

Short proofs of three nonlinear-mean inequalities of Nazarov, Vasyunin and Volberg

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Abstract

For the nonlinear mean $M[a, b] = (a + b)/\sqrt{4 + (a - b)^2}$, Nazarov, Vasyunin and Volberg [1] proved several inequalities by computer-assisted positivity tests of large polynomials. We give short computer-free proofs of three of them, I3, I4 and I6 of [1, §§4.3, 4.4, 4.6]. For I3, coordinates given by the hyperbolic distances between $x + i$, $y + i$, $z + i$ in the upper half-plane reduce the inequality to one polynomial identity and the triangle inequality; the same two facts give the reverse inequality when y and z lie on opposite sides of x . For I4 and I6 we use a chart, centred at a value u of M , in which the level set $\{M = u\}$ becomes an antidiagonal line: I4 follows from the concavity of one explicit function, and I6 from the endpoint values of a concave cubic.

1 Results

Throughout, $M[a, b] = (a + b)/\sqrt{4 + (a - b)^2}$ for real a, b .

Theorem 1 (I3, [1, §4.3]). *Let $x, y, z \geq 0$ with $x(y + z) \leq 2$.*

(a) *If $(x - y)(x - z) \geq 0$, then $M[M[y, x], M[z, x]] \geq M[M[y, z], x]$.*

(b) *If $(x - y)(x - z) \leq 0$, then $M[M[y, x], M[z, x]] \leq M[M[y, z], x]$.*

In particular (a) holds for $x, y, z \in [0, 1]$ with y, z on the same side of x , as stated in [1].

Theorem 2 (I4, [1, §4.4]). *If $0 \leq x \leq y \leq z \leq w \leq 1$, then*

$$M[M[x, y], M[z, w]] \leq \max\{M[x, w], M[y, z]\}.$$

Theorem 3 (I6, [1, §4.6]). *If $0 \leq z \leq y \leq 1$ and $y - z \leq \frac{3}{4}$, then*

$$M[z, y] \leq M\left[\frac{z + M[z, y]}{2}, M[M[z, y], y]\right].$$

2 Two lemmas

Lemma 1. $M[a, b] = M[b, a]$, $M[a, a] = a$, and $\partial_a M[a, b] = 2(2 + b^2 - ab)/(4 + (a - b)^2)^{3/2}$, as $4 + (a - b)^2 - (a + b)(a - b) = 2(2 + b^2 - ab)$. Hence $M[\cdot, b]$ is strictly increasing on every interval on which $ab < 2 + b^2$. In particular M is strictly increasing in each argument on $[0, 1]^2$, so M maps $[0, 1]^2$ into $[0, 1]$ and $a < M[a, b] < b$ for $0 \leq a < b \leq 1$.

The chart. Fix $u \in [0, 1)$, and let $k = \sqrt{1 - u^2}$ and $\varphi = \operatorname{artanh} u$, so that $k \cosh \varphi = 1$ and $u = k \sinh \varphi$. Let $\rho(t) = \operatorname{arsinh}(t/k)$ for $t \in \mathbb{R}$; thus ρ is strictly increasing, $t = k \sinh \rho(t)$ and $\rho(u) = \varphi$. For $d \in \mathbb{R}$ put $\tau = k \tanh d$ and, for $r \in \mathbb{R}$,

$$W_d(r) = 1 + \tau^2(\sinh^2 r - \sinh^2 \varphi), \quad \psi_d(r) = \operatorname{arsinh} \frac{\sinh r}{\sqrt{W_d(r)}};$$

here $W_d(r) \geq W_d(0) = 1 - u^2 \tanh^2 d > 0$.

Lemma 2. (a) *If $\rho(a) = r + d$ and $\rho(b) = r - d$, then $\rho(M[a, b]) = \psi_d(r)$.*

(b) $\psi_d(\varphi) = \varphi$, and ψ_d is strictly increasing on $\mathbb{R}$ and concave on $[0, \infty)$.

(c) For all real a, b , $M[a, b] - u$ has the sign of $\rho(a) + \rho(b) - 2\varphi$.

Proof. (a) Here $a + b = 2k \sinh r \cosh d$ and $a - b = 2k \cosh r \sinh d$. Since $k^2 \cosh^2 r - 1 = k^2 \sinh^2 r - u^2 = k^2(\sinh^2 r - \sinh^2 \varphi)$,

$$4 + (a - b)^2 = 4 \cosh^2 d (1 + \tanh^2 d (k^2 \cosh^2 r - 1)) = 4 \cosh^2 d W_d(r),$$

so $M[a, b] = k \sinh r / \sqrt{W_d(r)}$, which is (a).

(b) $W_d(\varphi) = 1$ gives $\psi_d(\varphi) = \varphi$. Write $W = W_d(r)$, $w_0 = W_d(0)$ and $\psi = \psi_d(r)$, so that $W = w_0 + \tau^2 \sinh^2 r$. Differentiating $\sinh \psi = \sinh r / \sqrt{W}$ gives $\cosh \psi \cdot \psi' = w_0 \cosh r / W^{3/2}$. Moreover $\cosh^2 \psi = (W + \sinh^2 r) / W$ and $W + \sinh^2 r = (1 + \tau^2) \cosh^2 r - \tanh^2 d$, because $1 + \tau^2 - w_0 = \tau^2 \cosh^2 \varphi = \tanh^2 d$. Therefore

$$\psi'_d(r) = \frac{w_0}{W_d(r) \sqrt{1 + \tau^2 - \tanh^2 d / \cosh^2 r}} > 0.$$

On $[0, \infty)$ both $W_d(r)$ and $\cosh r$ are nondecreasing, so ψ'_d is nonincreasing there.

(c) Write $\rho(a), \rho(b) = r \pm d$. Since ρ and ψ_d are strictly increasing and $\rho(u) = \varphi = \psi_d(\varphi)$, (a) shows that $M[a, b] - u$, $\psi_d(r) - \psi_d(\varphi)$ and $r - \varphi$ have the same sign. $\square$

In the notation of [1], where $X(u, t) = (u + t) / \sqrt{1 - t^2}$, we have $\rho(X(u, \tanh \theta)) = \varphi + \theta$.

3 Proof of Theorem 1

Fix $x \geq 0$. Since $\sinh$ maps $\mathbb{R}$ onto $\mathbb{R}$, we can write

$$y = x + 2 \sinh(\sigma + \delta), \quad z = x + 2 \sinh(\sigma - \delta) \quad (\sigma, \delta \in \mathbb{R}).$$

Put $s = \sinh \sigma$, $h = \sinh \delta$, $g = \cosh \delta$, $c = \cosh 2\delta = g^2 + h^2$, $K = (y + z)/2 = x + 2sg$, and

$$\varepsilon = s^2 - h^2 = \sinh(\sigma + \delta) \sinh(\sigma - \delta) = \frac{1}{4}(y - x)(z - x),$$

$$D = s^2 + g^2 = \cosh(\sigma + \delta) \cosh(\sigma - \delta) = c + \varepsilon.$$

Thus case (a) is $\varepsilon \geq 0$ and case (b) is $\varepsilon \leq 0$. Let $L = M[M[y, x], M[z, x]]$, $R = M[M[y, z], x]$ and $v = K/c$.

(i) Since $y - z = 4h \cosh \sigma$ and $1 + 4h^2 \cosh^2 \sigma = c^2 + 4h^2 \varepsilon$ (as $c = 1 + 2h^2$), we have $M[y, z] = K / \sqrt{c^2 + 4h^2 \varepsilon}$.

(ii) Since $y - x = 2 \sinh(\sigma + \delta)$, we have $M[y, x] = (x + \sinh(\sigma + \delta)) / \cosh(\sigma + \delta)$, and similarly for $M[z, x]$. Over the common denominator D , the numerators of $M[y, x] \pm M[z, x]$ are

$$x(\cosh(\sigma + \delta) + \cosh(\sigma - \delta)) + \sinh 2\sigma = 2 \cosh \sigma (xg + s),$$

$$x(\cosh(\sigma - \delta) - \cosh(\sigma + \delta)) + \sinh 2\delta = 2h(g - xs).$$

Hence

$$L^2 = \frac{(1 + s^2)(xg + s)^2}{E}, \quad E = D^2 + h^2(g - xs)^2.$$

(iii) Using $1 + h^2 = g^2$, we get $v + x = 2g(xg + s)/c$ and $v - x = 2(sg - xh^2)/c$, so

$$M[v, x] = \frac{g(xg + s)}{\sqrt{Q}}, \quad Q = c^2 + (sg - xh^2)^2.$$

(iv) The key identity is

$$(1 + s^2)Q - g^2E = \varepsilon h^2(c - xK). \quad (1)$$

Since $1 + s^2 - g^2 = \varepsilon$, the left side is $\varepsilon Q + g^2(Q - E)$. In $(sg - xh^2)^2 - h^2(g - xs)^2$ the cross terms cancel, leaving $\varepsilon(g^2 - x^2h^2)$; and $D^2 - c^2 = \varepsilon(2c + \varepsilon)$. Hence $Q - E = \varepsilon(g^2 - x^2h^2 - 2c - \varepsilon)$, and the left side of (1) is ε times

$$c^2 - 2cg^2 + g^4 + (sg - xh^2)^2 - \varepsilon g^2 - x^2 g^2 h^2 = h^4 + g^2 h^2 - 2xsg h^2 - x^2 h^2 = h^2(c - xK),$$

by $c - g^2 = h^2$, $s^2 - \varepsilon = h^2$, $g^2 - h^2 = 1$ and $c = g^2 + h^2$.

Step 1. By (ii)–(iv),

$$L^2 - M[v, x]^2 = \frac{(xg + s)^2 \varepsilon h^2 (c - xK)}{EQ},$$

where $E, Q > 0$ and $c \geq 1 \geq \frac{1}{2}x(y + z) = xK$. Since $x, y, z, v \geq 0$, both L and $M[v, x]$ are ≥ 0 . So $L \geq M[v, x]$ in case (a) and $L \leq M[v, x]$ in case (b).

Step 2. By (i), $M[y, z] \leq v$ in case (a) and $M[y, z] \geq v$ in case (b).

Step 3. Both $M[y, z]$ and v lie in $[0, K]$, since $c^2 + 4h^2\varepsilon = 1 + 4h^2 \cosh^2 \sigma \geq 1$ and $c \geq 1$. For $0 \leq a \leq K$ we have $ax \leq Kx \leq 1 < 2 + x^2$, so $M[\cdot, x]$ is increasing on $[0, K]$ by Lemma 1. Hence $L \geq M[v, x] \geq M[M[y, z], x] = R$ in case (a), and $L \leq M[v, x] \leq R$ in case (b). $\square$

Remark. For $P_t = t + i$ in the upper half-plane, $\sinh \frac{1}{2}d(P_a, P_b) = \frac{1}{2}|a - b|$; so (i) reads $\cosh \frac{1}{2}d(P_y, P_z) = \sqrt{c^2 + 4h^2\varepsilon}$, while $c = \cosh \frac{1}{2}|d(P_x, P_y) \mp d(P_x, P_z)|$ in case (a), resp. (b), and Step 2 is the hyperbolic triangle inequality.

4 Proof of Theorem 2

By Lemma 1, all values of M below lie in $[0, 1]$, where M is increasing in each argument. Let $u = \max\{M[x, w], M[y, z]\}$. If $u = 1$ there is nothing to prove, so let $u < 1$.

Reduction. If $M[x, w] < u$, then $M[x, w] < u = M[y, z] \leq M[y, w]$, so $M[x', w] = u$ for some $x' \in (x, y)$. If $M[y, z] < u$, then $M[y, z] < u = M[x, w] \leq M[y, w]$, so $M[y, z'] = u$ for some $z' \in (z, w)$. Replacing x by x' , or z by z' , preserves $x \leq y \leq z \leq w$ and u , and does not decrease $M[M[x, y], M[z, w]]$. So we may assume that $M[x, w] = M[y, z] = u$; then $y \leq M[y, z] = u$.

The critical case. Take the chart at u . By Lemma 2(c), $\rho(x) + \rho(w) = 2\varphi = \rho(y) + \rho(z)$. Write $\rho(x), \rho(y) = \varphi - \lambda \mp d$; then $\rho(w), \rho(z) = \varphi + \lambda \pm d$, and $0 \leq x \leq y \leq u$ gives $0 \leq d \leq \lambda \leq \varphi$. So the pairs (x, y) and (z, w) have chart midpoints $\varphi \mp \lambda \in [0, \infty)$ and the same half-difference d , and by Lemma 2(a),(b)

$$\rho(M[x, y]) + \rho(M[z, w]) = \psi_d(\varphi - \lambda) + \psi_d(\varphi + \lambda) \leq 2\psi_d(\varphi) = 2\varphi.$$

By Lemma 2(c), $M[M[x, y], M[z, w]] \leq u$. $\square$

The reduction in [1, §4.4] raises x and y , starting from $x \leq y \leq u \leq z \leq w$; but $u \leq z$ can fail (e.g. $x = y = z = 0 < w$). Raising x or z , as above, avoids this.

5 Proof of Theorem 3

Let $u = M[z, y]$. If $z = y$, both sides equal y . Otherwise $z < u < y \leq 1$ by Lemma 1, and we take the chart at u . By Lemma 2(c), $\rho(z) + \rho(y) = 2\varphi$; write $\rho(z), \rho(y) = \varphi \mp 2\beta$ with $\beta > 0$. Then $y - z = 2k \cosh \varphi \sinh 2\beta = 2 \sinh 2\beta$, so $\sinh 2\beta \leq \frac{3}{8}$, and $z \geq 0$ means $\varphi \geq 2\beta$. Let $A = (z + u)/2$,

$B = M[u, y]$, $S = \sinh \beta$ and $C = \cosh \beta$. Then $A/k = \frac{1}{2}(\sinh(\varphi - 2\beta) + \sinh \varphi) = C \sinh(\varphi - \beta)$. Since $\rho(u), \rho(y) = (\varphi + \beta) \mp \beta$ and $\sinh^2(\varphi + \beta) - \sinh^2 \varphi = S \sinh(2\varphi + \beta)$, Lemma 2(a) gives

$$B/k = \frac{\sinh(\varphi + \beta)}{\sqrt{W}}, \quad W = W_\beta(\varphi + \beta) = 1 + \tau^2 S \sinh(2\varphi + \beta) \geq 1, \quad \tau = k \tanh \beta.$$

By Lemma 2(c), the claim $M[A, B] \geq u$ means $\rho(A) + \rho(B) \geq 2\varphi$, i.e. $\sinh(\rho(A) + \rho(B)) \geq \sinh 2\varphi$. Now $\sinh \rho(A) = A/k \geq \sinh(\varphi - \beta) \geq 0$, $\cosh \rho(A) \geq \cosh(\varphi - \beta)$, $\sinh \rho(B) = B/k \geq 0$ and $\cosh \rho(B) = \sqrt{1 + B^2/k^2} \geq \cosh(\varphi + \beta)/\sqrt{W}$. Hence

$$\begin{aligned} \sqrt{W} \sinh(\rho(A) + \rho(B)) &\geq C \sinh(\varphi - \beta) \cosh(\varphi + \beta) + \cosh(\varphi - \beta) \sinh(\varphi + \beta) \\ &= \sinh 2\varphi + (C - 1) \sinh(\varphi - \beta) \cosh(\varphi + \beta). \end{aligned}$$

On the other hand, $\sqrt{W} \leq (1 + W)/2$ and $k^2 \sinh 2\varphi = 2 \tanh \varphi$ give $\sqrt{W} \sinh 2\varphi \leq \sinh 2\varphi + \tanh \varphi \tanh^2 \beta S \sinh(2\varphi + \beta)$. So it suffices that

$$(C - 1) \sinh(\varphi - \beta) \cosh(\varphi + \beta) \geq \tanh \varphi \tanh^2 \beta S \sinh(2\varphi + \beta).$$

Since $C - 1 = C^2 \tanh^2 \beta / (1 + C)$ and $2 \sinh(\varphi - \beta) \cosh(\varphi + \beta) = \sinh 2\varphi - \sinh 2\beta$, this is $F(\varphi) \geq 0$, where

$$F(\varphi) = C^2(\sinh 2\varphi - \sinh 2\beta) - 2S(1 + C) \tanh \varphi \sinh(2\varphi + \beta).$$

Let $t = \tanh \varphi \in [\tanh 2\beta, 1)$. As $(1 - t^2) \sinh 2\varphi = 2t$ and $(1 - t^2) \cosh 2\varphi = 1 + t^2$, we have $(1 - t^2)F(\varphi) = P(t)$, where

$$P(t) = C^2(2t - (1 - t^2) \sinh 2\beta) - 2tS(1 + C)(2tC + (1 + t^2)S).$$

We shall use $S \leq SC = \frac{1}{2} \sinh 2\beta \leq \frac{3}{16}$, $\tanh \beta \leq S$, $C^2 = 1 + S^2 < 1.21$ and $5\beta \leq \frac{5}{2} \sinh 2\beta \leq \frac{15}{16}$. The coefficients of t^3 and t^2 in P are $-2S^2(1 + C)$ and $2SC(C^2 - 2C - 2)$, both negative. So P is concave on $[0, \infty)$, and it suffices to show that $P(1) > 0$ and $P(\tanh 2\beta) > 0$. First,

$$P(1) = 2C^2 - 2 \sinh 2\beta (1 + C)(1 + \tanh \beta) > 2 - \frac{3}{4} \cdot 2.1 \cdot 1.2 > 0.$$

Second, $P(\tanh 2\beta) = (1 - \tanh^2 2\beta)F(2\beta)$, and $\sinh 4\beta - \sinh 2\beta = 2S \cosh 3\beta$, so

$$F(2\beta) = 2S(C^2 \cosh 3\beta - (1 + C) \tanh 2\beta \sinh 5\beta) \geq 2SC^2(1 - 2 \tanh 2\beta \sinh 5\beta) > 0,$$

because $1 + C \leq 2C^2$, $\cosh 3\beta \geq 1$, and $2 \tanh 2\beta \sinh 5\beta < \frac{3}{4} \sinh 1 < 1$ (as $e < 3$ gives $\sinh 1 < \frac{1}{2}(3 - \frac{1}{3}) = \frac{4}{3}$). $\square$

References

- [1] F. Nazarov, V. Vasyunin, A. Volberg, *On a Bellman function associated with the Chang–Wilson–Wolff theorem: a case study*, arXiv:2204.01336.