs)\,\mathrm ds, \qquad \mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}\,\mathrm du.$$

Let $R\to\infty$. The left side tends to $\mathcal H\rho_n(t) = U_n'(t)$. On the right, use dominated convergence: $|\,\mathrm{Si}\,| \le \mathrm{Si}(\pi) < 2$ and $\mathrm{Si}(Rs) \to \frac\pi2$.

Step 3. Fubini on $[0,t]\times(0,\infty)$ (dominated by $|\hat\rho_n|$) gives $U_n(t) - U_n(0) = \int_0^\infty \hat\rho_n(s)\frac{1-\cos ts}{s}\,\mathrm ds$.

Step 4. Expand $1-\cos ts = \sum_{r\ge1}(-1)^{r-1}(ts)^{2r}/(2r)!$. The absolute series is dominated by $\int_0^\infty |\hat\rho_n(s)|\frac{\cosh(ts)-1}{s}\,\mathrm ds < \infty$, so termwise integration is allowed. This gives $U_n(t) - U_n(0) = \sum_r \frac{(-1)^{r-1}t^{2r}}{(2r)!} M_r$ with $M_r = \int_0^\infty \hat\rho_n(s) s^{2r-1}\,\mathrm ds$, absolutely convergent.

Step 5. With $s^2 = 2u$, $M_r = 2^{r-1}\int_0^\infty u^{r-1}e^{-u/2}L_n(u)\,\mathrm du$. Expanding $L_n(u) = \sum_k \binom{n}{k}\frac{(-u)^k}{k!}$ and using $\int_0^\infty u^{r+k-1}e^{-u/2}\,\mathrm du = 2^{r+k}(r-1)!(r)_k$ gives $M_r = 2^{2r-1}(r-1)!\,F_r$. So $(-1)^{r-1}M_r/(2r)! = -\frac12 c_r F_r$.

Step 6. Put $t = x_{n,i}$ and sum over the n zeros: $\sum_i (U_n(x_{n,i}) - U_n(0)) = -\frac{n}{2}\sum_r c_r F_r \mu_{2r}(n)$. Lemma 2 turns this into (42). By Theorem B the left side is > 0 for $n \ge 2$. For $n = 1$ it is 0 (also $\mu_{2r}(1) = 0$). □

7 Remarks

Remark 2 (even n and the variance bound (7)). Let $\Delta_n = R_n - \frac{1}{2} - \frac{1}{2}\ln((2n+1)\pi)$, the difference between the right sides of (37) and (7). Then $\Delta_0 = 0$, $\Delta_n > 0.12$ for even $n \geq 2$, and $\Delta_n < 0$ for odd n. So for even n, (37) already follows from (7), as the paper says, while for odd n it is strictly stronger. Here $\Delta_n = \ln n! - n\ln 2 + n - n\psi(\frac{n+\varepsilon+1}{2}) - \frac{1}{2}\ln(2n+1)$. For even $n \geq 2$ we use three bounds: $\ln n! \geq (n+\frac{1}{2})\ln n - n + \frac{1}{2}\ln 2\pi$ (Robbins), $\psi(x) < \ln x - \frac{1}{2x}$, and $\ln(1+\frac{1}{n}) \leq \frac{1}{n}$. Together they give $\Delta_n > \frac{1}{2}\ln\frac{2\pi n}{2n+1} - 1 + \frac{n}{n+1} \geq \frac{1}{2}\ln\frac{4\pi}{5} - \frac{1}{3} > 0.12$. For odd $n \geq 3$ we use $\ln n! \leq (n+\frac{1}{2})\ln n - n + \frac{1}{2}\ln 2\pi + \frac{1}{12n}$, $\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}$ with $x = \frac{n+2}{2}$, and $\ln(1+u) \geq \frac{2u}{2+u}$ with $u = \frac{2}{n}$. Together they give $\Delta_n < \frac{1}{2}\ln\pi - \frac{n(n+3)}{(n+1)(n+2)} + \frac{1}{12n} + \frac{1}{3(n+2)}$, which is decreasing in n and ≈ -0.233 at $n = 3$. Finally, $\Delta_1 = \ln 2 - \frac{1}{2}\ln 3 + \gamma - 1 \approx -0.279$. The two digamma bounds are the first two truncations of the asymptotic series of ψ, whose remainders for real $x > 0$ have the sign of the first omitted term [DLMF §5.11(ii)].

Remark 3 (shape of the potential). By Lemmas 3 and 4, $U_n'' = \frac{2}{W}(y_1'y_2 + y_1y_2')$. So $U_n'' = -2$ at every zero of H_n, and $U_n'' = +2$ at every zero of G_n. The critical points of U_n are therefore non-degenerate. The n local maxima of U_n, which are the n local minima of V_n, sit exactly at the zeros of H_n. By Theorem B their values increase with $|x_{n,i}|$, which is the observation the paper made from its Fig. 1.

Remark 4 (numerical confirmation). An independent high-precision computation, written for the audit, computed S_n by direct quadrature. It uses Arb ball arithmetic and two tanh-sinh configurations that agree to 5×10^{-65} for $n \leq 100$. For $n \leq 100$, every value of U_n at the zeros was also computed in two further ways, from Lemma 3 and from the series of Proposition 8, and both agree with the direct values to 10^{-55}. $R_n - S_n$ is below 4×10^{-80} in absolute value at $n = 0, 1$. For $2 \leq n \leq 100$ it is at least $R_3 - S_3 = 0.06550110723553648320540\ldots$. A second run for $101 \leq n \leq 300$, accurate to about 30 digits, gives $R_n - S_n \geq 0.3396$ there, and all increments of Theorem B positive. Sample values:

n	S_n	$R_n - S_n$
0	1.07236494292470008707171367568	0 (computed: $\lvert\cdot\rvert < 10^{-79}$)
1	1.34272778838617825709545788722	0 (computed: $\lvert\cdot\rvert < 10^{-79}$)
2	1.49860923325172784063442781488	0.807628581155874
3	1.60971184130165311475002095212	0.0655011072355365
4	1.69655063068037526108474136697	0.968652857771516
5	1.76806125323833304526097684226	0.110276326442471
10	2.01017812546743775400062178521	1.12419609911581
11	2.04553787958448284496538379604	0.182491288655442
50	2.65650019904425762121356924398	1.28914902346022
51	2.66498075085802678686813894165	0.300404450926912
99	2.95442704229627490035867615 96	0.338644972858432
100	2.95888629151722967137538780257	1.33416905813723

If one uses the asymptotic formula (6) of the paper, $S_n = \ln(\pi\sqrt{2n}) - 1 + o(1)$, which is taken from the literature, Stirling's formula shows that the gap tends to $\frac{3}{2} - \varepsilon$; this is the paper's (44). At $n = 299$ and $n = 300$ the computed gaps are 0.3875 and 1.3860. So (37) is not asymptotically sharp, but it has the correct $\frac{1}{2}\ln n$ growth.

8 Changes relative to the submitted proof

The audit found no error in the submitted argument. The following points were added or made explicit here.

1. Lemma 3(a): the dominating function for differentiating U_n, both boundary terms of the integration by parts, and the identification of the folded integral with the principal value. The submitted text gave only the boundary term at 0.

2. Lemma 6: the existence of the smooth argument θ (by lifting) and the proof that θ advances by exactly π between consecutive zeros. The submitted text asserted both.

3. Lemma 5 is stated in the variable $\mu = \lambda/(1+\lambda)$. This makes the convergence and the monotonicity immediate, and gives the Wallis value $A_n(0)^2$ as a check of the constant. The submitted λ-form is the last display of Step 3.

4. Lemma 7: the only use of $a \geq 0$, and the boundary case $a = 0$ for odd n, are stated explicitly.

5. The paper's change of meaning of $x_{n,i}$ from Eq. (12) on is noted. It does not affect (30).

6. The digamma and Stirling inequalities in the remark on (7) are given with their source.

7. Two claims of the submission are not used and not repeated here: the unimodality of the Askey–Wimp weights, which depends on the literature's weight formula, and the $n^{-1/3}$ decay rate of $S_n - \ln(\pi\sqrt{2n}) + 1$, which is numerical only. The limits of the gap are stated as conditional on (6).

8. The formula for $|D_{-c}(ix)|^2$ is stated for every $c > 0$. The submission states it for $c > \frac{1}{2}$, because its Fubini bound drops the factor $(1+\lambda)^{-1/2}$.

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[5] NIST Digital Library of Mathematical Functions, §5.11 (asymptotic expansions of Γ and ψ), §7.2 (Dawson's integral) and §12.5 (integral representations of parabolic cylinder functions), `https://dlmf.nist.gov`.