

THE EXACT ROSENTHAL BOUND FOR INDEPENDENT ZERO-MEAN SUMMANDS WHEN $3 \leq p \leq 4$, AND IN THE TWO EXTREME REGIMES WHEN $4 < p < 5$

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This proof was generated on October 26, 2026, by Grok Bot using Cursor's agents, guided by several prompts from Paata Ivanisvili

ABSTRACT. For $p > 2$, $A, B > 0$, let $\mathcal{E}_{p;A,B}$ be the supremum of $\mathbb{E}|X_1 + \dots + X_n|^p$ over all n and all independent zero-mean random variables with $\sum \mathbb{E}X_i^2 = B$ and $\sum \mathbb{E}|X_i|^p = A$. Pinelis (Ann. Probab. 43 (2015)) computed $\mathcal{E}_{p;A,B}$ for $p \in (2, 3] \cup [5, \infty)$ and stated that the cases $p \in (3, 4)$ and $p \in (4, 5)$ remain open. We prove that $\mathcal{E}_{p;A,B} = A + \mathbb{E}|Z|^p B^{p/2}$ for all $p \in [3, 4]$, $Z \sim N(0, 1)$. In other words,

$$\mathbb{E}\left|\sum_{i=1}^n X_i\right|^p \leq \sum_{i=1}^n \mathbb{E}|X_i|^p + \mathbb{E}|Z|^p \left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{p/2}, \quad 3 \leq p \leq 4,$$

with the sharp constants; in the max form this confirms a hypothesis of Ibragimov and Sharakhmetov (2002) for $3 < t < 4$. The proof combines Utev's Poissonization with an elementary Fourier identity whose integrand is pointwise nonpositive; a second proof reduces the zero-mean case to the classical symmetric one via a symmetrization inequality for Lévy measures. Consequences include sharp versions for Hilbert-space-valued and L^p -valued summands and for a class of general moment functions. For $3 \leq p \leq 4$ the same argument gives the sharp bound $A + \mathbb{E}|X + B^{1/2}Z|^p$ in the presence of an extra independent summand X whose characteristic function is nonnegative. On the other hand, the extension with a general extra summand, suggested by Pinelis (his Theorem 1.5 for $p \in (3, 4)$), is false: it fails for $X \equiv 1$ and for some bounded zero-mean X . For $4 < p < 5$, with Π_λ a Poisson variable with mean λ and $c^p \lambda = A$, $c^2 \lambda = B$, we prove $c^p \mathbb{E}|\Pi_\lambda - \lambda|^p \leq \mathcal{E}_{p;A,B} \leq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p + C_p B^{p/2}$, and we prove that Pinelis' conjectured Poisson value is exact, $\mathcal{E}_{p;A,B} = c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$, whenever $B^{p/2}/A$ is sufficiently small (the threshold produced by the proof is very small), and likewise whenever $B^{p/2}/A$ is sufficiently large (again with a large threshold); so the conjecture holds near both ends of the parameter range. Quantitatively, for $\lambda \geq 1$ the supremum exceeds the Poisson value $B^{p/2}(m_p + K_p \lambda^{-1} + O(\lambda^{-2}))$, $K_p = \binom{p}{3} \mathbb{E}|Z|^p/12$, by at most $O(B^{p/2} \lambda^{-2})$. The Gaussian-regime results use a decomposition whose leading part is a linear functional of the Lévy measure maximized at the Poisson law, by a concavity (two-moment-problem) argument. We also reduce the full conjecture to a "skewness inequality" for compound Poisson laws, supported by numerics, and exhibit the non-convexity phenomenon that blocks second-variation arguments there.

1. STATEMENT OF RESULTS

Rosenthal's inequality [9] bounds $\mathbb{E}|\sum X_i|^p$ by a constant times $\max\{\sum \mathbb{E}|X_i|^p, (\sum \mathbb{E}X_i^2)^{p/2}\}$; here we are concerned with exact (sharp) forms. Throughout, $Z \sim N(0, 1)$, $m_p := \mathbb{E}|Z|^p = 2^{p/2} \Gamma(\frac{p+1}{2})/\sqrt{\pi}$, and $\binom{p}{k} := p(p-1) \dots (p-k+1)/k!$ for real p . For $p > 2$ and $A, B > 0$ let $\mathcal{X}_{p;A,B}$ be the set of all finite sequences $\mathbf{X} = (X_1, \dots, X_n)$ of independent random variables with $\mathbb{E}X_i = 0$, $\sum_i \mathbb{E}X_i^2 = B$, $\sum_i \mathbb{E}|X_i|^p = A$, and put $S_{\mathbf{X}} = X_1 + \dots + X_n$,

$$\mathcal{E}_{p;A,B} := \sup\{\mathbb{E}|S_{\mathbf{X}}|^p : \mathbf{X} \in \mathcal{X}_{p;A,B}\}.$$

Theorem A. *Let $3 \leq p \leq 4$. For every $n \geq 1$ and all independent zero-mean $X_1, \dots, X_n$ with $\mathbb{E}|X_i|^p < \infty$,*

$$(1) \quad \mathbb{E}|X_1 + \dots + X_n|^p \leq \sum_{i=1}^n \mathbb{E}|X_i|^p + m_p \left(\sum_{i=1}^n \mathbb{E}X_i^2 \right)^{p/2}.$$

Moreover $\mathcal{E}_{p;A,B} = A + m_p B^{p/2}$ for all $A, B > 0$.

Together with Pinelis' Theorem 1.5 [7] (the range $2 < p \leq 3$, proved there via Tyurin's zero-bias bound [10]) this gives $\mathcal{E}_{p;A,B} = A + m_p B^{p/2}$ on the whole range $2 < p \leq 4$. Consequently, the best constant Δ_p in $\|S_{\mathbf{X}}\|_p \leq \Delta_p \max\{(\sum \mathbb{E}X_i^2)^{1/2}, (\sum \mathbb{E}|X_i|^p)^{1/p}\}$ for independent zero-mean summands is $\Delta_p = (1 + m_p)^{1/p}$ for $2 < p \leq 4$. For $3 < p < 4$ this confirms the hypothesis stated at the end of [6]: for $2 < t < 4$, the exact constant $C(t)$ in $\mathbb{E}|S|^t \leq C(t) \max\{\sum \mathbb{E}|X_i|^t, (\sum \mathbb{E}X_i^2)^{t/2}\}$ for zero-mean summands coincides with the constant $1 + m_t$ for symmetric summands. The range $2 < t \leq 3$ follows from [7, Thm. 1.5]. This is the same value as for symmetric summands ([4, Thm. 6.1, Cor. 6.2], [5]). Indeed, by homogeneity $\mathcal{E}_{p;A,B} \leq (1 + m_p) \max\{A, B^{p/2}\}$ with equality at $A = B^{p/2}$.

Corollary B (Hilbert-space-valued summands). *Let $2 < p \leq 4$, let $\mathbb{H}$ be a separable real Hilbert space, and let $X_1, \dots, X_n$ be independent $\mathbb{H}$ -valued random vectors with $\mathbb{E}X_i = 0$ and $\mathbb{E}\|X_i\|^p < \infty$. Let Γ be a centered Gaussian vector in $\mathbb{H}$ whose covariance operator is the sum of the covariance operators of the X_i . Then*

$$\mathbb{E} \left\| \sum_i X_i \right\|^p \leq \sum_i \mathbb{E}\|X_i\|^p + \mathbb{E}\|\Gamma\|^p \leq \sum_i \mathbb{E}\|X_i\|^p + m_p \left(\sum_i \mathbb{E}\|X_i\|^2 \right)^{p/2}.$$

In particular the optimal constant in the max-form Rosenthal inequality for independent zero-mean summands in a Hilbert space is $(1 + m_p)^{1/p}$ for $2 < p \leq 4$.

Corollary C (L^p -valued summands). *Let $2 < p \leq 4$, let $(\Omega', \mathcal{F}', \nu)$ be a σ -finite measure space with $L^p(\nu)$ separable, and let $X_1, \dots, X_n$ be independent random elements of $L^p(\nu)$ (Borel measurable maps into $L^p(\nu)$), represented by jointly measurable functions $X_i(\omega, \omega')$, with $\mathbb{E}X_i(\cdot, \omega') = 0$ for ν -a.e. ω' and $\mathbb{E}\|X_i\|_p^p < \infty$. Then*

$$\mathbb{E} \left\| \sum_i X_i \right\|_{L^p(\nu)}^p \leq \sum_i \mathbb{E}\|X_i\|_{L^p(\nu)}^p + m_p \left\| \left(\sum_i \mathbb{E}|X_i|^2 \right)^{1/2} \right\|_{L^p(\nu)}^p,$$

and the constants 1 and m_p are optimal.

Proof. Independence of sections. For bounded continuous $\varphi_1, \dots, \varphi_n$ on $\mathbb{R}$ and $E \in \mathcal{F}'$ with $\nu(E) < \infty$, the map $\Psi(x_1, \dots, x_n) := \int_E \prod_i \varphi_i(x_i(\omega')) \nu(d\omega')$ is continuous on $L^p(\nu)^n$ (from every sequence converging in L^p extract an a.e. convergent subsequence and use dominated convergence). By independence, $\mathbb{E}\Psi(X_1, \dots, X_n)$ equals the integral of Ψ against the product of the laws of the X_i ; by Fubini (joint measurability) this reads $\int_E \mathbb{E} \prod_i \varphi_i(X_i(\cdot, \omega')) \nu(d\omega') = \int_E \prod_i \mathbb{E} \varphi_i(X_i(\cdot, \omega')) \nu(d\omega')$. As E is arbitrary and ν is σ -finite, the integrands agree ν -a.e. Applying this to the countably many tuples with $\varphi_i \in \{\cos(q \cdot), \sin(q \cdot) : q \in \mathbb{Q}\}$ shows that, for ν -a.e. ω' , the joint characteristic function of $(X_i(\cdot, \omega'))_i$ factorizes at rational points, hence everywhere by continuity; so for ν -a.e. ω' the real random variables $X_i(\cdot, \omega')$ are independent and zero-mean.

Conclusion. Apply Theorem A (for $3 \leq p \leq 4$) or [7, Thm. 1.5] (for $2 < p \leq 3$) to them, integrate in ω' against ν , and use Tonelli's theorem. The hypotheses of the scalar results hold for ν -a.e. ω' , because $\int \mathbb{E}|X_i(\cdot, \omega')|^p \nu(d\omega') = \mathbb{E}\|X_i\|_p^p < \infty$. Optimality follows by taking ν a point mass. $\square$

Theorem D. *Let $3 \leq p \leq 4$ and let X be a random variable with $\mathbb{E}|X|^p < \infty$ whose characteristic function is nonnegative, $\mathbb{E}e^{itX} \geq 0$ for all $t \in \mathbb{R}$ (e.g. $X = Y - Y'$ with Y' an independent copy*

of Y , or X centered Gaussian). If $X, X_1, \dots, X_n$ are independent and the X_i are zero-mean with $\sum \mathbb{E}X_i^2 = B$, $\sum \mathbb{E}|X_i|^p = A$, then

$$\mathbb{E}|X + X_1 + \dots + X_n|^p \leq A + \mathbb{E}|X + B^{1/2}Z|^p,$$

and the right-hand side is the supremum over $\mathcal{X}_{p;A,B}$ (with Z independent of X).

Pinelis [7, Sect. 1, after Cor. 1.7] wrote that “certain considerations suggest that Theorem 1.5 should hold for $p \in (3, 4)$ as well”. His Theorem 1.5 states, for $2 < p \leq 3$ and every X with $\mathbb{E}|X|^p < \infty$ (the condition $\mathbb{E}X = 0$ is not needed), that $\sup \mathbb{E}|X + S_{\mathbf{X}}|^p = A + \mathbb{E}|X + B^{1/2}Z|^p$, the supremum being over $\mathbf{X} \in \mathcal{X}_{p;A,B}$ independent of X (equivalently over the classes with $\leq A, \leq B$). Theorem A is the case $X = 0$. In the generality of an arbitrary extra summand the suggestion is false:

Theorem E. *Let $3 < p < 4$.*

- (a) *For every $x_0 \neq 0$ there is a zero-mean two-point random variable X_1 such that, with $A = \mathbb{E}|X_1|^p$, $B = \mathbb{E}X_1^2$,*

$$\mathbb{E}|x_0 + X_1|^p > A + \mathbb{E}|x_0 + B^{1/2}Z|^p.$$

- (b) *More generally, let X be a bounded random variable with $\mathbb{P}(|X| < \delta) = 0$ for some $\delta > 0$ and $\mathbb{E}[|X|^{p-3} \text{sign } X] \neq 0$ (for instance X zero-mean two-point, $\mathbb{P}(X = -1) = \frac{3}{4}$, $\mathbb{P}(X = 3) = \frac{1}{4}$). Then there is a zero-mean two-point X_1 , independent of X , with $\mathbb{E}|X + X_1|^p > \mathbb{E}|X_1|^p + \mathbb{E}|X + (\mathbb{E}X_1^2)^{1/2}Z|^p$.*

In particular the conclusion of [7, Thm. 1.5] fails for every $p \in (3, 4)$, already with one summand.

The proof of Theorem A uses only two properties of $|x|^p$, namely the Fourier representation of Lemma 2.1 and convexity of the second derivative. This answers, for a class of moment functions, the interest in general moment functions expressed in [7, Remark 1.4]:

Theorem F. *Let $\kappa \geq 0$, let μ be a nonnegative Borel measure on $(0, \infty)$ with $\int (t^2 \wedge t^4) \mu(dt) < \infty$, and put*

$$f(x) := \kappa x^4 + \int_{(0, \infty)} \left(\cos tx - 1 + \frac{t^2 x^2}{2} \right) \mu(dt), \quad x \in \mathbb{R}.$$

Assume that f'' is convex on $\mathbb{R}$. Then for all independent zero-mean $X_1, \dots, X_n$ with $\mathbb{E}X_i^4 < \infty$ and $B = \sum \mathbb{E}X_i^2$,

$$\mathbb{E}f(X_1 + \dots + X_n) \leq \sum_{i=1}^n \mathbb{E}f(X_i) + \mathbb{E}f(B^{1/2}Z).$$

Equivalently, f is even with $f(0) = f'(0) = 0$ and f'' is a convex, continuous, negative-definite function vanishing at 0: by the Lévy–Khinchine formula such f'' has the form $\sigma^2 x^2/2 + \int (1 - \cos tx) \nu(dt)$ with $\int (1 \wedge t^2) d\nu < \infty$, and then $\kappa = \sigma^2/24$, $\mu = t^{-2}\nu$, $\int (t^2 \wedge t^4) d\mu = \int (1 \wedge t^2) d\nu$. Examples: $|x|^p$ for $3 \leq p \leq 4$; finite mixtures $\int_{[3,4]} |x|^p \pi(dp)$ (for $p < 4$, μ has density $c_p^{-1} t^{-p-1}$ and $\int (t^2 \wedge t^4) c_p^{-1} t^{-p-1} dt = c_p^{-1} (\frac{1}{4-p} + \frac{1}{p-2})$ stays bounded as $p \uparrow 4$ because $c_p^{-1} \sim 24(4-p)$; the atom at $p = 4$ is the κx^4 term); and the function with $f''(x) = \sqrt{1+x^2} - 1 = \int_0^\infty (1 - \cos tx) \frac{2K_1(t)}{\pi t} dt$ (the normal-inverse-Gaussian exponent; K_1 a modified Bessel function), which is convex and gives $f(x) \sim |x|^3/6$.

Finally, the identity behind Theorem A reverses beyond $p = 4$:

Proposition G. *Let $4 < p < 6$ and let X_G be a compound Poisson random variable with finite Lévy measure G on $\mathbb{R} \setminus \{0\}$, $\int x G(dx) = 0$, $\int x^2 G(dx) = B$, $\int |x|^p G(dx) = A < \infty$. Then $\mathbb{E}|X_G|^p \geq A + m_p B^{p/2}$.*

For $4 < p < 5$, where Pinelis' Poisson conjecture is open, we obtain a uniform two-sided bound that is sharp to second order in the rare-jump regime:

Theorem H. *Let $4 < p < 5$, $A, B > 0$, and let $c, \lambda > 0$ be given by $c^p \lambda = A$, $c^2 \lambda = B$ (so $\lambda = (B^{p/2}/A)^{2/(p-2)}$ and $c^p \lambda^{p/2} = B^{p/2}$). Put $q_p := 2^{p-1} - p - 1$, $c'_p := -\Gamma(-p) \cos \frac{\pi p}{2} > 0$ and $\eta_p := \frac{12^{(6-p)/2}}{48(6-p)} + \frac{12^{(4-p)/2}}{4(p-4)}$. Then*

$$|c^p \mathbb{E}|\Pi_\lambda - \lambda|^p - A - q_p A \lambda| \leq \frac{\eta_p}{c'_p} B^{p/2}, \quad c^p \mathbb{E}|\Pi_\lambda - \lambda|^p \leq \mathcal{E}_{p,A,B} \leq A + q_p A \lambda + \frac{\eta_p}{c'_p} B^{p/2}.$$

In particular $\mathcal{E}_{p,A,B} \leq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p + C_p B^{p/2}$ with $C_p := 2\eta_p/c'_p$ ($C_{4.2} \approx 13.9$, $C_{4.5} \approx 16.9$, $C_{4.8} \approx 20.0$), and $|\mathcal{E}_{p,A,B} - A - q_p A \lambda| \leq \frac{1}{2} C_p B^{p/2}$. Since $B^{p/2} = A \lambda^{(p-2)/2} = o(A \lambda)$ as $\lambda \rightarrow 0$, the conjectured Poisson value is correct to second order as $B^{p/2}/A \rightarrow 0$, while the Gaussian-plus-far-jump value $A + m_p B^{p/2}$ misses the term $q_p A \lambda$. The bound is informative only in this rare-jump regime: C_p ranges over $[12, 22.05]$ on $(4, 5]$, about 3.5–4 times m_p .

Near the boundary $B^{p/2}/A = 0$ of the parameter domain the conjecture itself holds. Here, for a finite Borel measure G on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG < \infty$, $\tilde{X}_G$ denotes the centred compound Poisson variable, $\mathbb{E}e^{it\tilde{X}_G} = \exp \int (e^{itx} - 1 - itx)G(dx)$; for instance $\tilde{X}_{\lambda\delta_c} = c(\Pi_\lambda - \lambda)$.

Theorem I. *Let $4 < p < 5$. There is $\lambda_0(p) > 0$ such that $\mathcal{E}_{p,A,B} = c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ for all $A, B > 0$ with $\lambda = (B^{p/2}/A)^{2/(p-2)} \leq \lambda_0(p)$, that is, with $B^{p/2} \leq \lambda_0(p)^{(p-2)/2} A$. Moreover, for such A, B and every finite Borel measure $G \notin \{\lambda\delta_c, \lambda\delta_{-c}\}$ on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG = B$ and $\int |x|^p dG = A$, one has $\mathbb{E}|\tilde{X}_G|^p < c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$.*

The proof (Section 9) is perturbative: a quantitative stability estimate for the pair-interaction term of Proposition 9.6 is played off against a remainder that the two moment constraints make small to the same order. It yields an explicit $\lambda_0(p)$, but an astronomically small one (about $4 \cdot 10^{-28}$ at $p = 4.5$ and $4 \cdot 10^{-17}$ at $p = 4.8$; see Remark 9.9). The numerical evidence in Section 9 is consistent with the conjecture for all λ .

At the other end, $B^{p/2}/A \rightarrow \infty$, the conjectured value is confirmed to the first order beyond the Gaussian term:

Theorem J. *Let $4 < p < 5$ and $K_p := p(p-1)(p-2)m_p/72$. There is $\tilde{C}_p < \infty$ such that for all $A, B > 0$ with $\lambda = (B^{p/2}/A)^{2/(p-2)} \geq 1$,*

$$c^p \mathbb{E}|\Pi_\lambda - \lambda|^p \leq \mathcal{E}_{p,A,B} \leq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p + \tilde{C}_p B^{p/2} \lambda^{-2},$$

$$\left| c^p \mathbb{E}|\Pi_\lambda - \lambda|^p - B^{p/2} (m_p + K_p \lambda^{-1}) \right| \leq \tilde{C}_p B^{p/2} \lambda^{-2}.$$

Hence $\mathcal{E}_{p,A,B} = B^{p/2} (m_p + K_p \lambda^{-1} + O(\lambda^{-2}))$ as $\lambda \rightarrow \infty$. The Gaussian-plus-far-jump value $A + m_p B^{p/2} = B^{p/2} (m_p + \lambda^{-(p-2)/2})$ misses the term $K_p B^{p/2}/\lambda$.

The proof rests on a second exact decomposition of $\mathbb{E}|\tilde{X}_G|^p$. Its leading part is *linear* in G and, for every λ , is maximized exactly by the measures carried by $\{\pm c\}$, by the concavity of an explicit function (Lemma 9.12). This is the two-constraint moment problem of [3] in its simplest instance, a planar curve. The constant K_p is the Edgeworth coefficient $\frac{(p)_4}{24} m_{p-4} + \frac{(p)_6}{72} m_{p-6}$, with $(p)_k = p(p-1)\cdots(p-k+1)$ and $m_s = 2^{s/2} \Gamma(\frac{s+1}{2})/\sqrt{\pi}$ continued analytically, since $m_s = (s-1)m_{s-2}$. For instance $K_4 = 1$ and $K_6 = 25$ are the coefficients of λ in $\mathbb{E}(\Pi_\lambda - \lambda)^4 = \lambda + 3\lambda^2$ and of λ^2 in $\mathbb{E}(\Pi_\lambda - \lambda)^6 = \lambda + 25\lambda^2 + 15\lambda^3$, and $K_{4.5} = 2.36055\dots$. The second display is therefore an expansion of Edgeworth type (compare the classical expansions of absolute moments of normalized sums, e.g. [12, 2]); the proof below is self-contained. What is new in Theorem J is the upper bound for $\mathcal{E}_{p,A,B}$, uniform over all admissible summands, and Lemma 9.12. The proof works verbatim for $5 \leq p < 6$, where the first display is weaker than Pinelis' exact result.

A stability analysis of the same decomposition gives exactness in the Gaussian regime as well:

Theorem K. *Let $4 < p < 5$. There is $\lambda_1(p) < \infty$ such that $\mathcal{E}_{p;A,B} = c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ for all $A, B > 0$ with $\lambda \geq \lambda_1(p)$, that is, with $B^{p/2} \geq \lambda_1(p)^{(p-2)/2} A$. Moreover, for such A, B the last assertion of Theorem I holds: $\mathbb{E}|\tilde{X}_G|^p < c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ for every other admissible G . Consequently, Pinelis' conjecture holds for all $A, B > 0$ with $\lambda \notin (\lambda_0(p), \lambda_1(p))$.*

The threshold $\lambda_1(p)$ produced by the proof is about 10^{12} at $p = 4.5$, and between $10^{11.4}$ and 10^{13} for $p \in [4.05, 4.8]$. Unlike $\lambda_0(p)$ it stays bounded as $p \downarrow 4$, but it is far from optimal (Remark 9.17).

Remark 1.1 (Context). For $p = 4$, (1) follows from the identity $\mathbb{E}S^4 = \sum \mathbb{E}X_i^4 + 3(\sum \sigma_i^2)^2 - 3 \sum \sigma_i^4$. For $p \geq 5$ and $p \in \{4, 6, 8, \dots\}$ the extremal value is $c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ (Poisson) [7, 1, 6]. For $p \in (4, 5)$, Pinelis' Poisson conjecture remains open outside the two extreme regimes of Theorems I and K. That $\mathcal{E}_{p;A,B} \geq A + m_p B^{p/2}$ holds for all $2 < p < 6$ follows from the construction in Section 3; Proposition G adds that for $4 < p < 6$ every zero-mean compound Poisson law lies above this value. The pointwise comparison inside a Fourier integral that proves Theorem A does not by itself reach (4, 5). Theorems H and I combine the same identity with a separate extremal property of the Poisson law for the pair-interaction term (Proposition 9.6) and, for Theorem I, its stability (Lemma 9.7). Theorems J and K use a second decomposition adapted to the Gaussian regime (Lemmas 9.11–9.16). See Section 10.

2. THREE LEMMAS

Lemma 2.1 (Fourier representation). *Let $2 < p < 4$ and $c_p := \int_0^\infty (\cos s - 1 + \frac{s^2}{2})s^{-p-1} ds$. Then $0 < c_p < \infty$ and for every $x \in \mathbb{R}$*

$$(2) \quad |x|^p = \frac{1}{c_p} \int_0^\infty \left(\cos(tx) - 1 + \frac{t^2 x^2}{2} \right) t^{-p-1} dt.$$

Consequently, for every random variable Y with $\mathbb{E}Y^2 < \infty$, writing $\varphi_Y(t) = \mathbb{E}e^{itY}$,

$$(3) \quad \mathbb{E}|Y|^p = \frac{1}{c_p} \int_0^\infty \left(\operatorname{Re} \varphi_Y(t) - 1 + \frac{t^2 \mathbb{E}Y^2}{2} \right) t^{-p-1} dt \in [0, \infty].$$

Proof. Let $k(s) := \cos s - 1 + s^2/2$. Integrating $\sin s \leq s$ ($s \geq 0$) three times from 0 gives $1 - \cos s \leq s^2/2$, $\sin s \geq s - s^3/6$ and $1 - \cos s \geq s^2/2 - s^4/24$; by evenness $0 \leq k(s) \leq s^4/24$ for all $s \in \mathbb{R}$, and $k(s) > 0$ for $s \neq 0$. Hence $k(s)s^{-p-1} \leq s^{3-p}/24$ is integrable near 0 because $p < 4$. For $s \geq 1$, $k(s) \leq 2 + s^2/2 \leq \frac{5}{2}s^2$, so $k(s)s^{-p-1} \leq \frac{5}{2}s^{1-p}$ is integrable at ∞ because $p > 2$. Hence $0 < c_p < \infty$. For $x \neq 0$ substitute $s = t|x|$ (cosine is even) to get $\int_0^\infty k(tx)t^{-p-1} dt = |x|^p c_p$; for $x = 0$ both sides of (2) vanish. For (3), apply (2) with $x = Y$ and take expectations. Since the integrand $k(tY)t^{-p-1}$ is nonnegative, Tonelli's theorem allows us to exchange $\mathbb{E}$ and $\int dt$, and $\mathbb{E}k(tY) = \operatorname{Re} \varphi_Y(t) - 1 + t^2 \mathbb{E}Y^2/2$ because $\mathbb{E}Y^2 < \infty$. $\square$

Definition 2.2. For a finite nonnegative Borel measure G on $\mathbb{R} \setminus \{0\}$, let X_G denote a random variable with $\mathbb{E}e^{itX_G} = \exp \int (e^{itx} - 1)G(dx)$, i.e. $X_G = \sum_{k=1}^N \xi_k$ with $N \sim \text{Poisson}(G(\mathbb{R}))$ and ξ_k i.i.d. with law $G/G(\mathbb{R})$, independent of N (and $X_G = 0$ if $G = 0$).

Lemma 2.3 (Compound Poisson facts). *Let G be as above with $\int |x|^p G(dx) < \infty$ for some $p \geq 2$ and $\int x G(dx) = 0$. Then $\mathbb{E}|X_G|^p < \infty$, $\mathbb{E}X_G = 0$, $\mathbb{E}X_G^2 = \int x^2 G(dx)$, and*

$$\mathbb{E}e^{itX_G} = e^{-a(t)+ib(t)}, \quad a(t) := \int (1 - \cos tx)G(dx) \geq 0, \quad b(t) := \int \sin(tx)G(dx).$$

In particular $\operatorname{Re} \mathbb{E}e^{itX_G} = e^{-a(t)} \cos b(t)$. Moreover $a(t) \leq \frac{t^2}{2} \int x^2 G(dx)$. If $G_1, \dots, G_n$ are such measures and $X_{G_1}, \dots, X_{G_n}$ are independent, then $X_{G_1} + \dots + X_{G_n} \stackrel{d}{=} X_{G_1 + \dots + G_n}$.

Proof. Let $q = G(\mathbb{R}) > 0$ and $\xi \sim G/q$. Then $\mathbb{E}|\xi|^p < \infty$, and by convexity of $|\cdot|^p$, $|\sum_{k \leq N} \xi_k|^p \leq N^{p-1} \sum_{k \leq N} |\xi_k|^p$, whence $\mathbb{E}|X_G|^p \leq \mathbb{E}N^p \mathbb{E}|\xi|^p < \infty$. Conditioning on N gives $\mathbb{E}e^{itX_G} = \mathbb{E}(\varphi_\xi(t))^N = \exp(q(\varphi_\xi(t) - 1)) = \exp \int (e^{itx} - 1)G(dx)$, which has the stated real and imaginary parts. Conditioning on N and using $\mathbb{E}\xi = q^{-1} \int x dG = 0$: $\mathbb{E}X_G = \mathbb{E}N \mathbb{E}\xi = 0$ and $\mathbb{E}X_G^2 = \mathbb{E}N \mathbb{E}\xi^2 = \int x^2 dG$. The bound on a follows from $1 - \cos y \leq y^2/2$. The last claim holds because characteristic functions multiply under independent sums and the exponents add. $\square$

Lemma 2.4 (Poissonization; Utev [11], see also [8] and [7, Thm. A]). *Let $p \geq 2$ and let $h \in C^2(\mathbb{R})$ with h'' convex and, for some constant C , $|h(x)| \leq C(1 + |x|^p)$ and $|h''(x)| \leq C(1 + |x|^{p-2})$ for all x . Let X be a random variable with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$, and let G_X be the restriction of the law of X to $\mathbb{R} \setminus \{0\}$. Then $\mathbb{E}h(X) \leq \mathbb{E}h(X_{G_X})$.*

We include the short proof for completeness.

Proof. Let $q = \mathbb{P}(X \neq 0)$. If $q = 0$ both sides equal $h(0)$. Otherwise let $\xi, \xi_1, \xi_2, \dots$ be i.i.d. with law $\nu := G_X/q$ (the law of X given $X \neq 0$), $S_0 := 0$, $S_k := \xi_1 + \dots + \xi_k$, and $N \sim \text{Poisson}(q)$ independent, so that $X_{G_X} \stackrel{d}{=} S_N$. Note $\mathbb{E}\xi = \mathbb{E}[X\mathbf{1}_{X \neq 0}]/q = \mathbb{E}X/q = 0$ and $\mathbb{E}|\xi|^p = \mathbb{E}|X|^p/q < \infty$.

Step 1: $u(s) := \mathbb{E}h(s + \xi) - h(s)$ is convex. Integrating the bound on h'' gives $|h'(x)| \leq |h'(0)| + C(|x| + |x|^{p-1}) \leq C'(1 + |x|^{p-1})$, and for s in a bounded interval I the random variables $|h(s + \xi)|$, $|h'(s + \xi)|$ and $|h''(s + \xi)|$ are dominated by $C''(1 + |\xi|^p)$, which is integrable. (We use $|a + b|^r \leq 2^{(r-1)+}(|a|^r + |b|^r)$ and $|\xi|^r \leq 1 + |\xi|^p$ for $0 \leq r \leq p$.) Hence we may differentiate twice under the expectation, so $u \in C^2$ and $u''(s) = \mathbb{E}h''(s + \xi) - h''(s)$. Since h'' is convex and $\mathbb{E}\xi = 0$, Jensen's inequality gives $\mathbb{E}h''(s + \xi) \geq h''(s)$. Thus $u'' \geq 0$.

Step 2: $\mathbb{E}h(S_k) \geq h(0) + k u(0)$ for all $k \geq 0$. The case $k = 0$ is trivial. For $k \geq 0$, independence of ξ_{k+1} and S_k and Fubini give $\mathbb{E}h(S_{k+1}) - \mathbb{E}h(S_k) = \mathbb{E}u(S_k)$; all expectations are finite since $\mathbb{E}|S_k|^p \leq k^p \mathbb{E}|\xi|^p$. By Jensen's inequality for the convex u and $\mathbb{E}S_k = 0$, $\mathbb{E}u(S_k) \geq u(0)$. Induct on k .

Step 3. Since $\sum_k \mathbb{P}(N = k) \mathbb{E}|h(S_k)| \leq C \sum_k \mathbb{P}(N = k)(1 + k^p \mathbb{E}|\xi|^p) < \infty$,

$$\mathbb{E}h(S_N) = \sum_{k \geq 0} \mathbb{P}(N = k) \mathbb{E}h(S_k) \geq \sum_{k \geq 0} \mathbb{P}(N = k)(h(0) + k u(0)) = h(0) + q(\mathbb{E}h(\xi) - h(0)) = \mathbb{E}h(X),$$

because the law of X is $(1 - q)\delta_0 + q\nu$. $\square$

Corollary 2.5. *Let $p \geq 3$, let W be a random variable with $\mathbb{E}|W|^p < \infty$, and let $W, X_1, \dots, X_n$ be independent with $\mathbb{E}X_i = 0$, $\mathbb{E}|X_i|^p < \infty$. Put $G := \sum_i G_{X_i}$ and let X_G be independent of W . Then $\mathbb{E}|W + X_1 + \dots + X_n|^p \leq \mathbb{E}|W + X_G|^p$.*

Proof. Let $X_{G_{X_1}}, \dots, X_{G_{X_n}}$ be independent compound Poisson variables, independent of everything else. We replace $X_1, \dots, X_n$ one at a time. Suppose $X_1, \dots, X_{j-1}$ have been replaced, and let $V := W + \sum_{i < j} X_{G_{X_i}} + \sum_{i > j} X_i$. Then V is independent of X_j and of $X_{G_{X_j}}$, and $\mathbb{E}|V|^p < \infty$ by Lemma 2.3. For fixed v , the function $h_v(x) := |v + x|^p$ is C^2 with $h_v''(x) = p(p-1)|v + x|^{p-2}$, which is convex because $p-2 \geq 1$, and it satisfies the growth bounds of Lemma 2.4 with $C = C_p(1 + |v|^p)$. Lemma 2.4 gives $\mathbb{E}h_v(X_j) \leq \mathbb{E}h_v(X_{G_{X_j}})$. Integrating over the law of V (Fubini; both sides are finite) yields $\mathbb{E}|V + X_j|^p \leq \mathbb{E}|V + X_{G_{X_j}}|^p$. After n steps, use $\sum_i X_{G_{X_i}} \stackrel{d}{=} X_G$ (Lemma 2.3). $\square$

3. PROOF OF THEOREM A

The case $p = 4$. Expanding, independence and $\mathbb{E}X_i = 0$ give $\mathbb{E}S^4 = \sum_i \mathbb{E}X_i^4 + 6 \sum_{i < j} \mathbb{E}X_i^2 \mathbb{E}X_j^2 = A + 3B^2 - 3 \sum_i (\mathbb{E}X_i^2)^2 \leq A + m_4 B^2$, since $m_4 = 3$.

The upper bound for $3 \leq p < 4$. Let $X_1, \dots, X_n$ be as in the theorem, $G = \sum_i G_{X_i}$. Then G is a finite measure on $\mathbb{R} \setminus \{0\}$ with $\int x dG = \sum \mathbb{E}X_i = 0$, $\int x^2 dG = B$ and $\int |x|^p dG = A$. By Corollary 2.5 with $W = 0$, $\mathbb{E}|S|^p \leq \mathbb{E}|X_G|^p$. It remains to show that

$$(4) \quad \mathbb{E}|X_G|^p \leq A + m_p B^{p/2} \quad (2 < p < 4).$$

Write $g(t) := t^2 B/2$ and let a, b be as in Lemma 2.3. Each of the three quantities in (4) is finite. Lemma 2.1 applies to each of them:

$$\begin{aligned} c_p \mathbb{E}|X_G|^p &= \int_0^\infty (e^{-a(t)} \cos b(t) - 1 + g(t)) t^{-p-1} dt, \\ c_p A &= \int_0^\infty \int \left(\cos tx - 1 + \frac{t^2 x^2}{2} \right) G(dx) t^{-p-1} dt = \int_0^\infty (-a(t) + g(t)) t^{-p-1} dt, \\ c_p m_p B^{p/2} &= c_p \mathbb{E}|B^{1/2} Z|^p = \int_0^\infty (e^{-g(t)} - 1 + g(t)) t^{-p-1} dt. \end{aligned}$$

The first line is (3) combined with Lemma 2.3. In the second line we integrated (2) against G and used Tonelli for the nonnegative integrand. In the third we used $\mathbb{E}e^{itB^{1/2}Z} = e^{-g(t)}$. All three integrands are nonnegative and the integrals are finite, so we may subtract:

$$(5) \quad c_p (\mathbb{E}|X_G|^p - A - m_p B^{p/2}) = \int_0^\infty (e^{-a(t)} \cos b(t) + a(t) - e^{-g(t)} - g(t)) t^{-p-1} dt.$$

Let $F(x) := e^{-x} + x$. Then $F'(x) = 1 - e^{-x} \geq 0$ for $x \geq 0$, so F is nondecreasing on $[0, \infty)$. By Lemma 2.3, $0 \leq a(t) \leq g(t)$ for every t . Hence, pointwise in t ,

$$e^{-a} \cos b + a - e^{-g} - g \leq F(a) - F(g) \leq 0.$$

Since $c_p > 0$, (5) yields (4), and hence (1).

Sharpness. Fix $A, B > 0$ and $2 < p \leq 4$. We construct $\mathbf{X}^{(n)} \in \mathcal{X}_{p;A,B}$ with $\mathbb{E}|S_{\mathbf{X}^{(n)}}|^p \rightarrow A + m_p B^{p/2}$. (This is essentially [7, Prop. 1.6]; we give a self-contained version with exact constraints.) Let $\chi(\varepsilon) := \varepsilon(1 - \varepsilon)((1 - \varepsilon)^{p-1} + \varepsilon^{p-1})$, so that for $J = M(\beta - \varepsilon)$ with $\beta \sim \text{Bernoulli}(\varepsilon)$ we have $\mathbb{E}J = 0$, $\mathbb{E}J^2 = M^2\varepsilon(1 - \varepsilon)$ and $\mathbb{E}|J|^p = M^p\chi(\varepsilon)$. Let $r_1, r_2, \dots$ be i.i.d. Rademacher signs independent of β . For $n \in \mathbb{N}$, $\varepsilon \in (0, \frac{1}{2})$ and $0 \leq M \leq M_{\max} := (B/(\varepsilon(1 - \varepsilon)))^{1/2}$ let $s := (B - M^2\varepsilon(1 - \varepsilon))^{1/2}$ and

$$\mathbf{X} = (J, sn^{-1/2}r_1, \dots, sn^{-1/2}r_n), \quad \sum \mathbb{E}X_i^2 = B, \quad \sum \mathbb{E}|X_i|^p = \phi(M) := M^p\chi(\varepsilon) + s^p n^{1-p/2}.$$

Now ϕ is continuous on $[0, M_{\max}]$, with $\phi(0) = B^{p/2}n^{1-p/2}$ and $\phi(M_{\max}) = M_{\max}^p\chi(\varepsilon) \geq (B/\varepsilon)^{p/2}\varepsilon(1 - \varepsilon)^p \geq B^{p/2}2^{-p}\varepsilon^{1-p/2}$. Choose n so large that $B^{p/2}n^{1-p/2} < A$, and then $\varepsilon = \varepsilon_n \in (0, 1/n)$ so small that $\phi(M_{\max}) > A$. By the intermediate value theorem there is $M = M_n$ with $\phi(M_n) = A$, so $\mathbf{X}^{(n)} \in \mathcal{X}_{p;A,B}$. As $n \rightarrow \infty$: $M_n^p\chi(\varepsilon_n) = A - s^p n^{1-p/2} \rightarrow A$. Since $\chi(\varepsilon)/\varepsilon \rightarrow 1$, also $M_n^p\varepsilon_n \rightarrow A$, whence $M_n \rightarrow \infty$, $M_n\varepsilon_n = (M_n^p\varepsilon_n)^{1/p}\varepsilon_n^{1-1/p} \rightarrow 0$ and $M_n^2\varepsilon_n = (M_n^p\varepsilon_n)^{2/p}\varepsilon_n^{1-2/p} \rightarrow 0$, so $s = s_n \rightarrow B^{1/2}$. Let $R_n := s_n n^{-1/2} \sum_{i \leq n} r_i$. Then $R_n \rightarrow B^{1/2}Z$ in distribution (CLT) and $\mathbb{E}R_n^6 \leq 15s_n^6$, so $(|R_n|^p)$ is uniformly integrable ($p \leq 4$) and $\|R_n\|_p \rightarrow m_p^{1/p}B^{1/2}$. Conditioning on β ,

$$\mathbb{E}|S|^p = (1 - \varepsilon_n)\mathbb{E}|R_n - M_n\varepsilon_n|^p + \varepsilon_n M_n^p \mathbb{E}|(1 - \varepsilon_n) + R_n/M_n|^p.$$

By Minkowski, $\|R_n - M_n\varepsilon_n\|_p - \|R_n\|_p \leq M_n\varepsilon_n \rightarrow 0$ and $\|(1 - \varepsilon_n) + R_n/M_n\|_p - (1 - \varepsilon_n) \leq \|R_n\|_p/M_n \rightarrow 0$. Hence $\mathbb{E}|S|^p \rightarrow m_p B^{p/2} + A$. Together with (1) this proves $\mathcal{E}_{p;A,B} = A + m_p B^{p/2}$ for $3 \leq p \leq 4$. $\square$

4. PROOF OF COROLLARY B

Finite dimension. Let $\mathbb{H} = \mathbb{R}^d$ and let σ be the normalized surface measure on S^{d-1} . By rotation invariance, $\int_{S^{d-1}} |\langle \theta, x \rangle|^p \sigma(d\theta) = \gamma_{d,p} \|x\|^p$ for all x , with $\gamma_{d,p} := \int |\theta_1|^p \sigma(d\theta) \in (0, \infty)$. For each fixed θ the real random variables $\langle \theta, X_i \rangle$ are independent, have mean zero, satisfy $\mathbb{E}|\langle \theta, X_i \rangle|^p \leq \mathbb{E}\|X_i\|^p < \infty$, and $\sum_i \mathbb{E}\langle \theta, X_i \rangle^2 = \langle \Sigma \theta, \theta \rangle$, where Σ is the covariance matrix of Γ . By Theorem A for $3 \leq p \leq 4$, and by [7, Thm. 1.5] with $X = 0$ for $2 < p \leq 3$,

$$\mathbb{E}|\langle \theta, S \rangle|^p \leq \sum_i \mathbb{E}|\langle \theta, X_i \rangle|^p + m_p \langle \Sigma \theta, \theta \rangle^{p/2} = \sum_i \mathbb{E}|\langle \theta, X_i \rangle|^p + \mathbb{E}|\langle \theta, \Gamma \rangle|^p,$$

because $\langle \theta, \Gamma \rangle \sim N(0, \langle \Sigma \theta, \theta \rangle)$. Integrating in θ against σ (Tonelli) and dividing by $\gamma_{d,p}$ gives the first inequality.

Separable $\mathbb{H}$. Let P_N be the orthogonal projection onto the span of the first N vectors of an orthonormal basis. The vectors $P_N X_i$ are independent and zero-mean in an N -dimensional space, and $P_N \Gamma$ is centered Gaussian with covariance $P_N \Sigma P_N = \sum_i \text{Cov}(P_N X_i)$. Hence $\mathbb{E}\|P_N S\|^p \leq \sum_i \mathbb{E}\|P_N X_i\|^p + \mathbb{E}\|P_N \Gamma\|^p \leq \sum_i \mathbb{E}\|X_i\|^p + \mathbb{E}\|\Gamma\|^p$. Since $\|P_N S\| \uparrow \|S\|$, monotone convergence gives the first inequality.

Second inequality. Write $\Gamma = \sum_j \sqrt{\lambda_j} Z_j e_j$ (Karhunen–Loève), with Z_j i.i.d. $N(0, 1)$ and $\sum_j \lambda_j = \mathbb{E}\|\Gamma\|^2 = \sum_i \mathbb{E}\|X_i\|^2$. Minkowski's inequality in $L^{p/2}$ ($p/2 \geq 1$) gives $\|\|\Gamma\|^2\|_{p/2} \leq \sum_j \lambda_j \|Z_j^2\|_{p/2} = m_p^{2/p} \sum_j \lambda_j$, i.e. $\mathbb{E}\|\Gamma\|^p \leq m_p (\sum_i \mathbb{E}\|X_i\|^2)^{p/2}$. The constants are optimal already for $\mathbb{H} = \mathbb{R}$ (Theorem A and [7, Thm. 1.5]). $\square$

5. PROOF OF THEOREM D

Let $\varphi_X(t) := \mathbb{E}e^{itX} \in [0, 1]$; φ_X is real, so X is symmetric, $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. For $p = 4$: $\mathbb{E}(X + S)^4 = \mathbb{E}X^4 + 6\mathbb{E}X^2 \mathbb{E}S^2 + \mathbb{E}S^4 \leq \mathbb{E}X^4 + 6B \mathbb{E}X^2 + A + 3B^2 = A + \mathbb{E}(X + B^{1/2}Z)^4$, using $\mathbb{E}X = \mathbb{E}S = 0$ and the case $p = 4$ of Theorem A. Let now $3 \leq p < 4$. Corollary 2.5 with $W = X$ gives $\mathbb{E}|X + S|^p \leq \mathbb{E}|X + X_G|^p$ with $G = \sum_i G_{X_i}$. By (3), using $\mathbb{E}(X + X_G)^2 = \mathbb{E}(X + B^{1/2}Z)^2 = \mathbb{E}X^2 + B$ and $\text{Re}(\varphi_X(t)e^{-a+ib}) = \varphi_X(t)e^{-a} \cos b$, the same computation as in (5) gives

$$c_p(\mathbb{E}|X + X_G|^p - A - \mathbb{E}|X + B^{1/2}Z|^p) = \int_0^\infty \left(\varphi_X(e^{-a} \cos b - e^{-g}) + a - g \right) t^{-p-1} dt.$$

Since $0 \leq \varphi_X \leq 1$ and $e^{-a} - e^{-g} \geq 0$, we get $\varphi_X(e^{-a} \cos b - e^{-g}) \leq \varphi_X(e^{-a} - e^{-g}) \leq e^{-a} - e^{-g}$, so the integrand is $\leq F(a) - F(g) \leq 0$. For sharpness, the sequence of the previous section satisfies $\mathbb{E}|X + S_{\mathbf{X}(n)}|^p \rightarrow A + \mathbb{E}|X + B^{1/2}Z|^p$. Indeed, the same conditioning on β gives $(1 - \varepsilon_n)\mathbb{E}|X + R_n - M_n \varepsilon_n|^p + \varepsilon_n M_n^p \mathbb{E}|(1 - \varepsilon_n) + (X + R_n)/M_n|^p$, and $X + R_n \rightarrow X + B^{1/2}Z$ in distribution with uniformly integrable p -th powers. $\square$

6. PROOF OF THEOREM E

(a). It suffices to treat $x_0 = 1$: if X_1 works for $x_0 = 1$, then $x_0 X_1$ (a zero-mean two-point variable) works for x_0 , by homogeneity of both sides and the symmetry of Z when $x_0 < 0$. Fix $u \in (0, 1)$ and $\lambda \in (0, \frac{1}{2}]$, and let $X_1 := u(\beta - \lambda)$ with $\beta \sim \text{Bernoulli}(\lambda)$. Then

$$L := \mathbb{E}|1 + X_1|^p = (1 - \lambda)(1 - u\lambda)^p + \lambda(1 + u - u\lambda)^p, \quad A = u^p(\lambda(1 - \lambda)^p + (1 - \lambda)\lambda^p), \quad B = u^2\lambda(1 - \lambda).$$

Lower bound for L . By Bernoulli's inequality $(1 - u\lambda)^p \geq 1 - pu\lambda$. By convexity of $y \mapsto y^p$, $(1 + u - u\lambda)^p \geq (1 + u)^p - pu\lambda(1 + u)^{p-1}$. Hence

$$L \geq 1 + \lambda((1 + u)^p - 1 - pu) - pu(1 + u)^{p-1}\lambda^2.$$

Upper bound for $R := A + \mathbb{E}|1 + B^{1/2}Z|^p$. We claim there is $\Theta_p < \infty$ with

$$(6) \quad |1 + y|^p \leq 1 + py + \binom{p}{2}y^2 + \binom{p}{3}y^3 + \Theta_p y^4 \quad (y \in \mathbb{R}).$$

For $y \geq -\frac{1}{2}$, Taylor's formula with Lagrange remainder gives remainder $\binom{p}{4}(1+\theta y)^{p-4}y^4$ for some $\theta \in (0,1)$, and $(1+\theta y)^{p-4} \leq 2^{4-p} \leq 2$. For $y < -\frac{1}{2}$ we have $|y|^k \leq 2^{4-k}y^4$ for $k \leq 4$, $|1+y|^p \leq (3|y|)^p \leq 3^p 2^{4-p}y^4$, and $-(1+py + \binom{p}{2}y^2 + \binom{p}{3}y^3) \leq (16+8p+4\binom{p}{2}+2\binom{p}{3})y^4$. This proves the claim. Taking $y = B^{1/2}Z$ in (6) and using $\mathbb{E}Z = \mathbb{E}Z^3 = 0$, $\mathbb{E}Z^4 = 3$, we get $\mathbb{E}|1+B^{1/2}Z|^p \leq 1 + \binom{p}{2}B + 3\Theta_p B^2$. With $A \leq u^p\lambda + u^p\lambda^p$ and $B \leq u^2\lambda$:

$$R \leq 1 + \lambda(u^p + \binom{p}{2}u^2) + u^p\lambda^p + 3\Theta_p u^4\lambda^2.$$

Conclusion. Let $D(u) := (1+u)^p - 1 - pu - \binom{p}{2}u^2 - u^p$. Then

$$L - R \geq \lambda D(u) - (pu(1+u)^{p-1} + 3\Theta_p u^4)\lambda^2 - u^p\lambda^p.$$

By Taylor's formula at order 4 (remainder $\binom{p}{4}(1+\theta u)^{p-4}u^4 \geq 0$, since $\binom{p}{4} > 0$ for $3 < p < 4$), $D(u) \geq \binom{p}{3}u^3 - u^p$. Moreover $\binom{p}{3} = \frac{p(p-1)(p-2)}{6} > 1$ and $u^{p-3} < 1$, so $D(u) > u^3(1-u^{p-3}) > 0$. Since $p > 2$, the right-hand side above is positive for all sufficiently small $\lambda > 0$. Thus $L > R$.

(b). Let $|X| \leq K$ and $|X| \geq \delta$ a.s., with $0 < \delta \leq K < \infty$. Put $f(x) = |x|^p$, $\tau := \mathbb{E}f'''(X) = p(p-1)(p-2)\mathbb{E}[|X|^{p-3}\text{sign } X] \neq 0$, and $\sigma := \text{sign } \tau$. For $0 < u \leq \delta/4$ and $0 < \lambda \leq \frac{1}{4}$ let $X_1 := \sigma u(\beta - \lambda)$ with $\beta \sim \text{Bernoulli}(\lambda)$ independent of X . Then $\mathbb{E}X_1 = 0$, $B := \mathbb{E}X_1^2 = u^2\lambda(1-\lambda)$ and $\mathbb{E}|X_1|^p \leq u^p(\lambda + \lambda^p)$.

Two Taylor bounds. Let $|x| \in [\delta, K]$. On $\{|y| \leq \delta/2\}$ the function $y \mapsto f(x+y)$ is C^∞ . Moreover $|f'''(z)| = p(p-1)(p-2)|z|^{p-4} \leq p(p-1)(p-2)(\delta/2)^{p-4}$ for $|z| \geq \delta/2$. Hence, with $C_1 := \binom{p}{4}(\delta/2)^{p-4}$ and $T_x(y) := f(x) + f'(x)y + \frac{1}{2}f''(x)y^2 + \frac{1}{6}f'''(x)y^3$,

$$(7) \quad |f(x+y) - T_x(y)| \leq C_1 y^4 \quad (|y| \leq \delta/2).$$

For $|y| > \delta/2$ we have $K \leq (2K/\delta)|y|$, so $f(x+y) \leq (K+|y|)^p \leq (1+2K/\delta)^p|y|^p \leq (1+2K/\delta)^p(2/\delta)^{4-p}y^4$. Also $|T_x(y)| \leq \sum_{k=0}^3 |f^{(k)}(x)||y|^k/k! \leq C'_2 y^4$ with $C'_2 := \sum_{k=0}^3 \frac{|f^{(k)}|_{\max}}{k!} (2/\delta)^{4-k}$, where $|f^{(k)}|_{\max} := \max_{\delta \leq |z| \leq K} |f^{(k)}(z)|$. Therefore, with $C_2 := \max\{C_1, (1+2K/\delta)^p(2/\delta)^{4-p} + C'_2\}$,

$$(8) \quad f(x+y) \leq T_x(y) + C_2 y^4 \quad (y \in \mathbb{R}).$$

Lower bound for $L := \mathbb{E}|X + X_1|^p$. Condition on $X = x$. The two values of X_1 are $y_1 = -\sigma u\lambda$ and $y_2 = \sigma u(1-\lambda)$, both of modulus $\leq u \leq \delta/4$. Using (7) and the identities $(1-\lambda)y_1 + \lambda y_2 = 0$, $(1-\lambda)y_1^2 + \lambda y_2^2 = B$ and $(1-\lambda)y_1^3 + \lambda y_2^3 = \sigma u^3\lambda(1-\lambda)(1-2\lambda)$,

$$(1-\lambda)f(x+y_1) + \lambda f(x+y_2) \geq f(x) + \frac{1}{2}f''(x)B + \frac{1}{6}f'''(x)\sigma u^3\lambda(1-\lambda)(1-2\lambda) - C_1 u^4\lambda(\lambda^3 + (1-\lambda)^4).$$

Taking expectation in x and using $\sigma\tau = |\tau|$, $\lambda^3 + (1-\lambda)^4 \leq 2$:

$$L \geq \mathbb{E}f(X) + \frac{1}{2}B\mathbb{E}f''(X) + \frac{1}{6}|\tau|u^3\lambda(1-\lambda)(1-2\lambda) - 2C_1 u^4\lambda.$$

Upper bound for $R := \mathbb{E}|X_1|^p + \mathbb{E}|X + B^{1/2}Z|^p$. By (8) with $y = B^{1/2}Z$, using $\mathbb{E}Z = \mathbb{E}Z^3 = 0$ and $\mathbb{E}Z^4 = 3$, $\mathbb{E}f(x + B^{1/2}Z) \leq f(x) + \frac{1}{2}f''(x)B + 3C_2 B^2$. Hence

$$R \leq u^p(\lambda + \lambda^p) + \mathbb{E}f(X) + \frac{1}{2}B\mathbb{E}f''(X) + 3C_2 u^4\lambda^2.$$

Conclusion. Since $(1-\lambda)(1-2\lambda) \geq \frac{3}{8}$ for $\lambda \leq \frac{1}{4}$,

$$\frac{L-R}{\lambda} \geq \frac{|\tau|}{16}u^3 - 2C_1 u^4 - u^p - u^p\lambda^{p-1} - 3C_2 u^4\lambda.$$

Because $p > 3$, we can first fix $u \in (0, \delta/4]$ with $\frac{|\tau|}{16}u^3 > 2(2C_1 u^4 + u^p)$, and then $\lambda \in (0, \frac{1}{4}]$ so small that $u^p\lambda^{p-1} + 3C_2 u^4\lambda < 2C_1 u^4 + u^p$. Then $L > R$. Part (a) is the special case $X \equiv x_0$; the separate proof of (a) above shows that there every $u \in (0, 1)$ works (after scaling to $x_0 = 1$). $\square$

Remark 6.1 (Numerical instances; 30-digit evaluation). The Gaussian terms were evaluated with $\mathbb{E}|x + \sigma Z|^p = \sigma^p 2^{p/2} \Gamma(\frac{p+1}{2}) \pi^{-1/2} {}_1F_1(-\frac{p}{2}; \frac{1}{2}; -\frac{x^2}{2\sigma^2})$. (i) In (a) with $p = 3.5$, $u = \frac{1}{2}$, $\lambda = \frac{1}{10}$: $L = 1.119202\dots$, $R = 1.104991\dots$, $L - R = 0.01421\dots$; the gap stays positive for $p = 3.05, 3.2, 3.8, 3.95$, being $0.0010, 0.0050, 0.0255$ and 0.0321 respectively. (ii) In (b) with $X \in \{-1, 3\}$ with probabilities $\frac{3}{4}, \frac{1}{4}$ (mean zero), $p = 3.5$, $X_1 = -\frac{1}{2}(\beta - \frac{1}{10})$: $L = 12.65025836\dots$, $R = 12.64955547\dots$ (this instance has $u = \frac{1}{2} > \delta/4$, so it is not produced by the recipe of the proof, and it is specific to the exponent: with the same X_1 the gap is negative at $p = 3.2$ and at $p = 3.95$; the recipe of the proof, with smaller u, λ , works at every $p \in (3, 4)$). These are single zero-mean two-point summands. For $p \leq 3$, Pinelis' Theorem 1.5 holds for every such configuration.

Remark 6.2 ($p = 4$). At $p = 4$ the extension still fails for non-centered X : for $X \equiv 1$ and any zero-mean X_1 , $\mathbb{E}(1 + X_1)^4 - \mathbb{E}X_1^4 - \mathbb{E}(1 + B^{1/2}Z)^4 = 4\mathbb{E}X_1^3 - 3B^2$, which equals $0.0345\dots$ for the instance (i). For centered X the extension holds at $p = 4$ (expand the fourth power), consistent with $\tau \rightarrow 0$ as $p \uparrow 4$ for centered X .

7. PROOF OF THEOREM F

Regularity of f . Let $k(s) = \cos s - 1 + s^2/2$. Since $0 \leq k(s) \leq \min(s^4/24, s^2/2)$, we have $0 \leq f(x) - \kappa x^4 \leq (x^2 + x^4) \int (t^2 \wedge t^4) d\mu$. Moreover $|\partial_x k(tx)| = t|tx - \sin tx| \leq t \min(|tx|^3/6, 2|tx|)$ and $0 \leq \partial_x^2 k(tx) = t^2(1 - \cos tx) \leq \min(t^4 x^2/2, 2t^2)$. These are dominated, locally uniformly in x , by multiples of $(t^2 \wedge t^4)$. By dominated convergence $f \in C^2$, with

$$f''(x) = 12\kappa x^2 + \int t^2(1 - \cos tx) \mu(dt), \quad 0 \leq f''(x) \leq C(1 + x^2), \quad 0 \leq f(x) \leq C(1 + x^4).$$

Poissonization. For fixed v , $h_v(x) := f(v + x)$ satisfies the hypotheses of Lemma 2.4 with exponent $p = 4$ (convex h_v'' ; $|h_v| \leq C_v(1 + x^4)$, $|h_v''| \leq C_v(1 + x^2)$). Hence the proof of Corollary 2.5, with $|\cdot|^p$ replaced by f and p -th moments by fourth moments, gives $\mathbb{E}f(S) \leq \mathbb{E}f(X_G)$ with $G = \sum_i G_{X_i}$. Here $\mathbb{E}X_G^4 < \infty$ by Lemma 2.3 with $p = 4$.

Fourier step. For any Y with $\mathbb{E}Y^4 < \infty$, Tonelli gives $\mathbb{E}f(Y) = \kappa \mathbb{E}Y^4 + \int (\operatorname{Re} \varphi_Y(t) - 1 + \frac{t^2}{2} \mathbb{E}Y^2) \mu(dt)$, all terms finite. For the quartic part, $\mathbb{E}X_G^4 = \int x^4 dG + 3B^2$ (the fourth cumulant of X_G is $\int x^4 dG$ and $\mathbb{E}X_G = 0$), so $\kappa \mathbb{E}X_G^4 = \kappa \int x^4 dG + \kappa \mathbb{E}(B^{1/2}Z)^4$. For the μ -part, exactly as in (5),

$$\int (\operatorname{Re} \varphi_{X_G} - 1 + g) d\mu - \int \int k(tx) G(dx) \mu(dt) - \int (e^{-g} - 1 + g) d\mu = \int (e^{-a} \cos b + a - e^{-g} - g) d\mu \leq 0.$$

Adding up, $\mathbb{E}f(X_G) \leq \int f dG + \mathbb{E}f(B^{1/2}Z) = \sum_i \mathbb{E}f(X_i) + \mathbb{E}f(B^{1/2}Z)$. $\square$

8. PROOF OF PROPOSITION G

For $4 < p < 6$ let $k_4(s) := 1 - \cos s - \frac{s^2}{2} + \frac{s^4}{24}$. Continuing the integration of $\sin s \leq s$ from the proof of Lemma 2.1 two more times gives $\sin s \leq s - s^3/6 + s^5/120$ and then $1 - \cos s \leq s^2/2 - s^4/24 + s^6/720$ for $s \geq 0$; together with $1 - \cos s \geq s^2/2 - s^4/24$ and evenness, $0 \leq k_4(s) \leq s^6/720$ for all s . Also $k_4(s) \leq s^4/24 + 3$, so $c'_p := \int_0^\infty k_4(s) s^{-p-1} ds \in (0, \infty)$ and $|x|^p = \frac{1}{c'_p} \int_0^\infty k_4(tx) t^{-p-1} dt$ for all x . For Y with $\mathbb{E}Y^4 < \infty$, Tonelli gives $c'_p \mathbb{E}|Y|^p = \int_0^\infty (1 - \operatorname{Re} \varphi_Y - \frac{t^2}{2} \mathbb{E}Y^2 + \frac{t^4}{24} \mathbb{E}Y^4) t^{-p-1} dt$. For $Y = X_G$ one has $\mathbb{E}X_G^4 = \int x^4 dG + 3B^2$, by the cumulant formula, the fourth cumulant being $\int x^4 dG$. Also $\mathbb{E}|B^{1/2}Z|^4 = 3B^2$ and $\int k_4(tx) G(dx) = a - g + \frac{t^4}{24} \int x^4 dG$. The terms with $\int x^4 dG$ cancel, and

$$\begin{aligned} c'_p (\mathbb{E}|X_G|^p - A - m_p B^{p/2}) &= \int_0^\infty (F(g) - a - e^{-a} \cos b) t^{-p-1} dt \\ &\geq \int_0^\infty (F(g) - F(a)) t^{-p-1} dt \geq 0. \end{aligned} \quad \square$$

9. SYMMETRIZATION, A SECOND PROOF, AND THE RANGE $4 < p < 5$

For a finite Borel measure G on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG < \infty$ let $\tilde{X}_G$ be the *centred* compound Poisson variable, $\mathbb{E}e^{it\tilde{X}_G} = \exp \int (e^{itx} - 1 - itx)G(dx)$. When $\int x dG = 0$, $\tilde{X}_G = X_G$. Let $G_s(E) := \frac{1}{2}(G(E) + G(-E))$ be the symmetrization of G . Then $\mathbb{E}e^{it\tilde{X}_G} = e^{-a(t)+ib(t)}$ with $a = \int (1 - \cos tx)dG$ and $b = \int (\sin tx - tx)dG$, and $\mathbb{E}e^{itX_{G_s}} = e^{-a(t)}$, with the same a . (If $\int x dG = 0$ this b agrees with that of Lemma 2.3; in general $\tilde{X}_G = X_G - \int x dG$.)

Proposition 9.1. *Let G be as above with $\int |x|^p dG < \infty$.*

- (a) *If $2 < p < 4$, then $\mathbb{E}|\tilde{X}_G|^p \leq \mathbb{E}|X_{G_s}|^p$.*
- (b) *If $4 < p < 6$, then $\mathbb{E}|\tilde{X}_G|^p \geq \mathbb{E}|X_{G_s}|^p$, and with c'_p as in the proof of Proposition G,*

$$c'_p(\mathbb{E}|\tilde{X}_G|^p - \mathbb{E}|X_{G_s}|^p) = \int_0^\infty e^{-a(t)}(1 - \cos b(t))t^{-p-1}dt.$$

Proof. Both variables have mean 0 and variance $B = \int x^2 dG$, and for $p > 4$ both have fourth moment $\int x^4 dG + 3B^2$, since the fourth cumulants agree. (a) By (3), $c_p(\mathbb{E}|\tilde{X}_G|^p - \mathbb{E}|X_{G_s}|^p) = \int_0^\infty e^{-a}(\cos b - 1)t^{-p-1}dt \leq 0$. (b) By the representation in the proof of Proposition G, $c'_p(\mathbb{E}|\tilde{X}_G|^p - \mathbb{E}|X_{G_s}|^p) = \int_0^\infty (e^{-a} - e^{-a} \cos b)t^{-p-1}dt \geq 0$. All integrals converge absolutely by the same bounds as before. $\square$

Remark 9.2 (A second proof of Theorem A). For $3 \leq p < 4$, Corollary 2.5, Proposition 9.1(a) and the symmetric exact bound ([4, Thm. 6.1 and Cor. 6.2], [5]) give $\mathbb{E}|S|^p \leq \mathbb{E}|X_G|^p \leq \mathbb{E}|X_{G_s}|^p \leq A + m_p B^{p/2}$. For the last step, X_{G_s} is the distributional limit of $S_n = \sum_{j \leq n} V_{j,n}$ with $V_{j,n}$ i.i.d. symmetric of law $(1 - G_s(\mathbb{R})/n)\delta_0 + G_s/n$. These satisfy $\sum_j \mathbb{E}V_{j,n}^2 = B$ and $\sum_j \mathbb{E}|V_{j,n}|^p = A$ exactly, so by Fatou's lemma for weak convergence $\mathbb{E}|X_{G_s}|^p \leq \liminf_n \mathbb{E}|S_n|^p \leq A + m_p B^{p/2}$. Thus for $3 \leq p \leq 4$ the zero-mean problem reduces to the symmetric one. For $4 < p < 6$ this reduction is impossible: for $p > 4$ the symmetric supremum is $c^p \mathbb{E}|\Pi_{\lambda/2} - \Pi'_{\lambda/2}|^p$ (Utev; [7, eq. (12)]), which by Proposition 9.1(b) applied to $G_* = \lambda\delta_c$ (the integrand $e^{-a}(1 - \cos b)$ being positive on a set of positive measure) is strictly below $c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$, a lower bound for $\mathcal{E}_{p,A,B}$ by [7, Lemma 2.1] and Fatou's lemma. Skewing the Lévy measure *increases* the p -th moment; this is why $p = 4$ is the threshold.

Proposition 9.3 (Conditional reduction for $4 < p < 5$). *Let $4 < p < 5$ and $A, B > 0$; let c, λ be given by $c^p \lambda = A$, $c^2 \lambda = B$, and $G_* = \lambda\delta_c$. Suppose that the following skewness inequality holds for every finite measure G on $\mathbb{R} \setminus \{0\}$ with $\int x dG = 0$, $\int x^2 dG = B$ and $\int |x|^p dG = A$:*

$$(9) \quad \mathbb{E}|\tilde{X}_G|^p - \mathbb{E}|X_{G_s}|^p \leq \mathbb{E}|\tilde{X}_{G_*}|^p - \mathbb{E}|X_{(G_*)_s}|^p = c^p(\mathbb{E}|\Pi_\lambda - \lambda|^p - \mathbb{E}|\Pi_{\lambda/2} - \Pi'_{\lambda/2}|^p).$$

Then $\mathcal{E}_{p,A,B} = c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$. In particular, if (9) holds for all $A, B > 0$ (by homogeneity: for all $\lambda > 0$), then Pinelis' conjecture holds for this p .

Proof. Let $\mathbf{X} \in \mathcal{X}_{p,A,B}$ and $G = \sum_i G_{X_i}$. Then $\int x dG = 0$, so $\tilde{X}_G = X_G$, and Corollary 2.5 gives $\mathbb{E}|S_{\mathbf{X}}|^p \leq \mathbb{E}|X_G|^p$. Write $\mathbb{E}|X_G|^p = (\mathbb{E}|X_G|^p - \mathbb{E}|X_{G_s}|^p) + \mathbb{E}|X_{G_s}|^p$. The first bracket is at most the right-hand side of (9). For the second, Utev's symmetric theorem [11] (see [7, eq. (12)]), valid for $p > 4$, combined with the approximation and Fatou argument of Remark 9.2, gives $\mathbb{E}|X_{G_s}|^p \leq c^p \mathbb{E}|\Pi_{\lambda/2} - \Pi'_{\lambda/2}|^p = \mathbb{E}|X_{(G_*)_s}|^p$. Adding up, $\mathbb{E}|S_{\mathbf{X}}|^p \leq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$. For the reverse inequality, [7, Lemma 2.1] gives $\mathbf{X}^{(n)} \in \mathcal{X}_{p,A,B}$ with $S_{\mathbf{X}^{(n)}} \rightarrow c(\Pi_\lambda - \lambda)$ in distribution, and Fatou's lemma for weak convergence gives $\liminf_n \mathbb{E}|S_{\mathbf{X}^{(n)}}|^p \geq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$. $\square$

Remark 9.4 (Numerical evidence for (9); not a proof). In Fourier form, (9) reads $\int e^{-a}(1 - \cos b)t^{-p-1}dt \leq \int e^{-a_*}(1 - \cos b_*)t^{-p-1}dt$ with $a_* = \lambda(1 - \cos ct)$, $b_* = \lambda(\sin ct - ct)$. We evaluated both sides with 30-digit quadrature and series-stabilized integrands for 180 random Lévy measures

with 1–3 atoms, at $p \in \{4.2, 4.5, 4.8, 5.5\}$. The identity of Proposition 9.1(b) matched direct summation to 7 digits, and the ratio of the two sides never exceeded 1; equality holds only for single (or coincident) atoms, and the largest ratio for genuinely multi-atom G was 0.9954. In the subfamily $G = \lambda_+ \delta_c + \lambda_- \delta_{-c}$, $\lambda_+ + \lambda_- = \lambda$, the symmetric part is fixed, so (9) there is *equivalent* to the restriction of the Poissonized conjecture $\mathbb{E}|\tilde{X}_G|^p \leq c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ to this subfamily. (These G have nonzero mean when $\lambda_+ \neq \lambda_-$; the centred laws $\tilde{X}_G$ enter the zero-mean problem through [7, Lemma 2.1].) Direct evaluation for $\lambda \in [0.1, 100]$ and $p \in \{4.2, 4.5, 4.8\}$ gives the maximum at $\lambda_- = 0$. Finally, for fixed G_s (180 random cases) the all-same-sign assignment maximized $\mathbb{E}|\tilde{X}_G|^p$ among all sign splittings. All scripts are in the accompanying repository (`numerics/split_45_stable.py` and others).

Remark 9.5 (Why second-variation arguments stop at $p = 5$; numerical). In the family $G_\theta = \frac{\lambda}{2}((1+\theta)\delta_1 + (1-\theta)\delta_{-1})$, $\theta \in [-1, 1]$, put $Y_\theta = \tilde{X}_{G_\theta}$. Proposition 9.1(b) and the identity $\mathbb{E}Y_\theta^4 = \lambda + 3\lambda^2$ give

$$c'_p \frac{d^2}{d\theta^2} \mathbb{E}|Y_\theta|^p = \int_0^\infty e^{-\lambda(1-\cos t)} \lambda^2 (t - \sin t)^2 \cos(\theta \lambda (t - \sin t)) t^{-p-1} dt.$$

For large λ the integrand is concentrated near $t \in 2\pi\mathbb{Z}$. The bump at $t = 0$ is positive and of order $\lambda^{(p-2)/2}$. The bumps at $t = 2\pi k$, $k \geq 1$, contribute $\approx \sqrt{2\pi} \lambda^{3/2} \sum_k (2\pi k)^{1-p} \cos(2\pi k \theta \lambda)$, of either sign. Hence for $4 < p < 5$ the map $\theta \mapsto \mathbb{E}|Y_\theta|^p$ is *not* convex when λ is large and $\theta \lambda \equiv \frac{1}{2} \pmod{1}$. A 30-digit evaluation at $p = 4.5$, $\lambda = 2 \cdot 10^4$, $\theta = 10000.5/\lambda$ (so $\theta \lambda \equiv \frac{1}{2}$) gives the value -3769 (bump at 0: $+6808$; lattice: -10577); at integer $\theta \lambda$ the lattice part is $+12848$. For $p > 5$ the bump at 0 dominates: at $p = 5.5$, same λ , the bump is $2.01 \cdot 10^6$ and the lattice part is $-1.75 \cdot 10^3$ at $\theta \lambda \equiv \frac{1}{2}$ ($+1.91 \cdot 10^3$ at integer $\theta \lambda$). The asymptotic description above is a Laplace-method heuristic without error bounds; the non-convexity is established numerically at the computed point. Nevertheless the endpoint $\theta = \pm 1$ (Poisson) remains the maximizer in all cases computed. At first order near $\theta = 1$ the $t = 0$ bump, of order $\lambda^{(p-2)/2}$, dominates the lattice terms, of order $\lambda^{1/2}$ (heuristically). Thus, for $4 < p < 5$, $H \mapsto \mathbb{E}|Y_H|^p$ is not convex along this segment of Lévy measures, hence not convex on the constraint set; arguments relying on such convexity cannot work there. This is consistent with the sign change of the factor $(q-4)(q-5)$ in the proofs of [7, Props. 2.9 and 2.11], and points to the lattice structure of the Poisson characteristic function as the source of the difficulty.

Proposition 9.6 (Second-order optimality of the Poisson law in the rare-jump regime). *For a finite Borel measure G on $\mathbb{R} \setminus \{0\}$ with $\int |x|^p dG < \infty$ put*

$$Q(G) := \frac{1}{2} \iint K(x, y) G(dx) G(dy), \quad K(x, y) := |x+y|^p - |x|^p - |y|^p - p|x|^{p-2}xy - p|y|^{p-2}yx.$$

- (a) (*Exact expansion.*) Let $p \geq 3$ and let G_1 be such a measure. As $\varepsilon \downarrow 0$, $\mathbb{E}|\tilde{X}_{\varepsilon G_1}|^p = \varepsilon \int |x|^p dG_1 + \varepsilon^2 Q(G_1) + O(\varepsilon^3)$.
- (b) (*Representation.*) Let $p > 4$ and let V, V' be i.i.d. with density $2(1-v)$ on $[0, 1]$. Then $K(x, y) = \frac{p(p-1)(p-2)(p-3)}{4} x^2 y^2 \mathbb{E}|xV + yV'|^{p-4}$ for all x, y .
- (c) (*Optimality.*) Let $4 < p \leq 5$, $\int x^2 dG = B > 0$, $\int |x|^p dG = A$, and let $c, \lambda > 0$ be defined by $c^p \lambda = A$, $c^2 \lambda = B$. Then $Q(G) \leq Q(\lambda \delta_c)$, with equality if and only if $G \in \{\lambda \delta_c, \lambda \delta_{-c}\}$.
- (d) (*Consequence.*) Let $4 < p \leq 5$ and let G_1 (with $\int x^2 dG_1 > 0$) not be a single atom. Then for all sufficiently small $\varepsilon > 0$, with $(A, B) = (\varepsilon \int |x|^p dG_1, \varepsilon \int x^2 dG_1)$, one has $\mathbb{E}|\tilde{X}_{\varepsilon G_1}|^p < c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$.

Proof. Write $f(x) = |x|^p$ and $q = G_1(\mathbb{R})$, $m = \int x dG_1$, ξ_k i.i.d. with law G_1/q , $N \sim \text{Poisson}(\varepsilon q)$, so $\tilde{X}_{\varepsilon G_1} \stackrel{d}{=} \sum_{k \leq N} \xi_k - \varepsilon m$.

(a) Condition on N . The event $N = 0$ contributes $e^{-\varepsilon q} f(-\varepsilon m) = O(\varepsilon^p) = O(\varepsilon^3)$ (here $p \geq 3$ is used). For $N = 1$, Taylor's formula with $f \in C^2$ and $\mathbb{E}|\xi|^p < \infty$ gives $\mathbb{E}f(\xi - \varepsilon m) = \mathbb{E}f(\xi) -$

$\varepsilon m \mathbb{E} f'(\xi) + O(\varepsilon^2)$. For $N = 2$, $\mathbb{E} f(\xi_1 + \xi_2 - \varepsilon m) = \mathbb{E} f(\xi_1 + \xi_2) + O(\varepsilon)$. Finally $\sum_{k \geq 3} \mathbb{P}(N = k) \mathbb{E} |f(S_k - \varepsilon m)| \leq \sum_{k \geq 3} \mathbb{P}(N = k) C(1 + k^p) = O(\varepsilon^3)$. Collecting terms,

$$\mathbb{E} f(\tilde{X}_{\varepsilon G_1}) = \varepsilon q \mathbb{E} f(\xi) + \varepsilon^2 \left[\frac{q^2}{2} \mathbb{E} f(\xi_1 + \xi_2) - q^2 \mathbb{E} f(\xi) - qm \mathbb{E} f'(\xi) \right] + O(\varepsilon^3),$$

and the bracket equals $\frac{1}{2} \iint [f(x+y) - f(x) - f(y) - yf'(x) - xf'(y)] dG_1 dG_1 = Q(G_1)$.

(b) For $p > 4$, $f \in C^4(\mathbb{R})$ with $f(0) = f'(0) = f''(0) = 0$ and $f'''(x) = p(p-1)(p-2)(p-3)|x|^{p-4}$. Put $\phi(x, y) = f(x+y) - f(x) - f(y) - yf'(x) - xf'(y)$; then $\phi(x, 0) = \phi(0, y) = 0$ and $\partial_x \partial_y \phi = \psi(x, y) := f''(x+y) - f''(x) - f''(y)$, while $\psi(u, 0) = \psi(0, w) = 0$ and $\partial_u \partial_w \psi = f'''(u+w)$. Hence $\phi(x, y) = \int_0^x \int_0^y \int_0^u \int_0^w f'''(a+b) db da dw du$ (oriented integrals). The substitution $u = \sigma x$, $w = \tau y$, $a = \sigma' u$, $b = \tau' w$ gives $\phi(x, y) = x^2 y^2 \int_{[0,1]^4} \sigma \tau f'''(\sigma \sigma' x + \tau \tau' y) d\sigma d\sigma' d\tau d\tau'$. Finally $\int_0^1 \int_0^1 \sigma g(\sigma \sigma') d\sigma' d\sigma = \int_0^1 (1-v)g(v)dv = \frac{1}{2} \mathbb{E} g(V)$ for bounded measurable g , applied in (σ, σ') and in (τ, τ') .

(c) Let $H = x^2 G$, a measure of mass B , and let Y_1, Y_2 be i.i.d. with law H/B , independent of (V, V') ; then $\mathbb{E}|Y|^{p-2} = A/B = c^{p-2}$. By (b), with $r := p-4 \in (0, 1]$ and $D_p := p(p-1)(p-2)(p-3)/8$,

$$\begin{aligned} Q(G) &= D_p B^2 \mathbb{E}|Y_1 V + Y_2 V'|^r \leq D_p B^2 \mathbb{E}(|Y_1|V + |Y_2|V')^r \\ &\leq D_p B^2 (\mathbb{E}|Y|)^r \mathbb{E}(V + V')^r \leq D_p B^2 c^r \mathbb{E}(V + V')^r. \end{aligned}$$

Here we used, in turn: the triangle inequality and $V, V' \geq 0$; Jensen's inequality conditionally on (V, V') for the concave function $(R_1, R_2) \mapsto (R_1 V + R_2 V')^r$ on $[0, \infty)^2$ ($0 < r \leq 1$; note $\mathbb{E}|Y| = \int |x|^3 dG/B < \infty$); and $\mathbb{E}|Y| \leq (\mathbb{E}|Y|^{p-2})^{1/(p-2)} = c$ (Lyapunov, $p-2 > 1$). The right-hand side is $Q(\lambda \delta_c)$, since then $Y \equiv c$. If equality holds, Lyapunov's inequality forces $|Y| = c$ a.s., and equality in the triangle inequality ($V, V' > 0$ a.s.) forces Y_1, Y_2 to have a.s. the same sign; as Y_1, Y_2 are i.i.d., $\mathbb{P}(Y > 0)^2 + \mathbb{P}(Y < 0)^2 = 1$, so Y has a constant sign. Thus $G = \lambda \delta_c$ or $\lambda \delta_{-c}$.

(d) The Poisson competitor of εG_1 is $\varepsilon G_{1,*}$ with $G_{1,*} = \lambda' \delta_c$ and fixed c ; so (a) applied to G_1 and to $G_{1,*}$ gives $\mathbb{E}|\tilde{X}_{\varepsilon G_1}|^p - \mathbb{E}|\tilde{X}_{\varepsilon G_{1,*}}|^p = \varepsilon^2 (Q(G_1) - Q(G_{1,*})) + O(\varepsilon^3)$, and $Q(G_1) < Q(G_{1,*})$ by (c). $\square$

Proof of Theorem H. By homogeneity ($X_i \mapsto X_i/c$) it suffices to take $c = 1$, so $A = B = \lambda$; then $B^{p/2} = \lambda^{p/2}$. The lower bound is [7, Lemma 2.1] with Fatou's lemma. For the upper bound let $\mathbf{X} \in \mathcal{X}_{p,\lambda,\lambda}$ and $G = \sum_i G_{X_i}$; by Corollary 2.5, $\mathbb{E}|\mathbf{S}_{\mathbf{X}}|^p \leq \mathbb{E}|X_G|^p$, where $\int x dG = 0$ and $\int x^2 dG = \int |x|^p dG = \lambda$. For any finite G on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG = \int |x|^p dG = \lambda$ (not necessarily centred), let $\psi(t) = \int (e^{itx} - 1 - itx)G(dx) = -a(t) + ib(t)$, the exponent of the centred law $\tilde{X}_G = X_G - \int x dG$, and $g(t) = \lambda t^2/2$. Being a shift of X_G , $\tilde{X}_G$ has a finite p -th moment; the proof of Lemma 2.3 gives $\mathbb{E}|X_G|^p < \infty$ without using $\int x dG = 0$. As in the proof of Proposition G (with $\mathbb{E}\tilde{X}_G = 0$, $\mathbb{E}\tilde{X}_G^2 = \lambda$, $\mathbb{E}\tilde{X}_G^4 = \int x^4 dG + 3\lambda^2$ and $\int x^4 dG \leq \int (x^2 + |x|^p) dG < \infty$),

$$c'_p(\mathbb{E}|\tilde{X}_G|^p - \lambda) = \int_0^\infty \left(1 - a - e^{-a} \cos b + \frac{g^2}{2}\right) t^{-p-1} dt = \int_0^\infty \left(\frac{g^2 - \operatorname{Re} \psi^2}{2} - \operatorname{Re} \rho(\psi)\right) t^{-p-1} dt,$$

where $\rho(z) = e^z - 1 - z - z^2/2$; here $1 - a - e^{-a} \cos b = -\operatorname{Re}(e^\psi - 1 - \psi)$. Since $|e^{iy} - 1 - iy| \leq y^2/2$, we have $|\psi(t)| \leq g(t)$. For $\operatorname{Re} z \leq 0$, $|\rho(z)| \leq |z|^3/6$ and $|\rho(z)| \leq |e^z - 1 - z| + |z|^2/2 \leq |z|^2$. Hence

$$\int_0^\infty |\rho(\psi)| t^{-p-1} dt \leq \int_0^\infty \min\left(\frac{g^3}{6}, g^2\right) t^{-p-1} dt = \eta_p \lambda^{p/2},$$

by splitting at $t^2 = 12/\lambda$. In particular the two parts of the integrand are separately integrable (near $t = 0$, $g^2 - \operatorname{Re} \psi^2 = O(t^6)$; at ∞ , $|g^2 - \operatorname{Re} \psi^2| \leq 2g^2$). The quadratic part equals $c'_p Q(G)$; apply the display to εG , in which ψ, g scale by ε ; the ρ -term is $O(\varepsilon^{p/2}) = o(\varepsilon^2)$; comparing with

Proposition 9.6(a) identifies the ε^2 -coefficients. We record the outcome, which is used again in the proofs of Theorems I–K: for every finite G on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG = \int |x|^p dG = \lambda$, centred or not,

$$(10) \quad \mathbb{E}|\tilde{X}_G|^p = \lambda + Q(G) - \frac{1}{c_p'} \int_0^\infty \operatorname{Re} \rho(\psi_G(t)) t^{-p-1} dt, \quad \psi_G(t) = \int (e^{itx} - 1 - itx) G(dx).$$

By Proposition 9.6(c), $Q(G) \leq Q(\lambda\delta_1) = (2^{p-1} - p - 1)\lambda^2$. Therefore, for our $G = \sum_i G_{X_i}$ (for which $\tilde{X}_G = X_G$),

$$\mathbb{E}|X_G|^p \leq \lambda + q_p \lambda^2 + \eta_p \lambda^{p/2} / c_p',$$

and taking the supremum over $\mathbf{X}$ gives the upper bound for $\mathcal{E}_{p;\lambda,\lambda}$. For $G_* = \lambda\delta_1$ (the centred Poisson law, $\tilde{X}_{G_*} = \Pi_\lambda - \lambda$) one has $Q(G_*) = q_p \lambda^2$, so the display gives $|\mathbb{E}|\Pi_\lambda - \lambda|^p - \lambda - q_p \lambda^2| \leq \eta_p \lambda^{p/2} / c_p'$. Scaling back ($c^p \lambda = A$, $c^p \lambda^2 = A\lambda$, $c^p \lambda^{p/2} = B^{p/2}$) gives the displays of the theorem, and the remaining assertions follow by combining them. Here $c_p' = \int_0^\infty k_4(s) s^{-p-1} ds$, and the closed form is the classical integral $\int_0^\infty (\cos s - 1 + \frac{s^2}{2} - \frac{s^4}{24}) s^{-p-1} ds = \Gamma(-p) \cos \frac{\pi p}{2}$ for $4 < p < 6$. It follows from $\int_0^\infty s^{w-1} \sin s ds = \Gamma(w) \sin \frac{\pi w}{2}$ ($-1 < \operatorname{Re} w < 1$) by repeated integration by parts. It is used only to evaluate constants, namely c_p' in Theorem H and K_p in Theorem J. An alternative to the scaling step is the pointwise identity $c_p' K(x, y) = \int_0^\infty [\frac{1}{4} t^4 x^2 y^2 - (1 - \cos tx)(1 - \cos ty) + (\sin tx - tx)(\sin ty - ty)] t^{-p-1} dt$. $\square$

The rare-jump regime: proof of Theorem I. By homogeneity we take $c = 1$, so $A = B = \lambda$. Let $\mathcal{G}_\lambda$ be the set of finite Borel measures G on $\mathbb{R} \setminus \{0\}$ with $\int x^2 dG = \int |x|^p dG = \lambda$. For $G \in \mathcal{G}_\lambda$ let $Y = Y_G$ be a random variable with law $x^2 G(dx) / \lambda$; then $\mathbb{E}|Y|^{p-2} = 1$ and $\mathbb{E}|Y| \leq 1$. Put $G_* := \lambda\delta_1$, $r := p - 4 \in (0, 1)$, $D_p := p(p-1)(p-2)(p-3)/8$, $e_r := \mathbb{E}(V + V')^r$ with V, V' as in Proposition 9.6, and

$$\omega_0(s) := \frac{p-3}{p-2} + \frac{s^{p-2}}{p-2} - s \quad (s \geq 0), \quad \hat{\omega}(y) := \omega_0(|y|) + \mathbf{1}_{\{y < 0\}}(1 + |y|^{p-2}) \quad (y \in \mathbb{R}).$$

By the weighted AM–GM inequality, $\omega_0 \geq 0$ with equality only at $s = 1$; hence $\hat{\omega} \geq 0$ with equality only at $y = 1$. Since $\omega_0(1) = \omega_0'(1) = 0$ and $\omega_0''(s) = (p-3)s^{p-4} \geq \frac{1}{2}$ on $[\frac{1}{2}, 2]$, we have $\omega_0(s) \geq \frac{1}{4}(s-1)^2$ there. Moreover $C_\omega := \sup\{(1 + s^{p-2})/\omega_0(s) : s \in [0, \frac{1}{2}] \cup (2, \infty)\} < \infty$, because ω_0 is continuous and positive on $[0, \frac{1}{2}] \cup [2, \infty)$ and $(1 + s^{p-2})/\omega_0(s) \rightarrow p-2$ as $s \rightarrow \infty$. Finally $\mathbb{E}\omega_0(|Y|) = 1 - \mathbb{E}|Y|$, since $\mathbb{E}|Y|^{p-2} = 1$.

Lemma 9.7 (Stability of Proposition 9.6(c)). *There is $c_1 = c_1(p) > 0$ such that for all $\lambda > 0$ and all $G \in \mathcal{G}_\lambda$ with $\mathbb{P}(Y \in [\frac{1}{2}, 2]) \geq \mathbb{P}(Y \in [-2, -\frac{1}{2}])$,*

$$Q(G_*) - Q(G) \geq c_1 \lambda^2 \mathbb{E}\hat{\omega}(Y).$$

Proof. Let Y_1, Y_2 be independent copies of Y , independent of (V, V') , and let $u := \mathbb{E}|Y|$. As in the proof of Proposition 9.6(c), $Q(G) = D_p \lambda^2 \mathbb{E}|Y_1 V + Y_2 V'|^r$, $Q(G_*) = D_p \lambda^2 e_r$, and $\mathbb{E}(|Y_1|V + |Y_2|V')^r \leq u^r e_r$ by Jensen's inequality, applied conditionally on (V, V') . Hence

$$Q(G_*) - Q(G) \geq D_p \lambda^2 [(1 - u^r) e_r + \Sigma], \quad \Sigma := \mathbb{E}[(|Y_1|V + |Y_2|V')^r - |Y_1 V + Y_2 V'|^r] \geq 0.$$

Since $u \mapsto 1 - u^r$ is convex on $[0, 1]$, $1 - u^r \geq r(1 - u) = r\mathbb{E}\omega_0(|Y|)$. For $x, y \in [0, 2]$ the mean value theorem gives $(x+y)^r - |x-y|^r \geq r4^{r-1}((x+y) - |x-y|) = 2r4^{r-1} \min(x, y)$. Applied with $x = aV$, $y = bV'$, $a, b \in [\frac{1}{2}, 2]$, and combined with $\mathbb{E} \min(V, V') = \int_0^1 \mathbb{P}(V > s)^2 ds = \int_0^1 (1-s)^4 ds = \frac{1}{5}$, this gives $\mathbb{E}[(aV + bV')^r - |aV - bV'|^r] \geq g_* := r4^{r-1}/5$. The integrand of Σ vanishes unless $Y_1 Y_2 < 0$, and the joint law and the integrand are invariant under exchanging (Y_1, V) with (Y_2, V') . Hence, with $\pi_\pm := \mathbb{P}(\pm Y \in [\frac{1}{2}, 2])$, conditioning on (Y_1, Y_2) gives $\Sigma \geq 2g_* \pi_+ \pi_-$. Next, $1 + |y|^{p-2} \leq 9$ for $|y| \leq 2$, and $1 \leq 1 + |y|^{p-2} \leq C_\omega \omega_0(|y|)$ for $|y| \notin [\frac{1}{2}, 2]$; therefore

$$\mathbb{E}\hat{\omega}(Y) \leq (1 + C_\omega) \mathbb{E}\omega_0(|Y|) + 9\pi_-, \quad \mathbb{P}(|Y| \notin [\frac{1}{2}, 2]) \leq C_\omega \mathbb{E}\omega_0(|Y|).$$

If $\mathbb{E}\omega_0(|Y|) \geq 1/(4C_\omega)$, then $9\pi_- \leq 36C_\omega\mathbb{E}\omega_0(|Y|)$, and $Q(G_*) - Q(G) \geq D_p\lambda^2re_r\mathbb{E}\omega_0(|Y|) \geq \frac{D_pre_r}{1+37C_\omega}\lambda^2\mathbb{E}\hat{\omega}(Y)$. Otherwise $\pi_+ + \pi_- > \frac{3}{4}$, hence $\pi_+ > \frac{3}{8}$ by hypothesis, so $\Sigma \geq \frac{3}{4}g_*\pi_-$ and

$$\mathbb{E}\hat{\omega}(Y) \leq M_1 \left(re_r\mathbb{E}\omega_0(|Y|) + \frac{3}{4}g_*\pi_- \right) \leq M_1 \frac{Q(G_*) - Q(G)}{D_p\lambda^2}, \quad M_1 := \max \left\{ \frac{1+C_\omega}{re_r}, \frac{12}{g_*} \right\}.$$

Both cases give the claim with $c_1 := D_p \min\{re_r/(1+37C_\omega), 1/M_1\}$. $\square$

Lemma 9.8 (A constraint-adapted bound for the Lévy exponent). *For $t > 0$ and $y \in \mathbb{R}$ let $h_t(y) := (e^{ity} - 1 - ity)/y^2$, $h_t(0) := -t^2/2$, and $e_t(y) := h_t(y) - h_t(1) - \frac{h'_t(1)}{p-2}(|y|^{p-2} - 1)$. Then $|e_t(y)| \leq C_M t^2 \hat{\omega}(y)$ for all $t > 0$ and $y \in \mathbb{R}$, where $C_M := \max\{968, 5 \max(1, C_\omega)\}$. Consequently, for $G \in \mathcal{G}_\lambda$ and $\psi_G(t) := \int (e^{itx} - 1 - itx)G(dx)$,*

$$|\psi_G(t) - \psi_{G_*}(t)| \leq C_M t^2 \lambda \mathbb{E}\hat{\omega}(Y) \quad (t > 0).$$

Proof. From $e^{iz} - 1 - iz = -z^2 \int_0^1 (1-u)e^{iuz} du$ we get $h_t(y) = -t^2 \int_0^1 (1-u)e^{ituy} du$. So h_t is smooth, with $|h_t| \leq t^2/2$, $|h'_t| \leq t^3/6$ and $|h''_t| \leq t^4/12$. Also, with $\phi(y) := e^{ity} - 1 - ity$, we have $h'_t(1) = \phi'(1) - 2\phi(1)$ and $h''_t = \phi''y^{-2} - 4\phi'y^{-3} + 6\phi y^{-4}$ for $y \neq 0$, where $|\phi''| = t^2$, $|\phi'| \leq 2t$ and $|\phi(y)| \leq 2 + t|y|$. Hence $|h'_t(1)| \leq 4t + 4$ and, on $[\frac{1}{2}, 2]$, $|h''_t| \leq 4t^2 + 256t + 192$. Using the first set of bounds for $t \leq 1$ and the second for $t > 1$, we get $|h'_t(1)| \leq 8t^2$, and $|h''_t| \leq 452t^2$ on $[\frac{1}{2}, 2]$, for all $t > 0$.

On $[\frac{1}{2}, 2]$ we have $e_t(1) = e'_t(1) = 0$ and $e''_t(y) = h''_t(y) - (p-3)y^{p-4}h'_t(1)$, with $0 < (p-3)y^{p-4} \leq 4$. Hence $|e''_t| \leq 452t^2 + 32t^2 = 484t^2$ there, and Taylor's formula with integral remainder gives $|e_t(y)| \leq 242t^2(y-1)^2 \leq 968t^2\omega_0(y) = 968t^2\hat{\omega}(y)$. Off $[\frac{1}{2}, 2]$, $|e_t(y)| \leq |h_t(y)| + |h_t(1)| + \frac{|h'_t(1)|}{p-2}(1 + |y|^{p-2}) \leq t^2 + 4t^2(1 + |y|^{p-2}) \leq 5t^2(1 + |y|^{p-2})$, and $1 + |y|^{p-2} \leq \max(1, C_\omega)\hat{\omega}(y)$ there (for $y < 0$ by the definition of $\hat{\omega}$, for $y \geq 0$ by that of C_ω).

For the consequence, let $H := x^2G$, so that $H(\mathbb{R}) = \int |x|^{p-2}dH = \lambda$, $\psi_G(t) = \int h_t dH$ and $\psi_{G_*}(t) = \lambda h_t(1)$. The two constraints give $\psi_G(t) - \psi_{G_*}(t) = \int e_t dH$, and $\int |e_t| dH \leq C_M t^2 \int \hat{\omega} dH = C_M t^2 \lambda \mathbb{E}\hat{\omega}(Y)$. $\square$

Proof of Theorem I. Let $G \in \mathcal{G}_\lambda$, $\Phi(G) := \mathbb{E}|\tilde{X}_G|^p$ and $\Omega := \mathbb{E}\hat{\omega}(Y_G)$. Replacing G by its reflection $G(-\cdot)$ changes neither $\Phi(G)$ nor $Q(G)$, and replaces Y by $-Y$; so we may assume $\mathbb{P}(Y \in [\frac{1}{2}, 2]) \geq \mathbb{P}(Y \in [-2, -\frac{1}{2}])$. By (10), $\Phi(G) = \lambda + Q(G) - \frac{1}{c'_p} \int_0^\infty \operatorname{Re} \rho(\psi_G) t^{-p-1} dt$ with $\rho(z) = e^z - 1 - z - z^2/2$, and likewise for G_* . Hence

$$\Phi(G) - \Phi(G_*) = -(Q(G_*) - Q(G)) - \frac{1}{c'_p} \int_0^\infty \operatorname{Re} (\rho(\psi_G) - \rho(\psi_{G_*})) t^{-p-1} dt.$$

For $\operatorname{Re} z \leq 0$, $\rho'(z) = e^z - 1 - z$ satisfies $|\rho'(z)| \leq \min(|z|^2/2, 2|z|)$. Every point z of the segment $[\psi_{G_*}(t), \psi_G(t)]$ has $\operatorname{Re} z \leq 0$ and $|z| \leq g(t) = \lambda t^2/2$. So, by Lemma 9.8,

$$|\rho(\psi_G(t)) - \rho(\psi_{G_*}(t))| \leq |\psi_G(t) - \psi_{G_*}(t)| \min\left(\frac{g^2}{2}, 2g\right) \leq C_M \lambda \Omega t^2 \min\left(\frac{\lambda^2 t^4}{8}, \lambda t^2\right).$$

The substitution $t = s/\sqrt{\lambda}$ gives

$$\begin{aligned} \int_0^\infty t^2 \min\left(\frac{\lambda^2 t^4}{8}, \lambda t^2\right) t^{-p-1} dt &= j_p \lambda^{(p-2)/2}, \quad \text{where} \\ j_p &:= \int_0^\infty s^2 \min\left(\frac{s^4}{8}, s^2\right) s^{-p-1} ds = \frac{8^{(6-p)/2}}{8(6-p)} + \frac{8^{(4-p)/2}}{p-4} < \infty. \end{aligned}$$

Together with Lemma 9.7 this gives

$$\Phi(G) - \Phi(G_*) \leq -c_1 \lambda^2 \Omega + \frac{C_M j_p}{c'_p} \lambda^{p/2} \Omega = \lambda^2 \Omega \left(\frac{C_M j_p}{c'_p} \lambda^{(p-4)/2} - c_1 \right).$$

Let $\lambda_0(p) := (c_1 c'_p / (2C_M j_p))^{2/(p-4)}$. For $\lambda \leq \lambda_0(p)$ the bracket is at most $-c_1/2$, so $\Phi(G) \leq \Phi(G_*)$. Equality forces $\Omega = 0$, i.e. $Y = 1$ a.s., i.e. $G = G_*$; before the reflection this means $G \in \{\lambda\delta_1, \lambda\delta_{-1}\}$. Since $\Phi(G_*) = \mathbb{E}|\Pi_\lambda - \lambda|^p$, this proves the second assertion (for $c = 1$). For the first, let $\mathbf{X} \in \mathcal{X}_{p;\lambda,\lambda}$ and $G := \sum_i G_{X_i} \in \mathcal{G}_\lambda$. This G is centred, so $\tilde{X}_G = X_G$, and Corollary 2.5 gives $\mathbb{E}|S_{\mathbf{X}}|^p \leq \Phi(G) \leq \mathbb{E}|\Pi_\lambda - \lambda|^p$. The reverse inequality for the supremum is [7, Lemma 2.1] with Fatou's lemma. Scaling back gives the theorem. $\square$

Remark 9.9 (Size of λ_0 ; numerical). Take g_* as above and $e_r = q_p/D_p$. The supremum defining C_ω is the limit as $s \downarrow 2$, as a grid evaluation confirms, so $C_\omega = (1 + 2^{p-2})/\omega_0(2) \approx 8.82, 7.72, 7.05$ at $p = 4.2, 4.5, 4.8$. With these values, the definition $\lambda_0 = (c_1 c'_p / (2C_M j_p))^{2/(p-4)}$ gives $\lambda_0 \approx 2 \cdot 10^{-74}, 4 \cdot 10^{-28}, 4 \cdot 10^{-17}$. The loss comes from the exponent $2/(p-4)$ and from crude constants. The best constant in Lemma 9.7 is at most about 0.13, 0.45, 0.96, versus the proved $c_1 \approx 0.0024, 0.010, 0.025$; these upper bounds are attained by an all-negative two-point Y just outside $[-2, -\frac{1}{2}]$. The best constant in Lemma 9.8 is about 1.5, 1.3, 1.1, versus $C_M = 968$. With these values the same argument would give $\lambda_0 \approx 5 \cdot 10^{-29}, 5 \cdot 10^{-10}, 8 \cdot 10^{-6}$ (`numerics/check_small_lambda.py`). As $p \downarrow 4$, c_1 is of order $p-4$, so λ_0 tends to 0 super-exponentially. This is intrinsic to the method: at $p = 4$ the constraints fix $\mathbb{E}S^4$ and the deficit of Q disappears. A substantially larger λ_0 would need a remainder bound better than $O(\lambda^{p/2}\Omega)$, for instance one separating the small jumps of G .

Remark 9.10. (i) The argument in (c) uses $p \leq 5$ only through the concavity of $t \mapsto t^{p-4}$; $p = 5$ is a threshold of this method only. For $p \geq 5$ the conclusion $Q(G) \leq Q(\lambda\delta_c)$ also holds: by [7, Lemma 2.1] and Fatou, $\mathbb{E}|\tilde{X}_{\varepsilon G}|^p \leq \mathcal{E}_{p;\varepsilon A, \varepsilon B}$, which equals the Poisson value by [7, Thm. 1.3]; expanding both sides by (a) gives the claim. (A sufficient pointwise condition, $\vartheta(1, s) \leq \vartheta(1, 1)((1 + |s|^{p-2})/2)^{(p-4)/(p-2)}$ with $\vartheta = K/(x^2 y^2)$, also holds on a numerical grid for $p \in [4.05, 5.9]$.) (ii) Proposition 9.6(d) is a statement along rays: the $O(\varepsilon^3)$ term depends on G_1 . The uniform statements are Theorem H, whose remainder $O(B^{p/2})$ is uniform but only of relative size $O(\lambda^{(p-4)/2})$ compared with the second-order term, and Theorem I. In the latter, Lemma 9.7 is the quantitative form of (c), and Lemma 9.8 makes the remainder vanish to the same order as the deficit, $\mathbb{E}\hat{\omega}(Y)$, near $\lambda\delta_1$. Without the subtraction of $h_t(1) + \frac{h'_t(1)}{p-2}(|y|^{p-2} - 1)$ the remainder would only be of the order of the distance of Y from 1, whereas the deficit is of the order of its square. (iii) The representation (b) extends to $3 < p < 4$ (then $f'''' \propto |x|^{p-4}$ is locally integrable and f''' is absolutely continuous, which suffices for the same integrations). There $r = p - 4 < 0$, and small jumps (Gaussian-like parts) make Q large: e.g. at $p = 3.5$, $G = \delta_1 + n^2\delta_{1/n}$ gives $Q(G)/Q(\lambda\delta_c) = 1.43, 2.88, 7.21$ for $n = 10, 100, 1000$. This is consistent with Theorem A.

The Gaussian regime: proof of Theorem J. Again $c = 1$, $G \in \mathcal{G}_\lambda$, $g(t) = \lambda t^2/2$, and $\psi_G(t) = \int(e^{itx} - 1 - itx)G(dx) = -a(t) + ib(t)$. Put

$$\begin{aligned} \varpi(u) &:= e^{iu} - 1 - iu + \frac{u^2}{2}, & k_2(u) &:= \operatorname{Re} \varpi(u) = \cos u - 1 + \frac{u^2}{2} \geq 0, \\ \psi_r(t) &:= \int \varpi(tx) G(dx) = \psi_G(t) + g(t), \end{aligned}$$

so that $\operatorname{Re} \psi_r = g - a \in [0, g]$ and $\operatorname{Im} \psi_r = b$. From $|\varpi(u)| \leq |u|^3/6$, $k_2(u) \leq u^4/24$ and the interpolation inequalities $\int |x|^3 dG \leq \lambda$, $\int x^4 dG \leq \lambda$ (weighted AM–GM: $|x|^q \leq \theta x^2 + (1-\theta)|x|^p$ with $q = 2\theta + p(1-\theta)$), we get

$$(11) \quad |\psi_r(t)| \leq \lambda t^3/6, \quad 0 \leq \operatorname{Re} \psi_r(t) \leq \lambda t^4/24, \quad |\psi_r(t)| \leq |\psi_G(t)| + g(t) \leq 2g(t).$$

Moreover $a(t) \geq \frac{11}{16}g(t)$ for $0 < t \leq \frac{1}{2}$. Indeed, $1 - \cos u \geq \frac{11}{24}u^2$ for $|u| \leq 1$, and $\int_{|x|>1/t} x^2 dG \leq t^{p-2} \int |x|^p dG \leq \lambda/4$, so $a(t) \geq \frac{11}{24}t^2 \cdot \frac{3}{4}\lambda$.

Lemma 9.11 (A second decomposition). *For $G \in \mathcal{G}_\lambda$ and $\Phi(G) = \mathbb{E}|\tilde{X}_G|^p$,*

$$c'_p(\Phi(G) - \lambda - m_p \lambda^{p/2}) = L(G) + N(G), \quad \text{where}$$

$$L(G) := \int_0^\infty \operatorname{Re} \psi_r (1 - e^{-g}) t^{-p-1} dt, \quad N(G) := - \int_0^\infty e^{-g} \operatorname{Re} (e^{\psi_r} - 1 - \psi_r) t^{-p-1} dt,$$

with absolutely convergent integrals.

Proof. The display in the proof of Theorem H gives $c'_p(\Phi(G) - \lambda) = \int_0^\infty (1 - a - e^{-a} \cos b + \frac{g^2}{2}) t^{-p-1} dt$, and the representation of $|x|^p$ by k_4 applied to $\lambda^{1/2} Z$ gives $c'_p m_p \lambda^{p/2} = \int_0^\infty (1 - g - e^{-g} + \frac{g^2}{2}) t^{-p-1} dt$. Both integrals converge absolutely. The difference of the integrands is $g - a - e^{-a} \cos b + e^{-g} = \operatorname{Re} \psi_r (1 - e^{-g}) - e^{-g} \operatorname{Re} (e^{\psi_r} - 1 - \psi_r)$, because $e^{-a} \cos b = \operatorname{Re} e^{\psi_r} = e^{-g} \operatorname{Re} e^{\psi_r}$. The first term is nonnegative and, by (11), at most $\min(\lambda t^4/24, g) \min(g, 1)$, which is integrable against t^{-p-1} for $4 < p < 6$. Hence the second term is integrable as well. $\square$

Lemma 9.12 (The linear part is maximized by the Poisson law). *For $\mu > 0$ let $\Psi(\mu) := \int_0^\infty k_2(u)(1 - e^{-\mu u^2/2}) u^{-p-1} du$, and for $s > 0$ let $\varphi_\lambda(s) := s \Psi(\lambda s^{-2/(p-2)})$. Then φ_λ is strictly concave on $(0, \infty)$, and $L(G) = \lambda \mathbb{E} \varphi_\lambda(|Y_G|^{p-2})$. Consequently $L(G) \leq \lambda \varphi_\lambda(1) = L(G_*)$ for every $G \in \mathcal{G}_\lambda$, with equality if and only if $|Y_G| = 1$ a.s.*

Proof. Ψ is finite, since $k_2(u)(1 - e^{-\mu u^2/2}) \leq \min(\mu u^6/48, u^2/2)$. Differentiation under the integral sign, justified by domination on $\mu \geq \mu_0 > 0$, gives $\Psi'(\mu) = \frac{1}{2} \int k_2(u) u^{1-p} e^{-\mu u^2/2} du > 0$ and $\Psi''(\mu) = -\frac{1}{4} \int k_2(u) u^{3-p} e^{-\mu u^2/2} du < 0$. With $\beta := 2/(p-2) \in (0, 1)$ and $\mu = \lambda s^{-\beta}$, so that $d\mu/ds = -\beta\mu/s$, we get $\varphi'_\lambda(s) = \Psi(\mu) - \beta\mu\Psi'(\mu)$ and $\varphi''_\lambda(s) = -\frac{\beta\mu}{s}((1-\beta)\Psi'(\mu) - \beta\mu\Psi''(\mu)) < 0$. By Tonelli and the substitution $u = t|x|$, $L(G) = \int \kappa_\lambda dG$ with $\kappa_\lambda(x) = \int_0^\infty k_2(tx)(1 - e^{-\lambda t^2/2}) t^{-p-1} dt = |x|^p \Psi(\lambda/x^2)$, i.e. $\kappa_\lambda(x) = x^2 \varphi_\lambda(|x|^{p-2})$. Hence $L(G) = \int \varphi_\lambda(|x|^{p-2}) x^2 G(dx) = \lambda \mathbb{E} \varphi_\lambda(|Y|^{p-2})$. Since $\mathbb{E}|Y|^{p-2} = 1$ and $\varphi_\lambda \geq 0$ is strictly concave, Jensen's inequality gives the claim and its equality case. $\square$

Lemma 9.13 (The nonlinear part). *With $\rho(z) = e^z - 1 - z - z^2/2$ as before, put*

$$N_s := \frac{1}{2} \int_0^\infty e^{-g} (\operatorname{Im} \psi_r)^2 t^{-p-1} dt, \quad N_c := \frac{1}{2} \int_0^\infty e^{-g} (\operatorname{Re} \psi_r)^2 t^{-p-1} dt,$$

$$N_3 := \int_0^\infty e^{-g} \operatorname{Re} \rho(\psi_r) t^{-p-1} dt,$$

and $N_s^+ := \frac{1}{2} \int_0^\infty e^{-g} (\lambda t^3/6)^2 t^{-p-1} dt = n_1 \lambda^{(p-2)/2}$ with $n_1 := 2^{(6-p)/2} \Gamma(\frac{6-p}{2})/144$. Then $N = N_s - N_c - N_3$ with absolutely convergent integrals. Moreover there is $C_N = C_N(p)$ such that for $\lambda \geq 1$ and all $G \in \mathcal{G}_\lambda$,

$$0 \leq N_s(G) \leq N_s^+, \quad N_s(G_*) \geq N_s^+ - C_N \lambda^{(p-4)/2},$$

$$0 \leq N_c(G) \leq C_N \lambda^{(p-4)/2}, \quad |N_3(G)| \leq C_N \lambda^{(p-4)/2}.$$

In particular $N(G) \leq N_s^+ + C_N \lambda^{(p-4)/2}$ and $N(G_) \geq N_s^+ - 3C_N \lambda^{(p-4)/2}$.*

Proof. The identity $N = N_s - N_c - N_3$ follows from $\operatorname{Re}(z^2) = (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2$. The integrals converge absolutely by the bounds below. By (11), $0 \leq N_s(G) \leq N_s^+$, and for every $G \in \mathcal{G}_\lambda$, $0 \leq N_c(G) \leq \frac{1}{2} \int e^{-g} (\lambda t^4/24)^2 t^{-p-1} dt = O(\lambda^{(p-4)/2})$. For G_* , $\operatorname{Im} \psi_r = -\lambda(t - \sin t)$ and $0 \leq (t^3/6)^2 - (t - \sin t)^2 \leq t^8/360$, so $N_s^+ - N_s(G_*) \leq \frac{\lambda^2}{720} \int e^{-g} t^{7-p} dt = O(\lambda^{(p-4)/2})$. For N_3 on $t \leq \frac{1}{2}$, write $\rho(z) = z^3/6 + \rho_4(z)$ with $|\rho_4(z)| \leq \frac{|z|^4}{24} e^{\max(\operatorname{Re} z, 0)}$. With $\xi := \operatorname{Re} \psi_r \in [0, \lambda t^4/24]$ and $|\eta| := |\operatorname{Im} \psi_r| \leq \lambda t^3/6$, the cubic term has $|\operatorname{Re}(\psi_r^3)| = |\xi^3 - 3\xi\eta^2| \leq \frac{\lambda^3 t^{12}}{13824} + \frac{\lambda^3 t^{10}}{288}$: its dominant part $(i\eta)^3$ is purely imaginary. For the quartic term, $e^{-g} e^{\operatorname{Re} \psi_r} = e^{-a} \leq e^{-11g/16}$ gives

$e^{-g}|\rho_4(\psi_r)| \leq e^{-11g/16}(\lambda t^3/6)^4/24$. On $t > \frac{1}{2}$ we use $e^{-g}|\rho(\psi_r)| \leq e^{-a} + e^{-g}(1 + 2g + 2g^2) \leq 3$. Hence, uniformly in $G \in \mathcal{G}_\lambda$,

$$|N_3(G)| \leq \int_0^\infty \left[\frac{e^{-g}}{6} \left(\frac{\lambda^3 t^{12}}{13824} + \frac{\lambda^3 t^{10}}{288} \right) + e^{-11g/16} \frac{\lambda^4 t^{12}}{31104} \right] t^{-p-1} dt + 3 \int_{1/2}^\infty t^{-p-1} dt = O(\lambda^{(p-4)/2}) + \frac{3 \cdot 2^p}{p}.$$

All $O(\cdot)$ terms are explicit multiples of the indicated powers (or of smaller ones), by $\int_0^\infty e^{-\alpha t^2} t^k dt = \frac{1}{2} \alpha^{-(k+1)/2} \Gamma(\frac{k+1}{2})$. Collecting the terms and using $\lambda \geq 1$ proves the lemma. $\square$

Proof of Theorem J. Take $c = 1$. Let $G \in \mathcal{G}_\lambda$. By Lemmas 9.11–9.13, $c'_p(\Phi(G) - \Phi(G_*)) = (L(G) - L(G_*)) + (N(G) - N(G_*)) \leq 4C_N \lambda^{(p-4)/2}$. As in the proof of Theorem I, Poissonization and [7, Lemma 2.1] turn this into $\mathbb{E}|\Pi_\lambda - \lambda|^p \leq \mathcal{E}_{p;\lambda,\lambda} \leq \mathbb{E}|\Pi_\lambda - \lambda|^p + 4C_N \lambda^{(p-4)/2}/c'_p$. For the expansion of $\Phi(G_*)$, write $k_2(u) = u^4/24 - k_4(u)$. With $\int_0^\infty u^{3-p}(1 - e^{-\mu u^2/2}) du = -\frac{1}{2} \Gamma(\frac{4-p}{2})(\frac{\mu}{2})^{(p-4)/2}$ (valid for $4 < p < 6$), this gives $\Psi(\mu) = l_1 \mu^{(p-4)/2} - c'_p + \int_0^\infty k_4(u) e^{-\mu u^2/2} u^{-p-1} du$ with $l_1 := -2^{(4-p)/2} \Gamma(\frac{4-p}{2})/48 > 0$. The last integral lies in $[0, \int \frac{u^6}{720} e^{-\mu u^2/2} u^{-p-1} du]$, so it is $O(\mu^{(p-6)/2})$. Hence $L(G_*) = \lambda \Psi(\lambda) = l_1 \lambda^{(p-2)/2} - c'_p \lambda + O(\lambda^{(p-4)/2})$, and by Lemma 9.13

$$c'_p(\Phi(G_*) - m_p \lambda^{p/2}) = c'_p \lambda + L(G_*) + N(G_*) = (l_1 + n_1) \lambda^{(p-2)/2} + O(\lambda^{(p-4)/2}).$$

Finally $(l_1 + n_1)/c'_p = K_p$. Indeed, Legendre's duplication formula for $\Gamma(-p)$ and the reflection formula for $\Gamma(\frac{1-p}{2})\Gamma(\frac{1+p}{2})$ give $c'_p m_p = -2^{-p/2-1} \Gamma(-\frac{p}{2})$. Together with $\Gamma(2 - \frac{p}{2}) = \frac{p(p-2)}{4} \Gamma(-\frac{p}{2})$ and $\Gamma(3 - \frac{p}{2}) = -\frac{p-4}{2} \Gamma(2 - \frac{p}{2})$ this yields $l_1 = c'_p m_p \frac{p(p-2)}{24}$ and $n_1 = c'_p m_p \frac{p(p-2)(p-4)}{72}$, whose sum is $c'_p m_p \frac{p(p-1)(p-2)}{72}$. Scaling back ($c^p \lambda^{p/2} = B^{p/2}$, $c^p \lambda^{(p-2)/2} = B^{p/2}/\lambda$, $c^p \lambda^{(p-4)/2} = B^{p/2} \lambda^{-2}$) proves the theorem. $\square$

Remark 9.14 (On the constants in Theorem J; numerical). Of the constant C_N , the part multiplying $\lambda^{(p-4)/2}$ is small; at $p = 4.5$ it is below 0.02. The bulk is the λ -independent term $3 \cdot 2^p/p \approx 15.1$ in the bound for N_3 , from $t > \frac{1}{2}$, which was absorbed using $\lambda \geq 1$. Keeping it separate, the upper bound of Theorem J at $p = 4.5$ falls below the first correction $K_p B^{p/2}/\lambda$ once $\lambda \gtrsim 10^2$. The function φ_λ of Lemma 9.12 satisfies $\varphi_\lambda(s) = \lambda^{1/\beta} \varphi_1(s \lambda^{-1/\beta})$, so its concavity for all λ is equivalent to the case $\lambda = 1$.

Exactness in the Gaussian regime: proof of Theorem K. We keep $c = 1$ and the notation of the two previous proofs; in particular $\hat{\omega}$, C_ω and C_M are as in Lemmas 9.7 and 9.8, $Y = Y_G$, and $\Omega = \Omega(G) := \mathbb{E} \hat{\omega}(Y_G)$. Write $\delta\psi := \psi_G - \psi_{G_*} = \int e_t dH$ (Lemma 9.8), and note $\psi_r(G) - \psi_r(G_*) = \delta\psi$.

Lemma 9.15. *For all $t > 0$ and $y \in \mathbb{R}$,*

$$|e_t(y)| \leq C_M \min(t^2, t^3) \hat{\omega}(y), \quad \text{hence} \quad |\delta\psi(t)| \leq C_M \min(t^2, t^3) \lambda \Omega.$$

Proof. For $t \geq 1$ this is Lemma 9.8. Let $t < 1$. On $[\frac{1}{2}, 2]$, $|e_t''| \leq |h_t''| + 4|h_t'(1)| \leq t^4/12 + 4t^3/6 \leq \frac{3}{4}t^3$, so $|e_t(y)| \leq \frac{3}{8}t^3(y-1)^2 \leq \frac{3}{2}t^3\omega_0(y)$. Off $[\frac{1}{2}, 2]$, $|h_t(y) - h_t(1)| \leq \frac{t^3}{6}|y-1| \leq \frac{t^3}{3}(1 + |y|^{p-2})$ and $\frac{|h_t'(1)|}{p-2}||y|^{p-2} - 1| \leq \frac{t^3}{12}(1 + |y|^{p-2})$. So $|e_t(y)| \leq \frac{5}{12}t^3(1 + |y|^{p-2}) \leq \frac{5}{12} \max(1, C_\omega) t^3 \hat{\omega}(y)$. Both constants are below C_M . $\square$

Lemma 9.16 (A one-sided bound for the imaginary part). *Let $t_1 := \min\{0.45, (10/(3C_\omega))^{1/2}\}$ and $c_B := \min\{1/108, 1/(24 \max(1, C_\omega))\}$. Then $\text{Im } e_t(y) \geq c_B t^3 \hat{\omega}(y)$ for all $0 < t \leq t_1$ and $y \in \mathbb{R}$.*

Proof. Let $S(u) := (u - \sin u)/u^2$ ($S(0) = 0$), an odd function, and $R(u) := u/6 - S(u)$. From $u - \sin u = u^3 \int_0^1 \frac{(1-v)^2}{2} \cos(uv) dv$ we get $R(u) = u \int_0^1 \frac{(1-v)^2}{2} (1 - \cos uv) dv$. Hence, for $u \geq 0$,

$0 \leq R(u) \leq \min(u^3/120, u/6)$, and, using $1 - \cos x \leq x^2/2$, $|\sin x| \leq |x|$ and $\int_0^1 \frac{(1-v)^2}{2} v^2 dv = \frac{1}{60}$, also $|R'(u)| \leq u^2/40$ and $|R''(u)| \leq u/20$. Since $\operatorname{Im} h_t(y) = (\sin ty - ty)/y^2 = -t^2 S(ty)$, we have

$$\operatorname{Im} e_t(y) = t^2 F_t(y), \quad F_t(y) := S(t) - S(ty) + tS'(t) \frac{|y|^{p-2} - 1}{p-2}.$$

For $y \geq 0$ (with $h_t(0) = -t^2/2$, so that $S(0) = R(0) = 0$), inserting $S = u/6 - R$ gives $F_t(y) = \frac{t}{6}\omega_0(y) + \Gamma_t(y)$ with $\Gamma_t(y) := R(ty) - R(t) - tR'(t) \frac{y^{p-2}-1}{p-2}$. On $[\frac{1}{2}, 2]$, $\Gamma_t(1) = \Gamma'_t(1) = 0$ and $|\Gamma''_t(y)| = |t^2 R''(ty) - tR'(t)(p-3)y^{p-4}| \leq \frac{t^3}{10} + \frac{t^3}{10}$, so $|\Gamma_t(y)| \leq \frac{t^3}{10}(y-1)^2 \leq \frac{2}{5}t^3\omega_0(y) \leq \frac{t}{12}\omega_0(y)$, since $t \leq 0.45$. Off $[\frac{1}{2}, 2]$, $R(ty) \geq 0$ gives $\Gamma_t(y) \geq -\frac{t^3}{120} - \frac{t^3}{80}(1+y^{p-2}) \geq -\frac{t^3}{40}(1+y^{p-2}) \geq -\frac{C_\omega t^3}{40}\omega_0(y) \geq -\frac{t}{12}\omega_0(y)$, since $t^2 \leq 10/(3C_\omega)$; at $y = 0$ this uses that the supremum defining C_ω includes $s = 0$. Thus $F_t(y) \geq \frac{t}{12}\omega_0(y)$ for $y \geq 0$. For $y < 0$, oddness of S gives $F_t(y) = F_t(|y|) + 2S(t|y|) \geq \frac{t}{12}\omega_0(|y|) + 2S(t|y|)$. Here $S \geq 0$ on $[0, \infty)$, and $2S(u) \geq \frac{u}{3} - \frac{u^3}{60} \geq \frac{u}{6}$ for $0 \leq u \leq 3$. For $|y| \in [\frac{1}{2}, 2]$ (so $t|y| \leq 1$) this gives $F_t(y) \geq \frac{t}{12}\omega_0(|y|) + \frac{t}{12} \geq \frac{t}{108}\hat{\omega}(y)$, because $1 + |y|^{p-2} \leq 9$. For $|y| \notin [\frac{1}{2}, 2]$, $F_t(y) \geq \frac{t}{24}\omega_0(|y|) + \frac{t}{24C_\omega}(1 + |y|^{p-2})$. In all cases $F_t(y) \geq c_B t \hat{\omega}(y)$. $\square$

Proof of Theorem K. Let $\lambda \geq \max(1, t_1^{-2})$ and $G \in \mathcal{G}_\lambda$. All constants $C_1, C_2, \dots$ below depend only on p . By Lemmas 9.11 and 9.12, and with $N = N_s - N_c - N_3$ as in Lemma 9.13,

$$(12) \quad c'_p(\Phi(G) - \Phi(G_*)) \leq (N_s(G) - N_s(G_*)) + |N_c(G) - N_c(G_*)| + |N_3(G) - N_3(G_*)|.$$

Step 1: variation of N_c and N_3 . By (11) and Lemma 9.15, $|\operatorname{Re} \psi_r(G)^2 - \operatorname{Re} \psi_r(G_*)^2| \leq |\delta\psi| \frac{\lambda t^4}{12} \leq \frac{C_M}{12} \lambda^2 \Omega t^7$, so $|N_c(G) - N_c(G_*)| \leq \frac{C_M}{24} \lambda^2 \Omega \int_0^\infty e^{-gt} t^{6-p} dt = C_1 \lambda^{(p-3)/2} \Omega$. For N_3 and $t \leq \frac{1}{2}$, every point z of the segment $[\psi_r(G_*), \psi_r(G)]$ has $|z| \leq \lambda t^3/6$ and $\operatorname{Re} z \leq g - \frac{11}{16}g$, by the bound $a \geq \frac{11}{16}g$ applied to both measures. Since $|\rho'(z)| = |e^z - 1 - z| \leq \frac{|z|^2}{2} e^{\max(\operatorname{Re} z, 0)}$, we get $e^{-g} |\rho(\psi_r(G)) - \rho(\psi_r(G_*))| \leq |\delta\psi| e^{-11g/16} \frac{\lambda^2 t^6}{72} \leq \frac{C_M}{72} \lambda^3 \Omega t^9 e^{-11g/16}$. For $t > \frac{1}{2}$ we write $e^{-g} \rho(\psi_r) = e^{\psi_G} - e^{-g}(1 + \psi_r + \psi_r^2/2)$. For $\operatorname{Re} z_0, \operatorname{Re} z_1 \leq 0$,

$$|e^{z_1} - e^{z_0}| \leq |z_1 - z_0| \int_0^1 e^{(1-s)\operatorname{Re} z_0} ds \leq |z_1 - z_0| \min(1, 1/|\operatorname{Re} z_0|).$$

We apply this with $z_0 = \psi_{G_*}$, $|\operatorname{Re} z_0| = a_*(t) = \lambda(1 - \cos t)$, and use $|\psi_r| \leq 2g$ for the remaining terms. This gives $e^{-g} |\rho(\psi_r(G)) - \rho(\psi_r(G_*))| \leq |\delta\psi| (\min(1, 1/a_*(t)) + e^{-g}(1+2g))$. Since $1 - \cos t \geq \frac{2}{\pi^2} d(t)^2$ with $d(t) = \operatorname{dist}(t, 2\pi\mathbb{Z})$, we have $\int_{\mathbb{R}} \min(1, \frac{\pi^2}{2\lambda u^2}) du = 2\sqrt{2} \pi \lambda^{-1/2}$. Summing over the neighbourhoods $[(2k-1)\pi, (2k+1)\pi]$ of $2\pi k$ gives

$$J(\lambda) := \int_{1/2}^\infty t^{1-p} \min\left(1, \frac{1}{a_*(t)}\right) dt \leq 2\sqrt{2} \pi^{2-p} \zeta(p-1) \lambda^{-1/2} + \frac{\pi^2 2^{p-1}}{p} \lambda^{-1}.$$

On $[\frac{1}{2}, \pi]$ we used $d(t) = t$. Moreover $e^{-g}(1+2g) \leq 3e^{-\lambda t^2/4}$. With $|\delta\psi| \leq C_M t^2 \lambda \Omega$, the part $t > \frac{1}{2}$ is at most $C_M \lambda \Omega (J(\lambda) + 3 \int_{1/2}^\infty e^{-\lambda t^2/4} t^{1-p} dt) = O((\lambda^{1/2} + 1)\Omega)$. Integrating, and using $\lambda^{1/2} \leq \lambda^{(p-3)/2}$ for $\lambda \geq 1$, we get $|N_3(G) - N_3(G_*)| \leq C_2 \lambda^{(p-3)/2} \Omega$.

Step 2: the case $\Omega(G) \leq \Omega_1 := 1/(7C_M)$. Put $\sigma(t) := t - \sin t$, so $\operatorname{Im} \psi_{G_*} = -\lambda\sigma$, and put $\varepsilon_t := \mathbb{E} \operatorname{Im} e_t(Y)$, so $\operatorname{Im} \psi_G = -\lambda\sigma + \lambda\varepsilon_t$. Then $(\operatorname{Im} \psi_{G_*})^2 - (\operatorname{Im} \psi_G)^2 = \lambda^2 \varepsilon_t (2\sigma - \varepsilon_t)$. For $t \leq t_1$, Lemmas 9.16 and 9.15 give $c_B t^3 \Omega \leq \varepsilon_t \leq C_M t^3 \Omega$. Moreover $\sigma \geq t^3/6 - t^5/120 \geq t^3/7$, so $2\sigma - \varepsilon_t \geq t^3/7$ and $(\operatorname{Im} \psi_{G_*})^2 - (\operatorname{Im} \psi_G)^2 \geq \frac{c_B}{7} \lambda^2 \Omega t^6$. For $t > t_1$, $|\lambda^2 \varepsilon_t (2\sigma - \varepsilon_t)| \leq \lambda^2 C_M t^3 \Omega (\frac{t^3}{3} + C_M t^3 \Omega_1) \leq \frac{C_M}{2} \lambda^2 \Omega t^6$. Hence, using $\int_0^{t_1} e^{-\lambda t^2/2} t^{5-p} dt \geq \lambda^{-(6-p)/2} e^{-1/2}/(6-p)$ (valid as $t_1 \sqrt{\lambda} \geq 1$) and $e^{-\lambda t^2/2} \leq e^{-\lambda t_1^2/4} e^{-t^2/4}$ for $t \geq t_1$, $\lambda \geq 1$,

$$N_s(G_*) - N_s(G) \geq c_s \lambda^{(p-2)/2} \Omega - C_3 \lambda^2 e^{-\lambda t_1^2/4} \Omega, \quad c_s := \frac{c_B e^{-1/2}}{14(6-p)}.$$

With (12) and Step 1, $c'_p(\Phi(G) - \Phi(G_*)) \leq \lambda^{(p-2)/2} \Omega(-c_s + C_3 \lambda^{(6-p)/2} e^{-\lambda t_1^2/4} + (C_1 + C_2) \lambda^{-1/2})$. Fix $\lambda_{\text{near}}(p) \geq \max(1, t_1^{-2})$ such that the bracket is negative for all $\lambda \geq \lambda_{\text{near}}(p)$; such a number exists because the bracket tends to $-c_s$. For $\lambda \geq \lambda_{\text{near}}(p)$ we get $\Phi(G) \leq \Phi(G_*)$, with equality only if $\Omega = 0$, i.e. $G = G_*$.

Step 3: the remaining case. Let $\check{G} := G(-\cdot)$; then $\Phi(\check{G}) = \Phi(G)$ and $Y_{\check{G}} = -Y_G$. If $\Omega(G) \leq \Omega_1$ or $\Omega(\check{G}) \leq \Omega_1$, Step 2 applies to G or to $\check{G}$. Otherwise choose $G' \in \{G, \check{G}\}$ with $\pi_+ \geq \pi_-$, where $\pi_{\pm} := \mathbb{P}(\pm Y' \in [\frac{1}{2}, 2])$ and $Y' = Y_{G'}$. Then $\Omega(G') > \Omega_1$. Since $\mathbb{E}Y'_{\pm} \geq \pi_{\pm}/2$,

$$|\mathbb{E}Y'| = \mathbb{E}|Y'| - 2\min(\mathbb{E}Y'_+, \mathbb{E}Y'_-) \leq 1 - \mathbb{E}\omega_0(|Y'|) - \pi_-,$$

and, as in the proof of Lemma 9.7, $\Omega(G') \leq (1 + C_{\omega})\mathbb{E}\omega_0(|Y'|) + 9\pi_-$. Hence $1 - (\mathbb{E}Y')^2 \geq 1 - |\mathbb{E}Y'| \geq \Omega(G')/(10 + C_{\omega}) > \Omega_1/(10 + C_{\omega})$. Next, with $\sigma_t(y) := (ty - \sin ty)/y^2$ we have $\text{Im} \psi_{G'}(t) = -\lambda \mathbb{E}\sigma_t(Y')$ and $\sigma_t(y) - t^3 y/6 = -t^2 R(ty)$. Interpolating $R(u) \leq \min(u^3/120, u/6)$ ($u \geq 0$; R is odd, so $|R(u)| = R(|u|)$) with exponent $\theta = (p-3)/2$ gives $t^2 |R(ty)| \leq C_4 t^p |y|^{p-2}$, so $|\mathbb{E}\sigma_t(Y')| \leq \frac{t^3}{6} |\mathbb{E}Y'| + C_4 t^p$. Squaring and integrating, $N_s(G') \leq (\mathbb{E}Y')^2 N_s^+ + C_5 \lambda^{1/2}$. Lemma 9.13 gives $N_s(G_*) \geq N_s^+ - C_N \lambda^{(p-4)/2}$, together with the bounds $0 \leq N_c(\cdot) \leq C_N \lambda^{(p-4)/2}$ and $|N_3(\cdot)| \leq C_N \lambda^{(p-4)/2}$, valid for every measure in $\mathcal{G}_{\lambda}$. So (12) gives

$$c'_p(\Phi(G') - \Phi(G_*)) \leq -\frac{\Omega_1 n_1}{10 + C_{\omega}} \lambda^{(p-2)/2} + C_5 \lambda^{1/2} + 5C_N \lambda^{(p-4)/2},$$

which is negative for all $\lambda \geq \lambda_{\text{far}}(p)$, for a suitable $\lambda_{\text{far}}(p) < \infty$. Hence $\Phi(G) = \Phi(G') < \Phi(G_*)$.

With $\lambda_1(p) := \max\{1, t_1^{-2}, \lambda_{\text{near}}(p), \lambda_{\text{far}}(p)\}$ we have shown that $\Phi(G) \leq \Phi(G_*)$ for all $G \in \mathcal{G}_{\lambda}$, with equality only for $G \in \{\lambda \delta_1, \lambda \delta_{-1}\}$. Poissonization and [7, Lemma 2.1] conclude the proof, as for Theorem I. $\square$

Remark 9.17 (Size of λ_1 ; numerical). We evaluated the explicit constants of the proof with the script `numerics/lambda1_constants.py`. At $p = 4.05, 4.2, 4.5, 4.8$ they give $\lambda_{\text{near}} \approx 10^{13.0}, 10^{12.3}, 10^{11.8}, 10^{11.4}$ and $\lambda_{\text{far}} \approx 10^{9.3}, 10^{8.1}, 10^{6.8}, 10^{6.0}$. So $\lambda_1(p) \approx 10^{11}-10^{13}$, and it stays bounded as $p \downarrow 4$, unlike $\lambda_0(p)$. The refined Lipschitz bound in Step 1 matters here. With $|e^{z_1} - e^{z_0}| \leq |z_1 - z_0|$ alone, a term of order $\lambda \Omega$ would remain; that gives $\lambda_1 \approx 10^{29}$ at $p = 4.5$, with $\lambda_1 \rightarrow \infty$ as $p \downarrow 4$. The remaining loss comes from the constants C_M and c_B . Numerically, $\text{Im} e_t(y)/(t^3 \hat{\omega}(y))$ stays above 0.04 for $t \leq t_1$ (`numerics/check_K1.py`), whereas $c_B \approx 0.005$.

Remark 9.18 (A conditional reduction for all λ ; numerical). By Lemmas 9.11 and 9.12, Pinelis' conjecture for a given λ follows from the inequality $N(G) \leq N(G_*)$ for all centred $G \in \mathcal{G}_{\lambda}$; Poissonization only produces centred measures. We tested it at $p = 4.5$ on two families. The first is 150 random two- and three-atom G for each $\lambda \in \{0.05, 0.3, 1, 3, 10\}$. The second is a grid of 343 two-atom G for each λ , of which 7 coincide with $\lambda \delta_{-1}$. For $\lambda \geq 0.3$ the inequality held in every non-degenerate case. At $\lambda = 0.05$ it failed for 64 of the 150 random and 184 of the 343 grid measures, although $\Phi(G) < \Phi(G_*)$ held for all non-degenerate measures of both families. So it is not a substitute for Theorem I when λ is small. The scripts are `numerics/check_N_sign.py` (outputs `nsign*.txt`) and `numerics/check_large_lambda.py`. The latter also checks Lemma 9.11 against exact Poisson summation, to ten digits, and the concavity of φ_1 on a logarithmic grid.

10. DISCUSSION; WHAT REMAINS OPEN

- (1) **Where $p \geq 3$ is used.** Only in Corollary 2.5, through convexity of $|v + x|^{p-2}$, which is needed to apply Lemma 2.4. The bound (4) for compound Poisson laws holds for all $2 < p < 4$.
- (2) **Why $p = 4$ is the threshold.** For $2 < p < 4$ the representing kernel $\cos s - 1 + s^2/2$ is nonnegative and the comparison $F(a) \leq F(g)$ goes the right way. For $4 < p < 6$ the kernel changes to $-(\cos s - 1 + s^2/2 - s^4/24)$ and the same comparison reverses (Proposition G).

- (3) **Open:** $4 < p < 5$ **with** $B^{p/2}/A$ **not small.** Pinelis' conjecture $\mathcal{E}_{p;A,B} = c^p \mathbb{E}|\Pi_\lambda - \lambda|^p$ is proved here only for $\lambda \leq \lambda_0(p)$ (Theorem I), with a tiny $\lambda_0(p)$, and for $\lambda \geq \lambda_1(p)$ (Theorem K), with $\lambda_1(p) \approx 10^{12}$. It remains open for $\lambda \in (\lambda_0(p), \lambda_1(p))$. For $1 \leq \lambda < \lambda_1(p)$, Theorem J confirms it up to $O(B^{p/2}\lambda^{-2})$, and Remark 9.18 gives a sufficient condition, $N(G) \leq N(G_*)$, that held in all our tests with $\lambda \geq 0.3$. Making λ_0, λ_1 reasonable would need sharper constants in Lemmas 9.7, 9.8 and 9.16; closing the gap requires a non-perturbative idea. In particular the companion hypothesis of [6] for $t \geq 4$, $C(t) = \mathbb{E}|\Pi_1 - 1|^t$, remains open for $4 < t < 5$. Under Pinelis' conjecture it is the case $\lambda = 1$: numerically, $\mathbb{E}|\Pi_\lambda - \lambda|^t / \max(\lambda, \lambda^{t/2})$ is maximal at $\lambda = 1$. That value lies inside the gap $(\lambda_0(p), \lambda_1(p))$. The other partial results are Theorem H (the conjecture holds up to an additive $C_p B^{p/2}$, which is of lower order than both sides when $B^{p/2}/A \rightarrow 0$), Proposition 9.6 (second-order optimality of the Poisson law along every ray in the rare-jump regime), and Proposition 9.3 (a reduction to a “skewness inequality” bounding $\mathbb{E}|\tilde{X}_G|^p - \mathbb{E}|X_{G_s}|^p$ by its Poisson value). After Poissonization, $c'_p(\mathbb{E}|X_G|^p - A) = \int_0^\infty (1 - a - e^{-a} \cos b + g^2/2)t^{-p-1}dt$. The Poisson law is not a pointwise maximizer of this integrand, so a variational argument is needed. Numerically (grid scans of two-atom Lévy measures at $p = 4.5$ and $\lambda_{\text{cand}} \in \{0.05, 0.3, 1, 3, 10, 40\}$; random and differential-evolution searches over up to three atoms plus a Gaussian part at $p \in \{4.1, \dots, 4.9\}$) the Poisson value was never beaten; relative margins of the best two-atom competitors ranged from 10^{-9} (nearly merged same-side atoms) to $5 \cdot 10^{-3}$ (two-sided).
- (4) **Role of the moment-curve viewpoint (heuristic).** The first-order (KKT) conditions identify the support of an optimal Lévy measure with the contact set of the planar curve $v \mapsto (v, \Psi_W(\pm v^{1/(p-2)}))$ with its upper hull, in the sense of [3]. Here $\Psi_W(u) = \mathbb{E}[f(W+u) - f(W) - uf'(W)]/u^2$. Convexity of this curve on the skew side and its failure on the anti-skew side (observed numerically, not proved) is consistent with the failure of the extension with an extra skewed summand (Theorem E), whose proof is a direct Taylor argument. The final proof of Theorem A bypasses the curve analysis entirely.
- (5) **Open: the extra-summand version for $3 < p < 4$.** Theorems D and E leave open: symmetric X whose characteristic function changes sign (e.g. Rademacher X); non-symmetric X with $\mathbb{E}[|X|^{p-3} \text{sign } X] = 0$; and X that is unbounded or not bounded away from 0 (Theorem E(b) needs both). For $X = \pm x_0$ Rademacher we found no violation in about 900 random compound Poisson configurations at $p \in \{3.2, 3.5, 3.8\}$ (the gap approached 0 only in degenerate limits), which suggests that the extension holds for all symmetric X .

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