

Computer-free proofs of three nonlinear-mean inequalities of Nazarov, Vasyunin and Volberg

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Abstract

For the nonlinear mean $M[a, b] = \frac{a+b}{\sqrt{4+(a-b)^2}}$, Nazarov, Vasyunin and Volberg [1, §4] verified several inequalities by computer-assisted positivity tests for large polynomials. We give short proofs, without computer assistance, of three of them: the “same-side” inequality I3 ([1, §4.3]), the four-point inequality I4 ([1, §4.4]), and the inequality I6 ([1, §4.6]). Two coordinate systems, both coming from the hyperbolic geometry hidden in M , do all the work. For I3 we use horocyclic coordinates centred at x . In them the inequality splits into the hyperbolic triangle inequality and one polynomial identity, and the same identity proves the reverse inequality when y, z lie on opposite sides of x . For I4 and I6 we use a chart centred at a value u of M , in which the level set $\{M = u\}$ becomes a reflection. I4 then follows from one application of Cauchy–Schwarz and I6 from the endpoint values of a concave cubic. Every identity used is elementary and is derived in the text. Numerical checks were used to find the arguments and to double-check them, but no step of any proof depends on them.

1 Results

Throughout, $M[a, b] := (a+b)/\sqrt{4+(a-b)^2}$ for real a, b .

Theorem 1 (I3, [1, §4.3]). *Let $x, y, z \geq 0$ with $x(y+z) \leq 2$.*

(a) *If $(x-y)(x-z) \geq 0$, then $M[M[y, x], M[z, x]] \geq M[M[y, z], x]$.*

(b) *If $(x-y)(x-z) \leq 0$, then $M[M[y, x], M[z, x]] \leq M[M[y, z], x]$.*

In particular (a) holds for all $x, y, z \in [0, 1]$ with y, z on the same side of x ; this is the inequality of [1, §4.3].

Theorem 2 (I4, [1, §4.4]). *For $0 \leq x \leq y \leq z \leq w \leq 1$,*

$$M[M[x, y], M[z, w]] \leq \max\{M[x, w], M[y, z]\}.$$

Theorem 3 (I6, [1, §4.6]). *For $0 \leq z \leq y \leq 1$ with $y-z \leq 3/4$,*

$$M[z, y] \leq M\left[\frac{z+M[z, y]}{2}, M[M[z, y], y]\right].$$

The proof of Theorem 1 (§3) is the triangle inequality plus the identity (3.2). Theorem 2 (§5) follows from the reduction of [1] (with a missing case supplied by Theorem 1(b)) and Lemma 5.1, whose proof is one Cauchy–Schwarz inequality in the chart of §4. Theorem 3 (§6) uses the same chart and the concavity of a cubic polynomial. Section 7 records sharper or extended forms, the routes that failed (and why), and the numerical evidence, which is not used in any proof.

Notation. $\sinh, \cosh, \tanh$ are the hyperbolic functions; $\operatorname{arsinh}, \operatorname{artanh}$ their inverses.

2 Preliminaries

Lemma 2.1. (a) $M[a, b] = M[b, a]$, $M[a, a] = a$, $M[-a, -b] = -M[a, b]$.

(b) $\partial_a M[a, b] = 2(2 + b^2 - ab)(4 + (a - b)^2)^{-3/2}$. Hence $a \mapsto M[a, b]$ is strictly increasing on $\{a : ab < 2 + b^2\}$; in particular on $[0, 1/b]$ if $b > 0$, on $\mathbb{R}$ if $b = 0$, and on $[0, 1]$ if $b \in [0, 1]$.

(c) If $0 \leq a \leq b$ then $M[a, b] \leq b$, with equality only if $a = b$; if moreover $a \leq 1$, then $M[a, b] \geq a$.

Proof. (a) is clear. (b) $\partial_a M = [(4 + (a - b)^2) - (a + b)(a - b)](4 + (a - b)^2)^{-3/2}$ and $4 + (a - b)^2 - (a^2 - b^2) = 2(2 + b^2 - ab)$. (c) $(a + b)^2 \leq 4b^2 \leq b^2(4 + (b - a)^2)$, and the first inequality is strict if $a < b$. With $d = b - a \geq 0$, $(a + b)^2 - a^2(4 + d^2) = d(4a + (1 - a^2)d) \geq 0$ for $0 \leq a \leq 1$. $\square$

The hyperbolic picture. This explains the coordinates used below; the proofs do not depend on it. In the upper half-plane $\mathbb{H}$ put $P_a = a + i$, so $a \mapsto P_a$ parametrizes the horocycle $H = \{\text{Im } \zeta = 1\}$. The geodesic through P_a, P_b is the half-circle with centre $\frac{a+b}{2}$ and radius $\sqrt{1 + (a - b)^2/4}$. By the reflection symmetry in $\text{Re } \zeta = \frac{a+b}{2}$, its top point $m_{ab} = \frac{a+b}{2} + i\sqrt{1 + (a - b)^2/4}$ is the hyperbolic midpoint of P_a, P_b . Therefore

$$M[a, b] = \frac{\text{Re } m_{ab}}{\text{Im } m_{ab}}, \quad d(P_a, P_b) = 2 \text{arsinh } \frac{|a - b|}{2}, \quad (2.1)$$

the second formula because $\cosh d(P_a, P_b) = 1 + |P_a - P_b|^2/2$. The function $\text{Re } \zeta / \text{Im } \zeta$ is the sinh of the signed distance from ζ to the geodesic $\gamma_0 = i\mathbb{R}_{>0}$, and it is constant on the rays from 0 (the hypercycles of γ_0). So $M[a, b]$ is obtained in two steps: take the hyperbolic midpoint of P_a and P_b , then slide it along its hypercycle back to H . In the hyperboloid model, Re / Im is a linear functional ℓ , and $M[a, b] = \ell(P_a + P_b) / \|P_a + P_b\|$. The boost $X(u, \tau) = (u + \tau) / \sqrt{1 - \tau^2}$ of [1] satisfies $X(M[a, b], \pm\tau) = a, b$ for $\tau = (a - b) / \sqrt{4 + (a - b)^2}$.

3 Proof of Theorem 1: horocyclic coordinates

Fix $x \geq 0$ and write (every real number is of the form $x + 2 \sinh \lambda$)

$$\begin{aligned} y &= x + 2 \sinh \lambda, & z &= x + 2 \sinh \mu, & \sigma &= \frac{\lambda + \mu}{2}, & \delta &= \frac{\lambda - \mu}{2}, \\ s &= \sinh \sigma, & g &= \cosh \delta, & \alpha &= s^2, & \beta &= g^2 - 1 = \sinh^2 \delta, \\ c &= \cosh 2\delta = 1 + 2\beta, & K &= \frac{y+z}{2} = x + 2sg. \end{aligned}$$

By (2.1), $2|\lambda| = d(P_x, P_y)$ and $2|\mu| = d(P_x, P_z)$. Moreover

$$\alpha - \beta = \sinh^2 \sigma - \sinh^2 \delta = \sinh \lambda \sinh \mu = \frac{1}{4}(y - x)(z - x), \quad (3.1)$$

so y, z lie on the same side of x if and only if $\alpha \geq \beta$.

Lemma 3.1. (i) $M[y, x] = (x + \sinh \lambda) / \cosh \lambda$ and $M[z, x] = (x + \sinh \mu) / \cosh \mu$.

(ii) $L := M[M[y, x], M[z, x]]$ satisfies $L^2 = \frac{(1 + \alpha)(xg + s)^2}{E}$, where $E := (\alpha + g^2)^2 + \beta(g - xs)^2$.

(iii) $M[y, z] = K / \cosh \nu$, where $\cosh \nu := \sqrt{1 + (\sinh \lambda - \sinh \mu)^2}$ (so $2\nu = d(P_y, P_z)$), and $\cosh^2 \nu - c^2 = 4\beta(\alpha - \beta)$.

$$(iv) \quad M[K/c, x] = \frac{g(xg + s)}{\sqrt{c^2 + (sg - x\beta)^2}}.$$

Proof. (i) $y + x = 2(x + \sinh \lambda)$, $y - x = 2 \sinh \lambda$ and $\sqrt{4 + 4 \sinh^2 \lambda} = 2 \cosh \lambda$.

(ii) Let $a_1 = (x + \sinh \lambda) / \cosh \lambda$ and $a_2 = (x + \sinh \mu) / \cosh \mu$. From

$$\begin{aligned} \cosh \lambda + \cosh \mu &= 2 \cosh \sigma \cosh \delta, & \cosh \mu - \cosh \lambda &= -2 \sinh \sigma \sinh \delta, \\ \sinh(\lambda + \mu) &= 2 \sinh \sigma \cosh \delta, & \sinh(\lambda - \mu) &= 2 \sinh \delta \cosh \delta \end{aligned}$$

we get

$$a_1 + a_2 = \frac{2 \cosh \sigma (xg + s)}{\cosh \lambda \cosh \mu}, \quad a_1 - a_2 = \frac{2 \sinh \delta (g - xs)}{\cosh \lambda \cosh \mu},$$

and $\cosh \lambda \cosh \mu = \frac{1}{2}(\cosh 2\sigma + \cosh 2\delta) = \alpha + g^2$, $\cosh^2 \sigma = 1 + \alpha$. Insert these into $M[a_1, a_2]^2 = (a_1 + a_2)^2 / (4 + (a_1 - a_2)^2)$.

(iii) $y + z = 2K$ and $(y - z)/2 = \sinh \lambda - \sinh \mu$, so $M[y, z] = 2K / \sqrt{4 + 4(\sinh \lambda - \sinh \mu)^2} = K / \cosh \nu$. Since $\sinh \lambda - \sinh \mu = 2 \cosh \sigma \sinh \delta$, $\cosh^2 \nu = 1 + 4(1 + \alpha)\beta$, while $c^2 = 1 + 4\beta + 4\beta^2$.

(iv) $K/c + x = 2(x(1 + \beta) + sg)/c = 2g(xg + s)/c$ and $K/c - x = 2(sg - x\beta)/c$. $\square$

Lemma 3.2 (key identity).

$$(1 + \alpha)(c^2 + (sg - x\beta)^2) - g^2 E = \beta(\alpha - \beta)(c - xK). \quad (3.2)$$

Proof. Both sides are quadratic polynomials in x ; we compare coefficients, using $s^2 = \alpha$, $g^2 = 1 + \beta$, $c = 1 + 2\beta$. Write $A = 1 + \alpha$. The coefficients of the left side are:

$$\begin{aligned} x^0 : & \quad A(c^2 + \alpha g^2) - g^2((\alpha + g^2)^2 + \beta g^2) \\ & \quad = A(A + 3\beta + A\beta + 4\beta^2) - (1 + \beta)(A^2 + 2A\beta + \beta + 2\beta^2) \\ & \quad = \beta[(A - 1)(1 + 2\beta) - \beta(1 + 2\beta)] = \beta(\alpha - \beta)c, \\ x^1 : & \quad -2Asg\beta + 2\beta g^3 s = 2\beta sg(g^2 - 1 - \alpha) = -\beta(\alpha - \beta) \cdot 2sg, \\ x^2 : & \quad A\beta^2 - \beta g^2 s^2 = \beta((1 + \alpha)\beta - (1 + \beta)\alpha) = -\beta(\alpha - \beta). \end{aligned}$$

Summing, the left side equals $\beta(\alpha - \beta)(c - x(2sg + x)) = \beta(\alpha - \beta)(c - xK)$. $\square$

Proof of Theorem 1. Let $L = M[M[y, x], M[z, x]]$, $R = M[M[y, z], x]$, and $w^* := K/c = (y + z)/(2 \cosh(\lambda - \mu))$. All of $x, y, z, M[y, x], M[z, x], K, w^*$ are ≥ 0 , hence $L \geq 0$ and $M[w^*, x] \geq 0$.

Step 1 (the identity). By Lemma 3.1(ii),(iv) and (3.2),

$$L^2 - M[w^*, x]^2 = \frac{(xg + s)^2 \beta(\alpha - \beta)(c - xK)}{E(c^2 + (sg - x\beta)^2)}.$$

Here $E \geq 1$ and $c \geq 1 \geq \frac{x(y+z)}{2} = xK$. So by (3.1), $L \geq M[w^*, x]$ in case (a) and $L \leq M[w^*, x]$ in case (b).

Step 2 (the triangle inequality). By Lemma 3.1(iii) and (3.1), $\cosh \nu \geq c$ in case (a) and $\cosh \nu \leq c$ in case (b). Geometrically: $2\nu = d(P_y, P_z)$ and $c = \cosh(\lambda - \mu)$. If y, z are on the same side of x , then $|\lambda - \mu| = ||\lambda| - |\mu||$, and $\nu \geq |\lambda - \mu|$ is the triangle inequality $d(P_y, P_z) \geq |d(P_x, P_y) - d(P_x, P_z)|$. If they are on opposite sides, then $|\lambda - \mu| = |\lambda| + |\mu|$, and $\nu \leq |\lambda - \mu|$ is $d(P_y, P_z) \leq d(P_x, P_y) + d(P_x, P_z)$. Consequently $M[y, z] = K / \cosh \nu \leq w^*$ in case (a) and $\geq w^*$ in case (b).

Step 3 (monotonicity). Both $M[y, z]$ and w^* lie in $[0, K] \subset [0, 1/x]$, on which $M[\cdot, x]$ is increasing (Lemma 2.1(b)). Hence in case (a) $L \geq M[w^*, x] \geq M[M[y, z], x] = R$, and in case (b) $L \leq M[w^*, x] \leq R$. $\square$

Remark 3.3 (meaning of the proof). The intermediate value $M[w^*, x]$, with $w^* = (y + z)/(2 \cosh(\lambda - \mu))$, is what $M[M[y, z], x]$ would be if the triangle $P_x P_y P_z$ were flattened, so that $d(P_y, P_z) = |d(P_x, P_y) \mp d(P_x, P_z)|$. Theorem 1 then splits into the triangle inequality (Step 2) and the sign of one explicit product (Step 1). The two steps point the same way exactly when $(x - y)(x - z)$ has a fixed sign, which explains both the hypothesis of I3 and its reverse (b).

Remark 3.4 ($x = 0$). For $x = 0$ no intermediate value is needed: with $E_0 = (\alpha + g^2)^2 + \beta g^2$, for all $y, z \geq 0$,

$$L^2 - R^2 = \frac{s^2 \beta (\alpha - \beta) (5 + 4\alpha + 2\beta)}{E_0 (\cosh^2 \nu + \alpha g^2)}.$$

This factorization is how the identity (3.2) was found.

4 A chart centred at a value of M

Fix $u \in [0, 1)$ and put

$$k = \sqrt{1 - u^2}, \quad \varphi = \operatorname{artanh} u \quad (\text{so } u = k \sinh \varphi, \quad k \cosh \varphi = 1), \\ \rho(t) = \rho_u(t) := \operatorname{arsinh}(t/k) \quad (t \in \mathbb{R}).$$

Thus $t = k \sinh \rho(t)$, and $\rho(u) = \varphi$.

Lemma 4.1. *Let $a, b \in \mathbb{R}$, $\bar{r} = \frac{\rho(a) + \rho(b)}{2}$, $\delta = \frac{\rho(a) - \rho(b)}{2}$, $m = k \sinh \bar{r}$ and $T = \tanh \delta$.*

$$(a) \quad M[a, b] = \frac{m}{\sqrt{1 + T^2(m^2 - u^2)}}, \text{ where } m^2 - u^2 = k^2 \sinh(\bar{r} - \varphi) \sinh(\bar{r} + \varphi) \text{ and } 1 + T^2(m^2 - u^2) \geq 1 - T^2 u^2 > 0.$$

$$(b) \quad M[a, b] - u \text{ has the sign of } \rho(a) + \rho(b) - 2\varphi. \text{ In particular } M[a, b] = u \iff \rho_u(a) + \rho_u(b) = 2\rho_u(u).$$

$$(c) \quad M[a, b] \leq u \iff a\sqrt{k^2 + b^2} + b\sqrt{k^2 + a^2} \leq 2u, \text{ and the same holds with } \geq \text{ in both places.}$$

Proof. (a) $a + b = 2k \sinh \bar{r} \cosh \delta = 2m \cosh \delta$ and $a - b = 2k \cosh \bar{r} \sinh \delta$, where $k^2 \cosh^2 \bar{r} = k^2 + m^2 = 1 - u^2 + m^2$. Hence $4 + (a - b)^2 = 4(\cosh^2 \delta + \sinh^2 \delta (m^2 - u^2))$, so $M[a, b] = m \cosh \delta / \sqrt{\cosh^2 \delta + \sinh^2 \delta (m^2 - u^2)}$. This is (a). Also $m^2 - u^2 = k^2(\sinh^2 \bar{r} - \sinh^2 \varphi)$.

(b) By (a), $M[a, b]$ has the sign of m , and $M[a, b]^2 - u^2 = (m^2 - u^2)(1 - T^2 u^2)/(1 + T^2(m^2 - u^2))$. So if $m \geq 0$, then $M[a, b] - u$ has the sign of $m - u$. If $m < 0$, then $M[a, b] < 0 \leq u$ and $m < u$. Finally, $m - u = k(\sinh \bar{r} - \sinh \varphi)$ has the sign of $\bar{r} - \varphi$.

(c) By (b), $M[a, b] \leq u \iff \rho(a) + \rho(b) \leq 2\varphi \iff \sinh(\rho(a) + \rho(b)) \leq \sinh 2\varphi$. By the addition theorem, $\sinh(\rho(a) + \rho(b)) = \frac{a}{k} \sqrt{1 + \frac{b^2}{k^2}} + \frac{b}{k} \sqrt{1 + \frac{a^2}{k^2}}$, and $\sinh 2\varphi = 2 \sinh \varphi \cosh \varphi = 2u/k^2$. $\square$

Remark 4.2. (1) Since $X(u, \tanh \theta) = u \cosh \theta + \sinh \theta = k \sinh(\varphi + \theta)$, we have $\rho_u(X(u, \tanh \theta)) = \varphi + \theta$. So ρ_u is the additive parameter of the boost of $[1]$, and Lemma 4.1(b) says that the involution $b \mapsto a$ defined by $M[a, b] = u$ is the reflection $\rho \mapsto 2\varphi - \rho$ in this chart. For $u = 0$, $\rho_0(t) = \operatorname{arsinh} t$ is the signed distance from P_t to γ_0 , and the involution is the hyperbolic reflection in γ_0 . (2) Equivalently: for $a, b \geq 0$ with $M[a, b] < 1$, $M[a, b]$ is the unique $u \in [0, 1)$ such that the quasi-arithmetic mean with generator ρ_u , namely $\rho_u^{-1}(\frac{\rho_u(a) + \rho_u(b)}{2})$, equals u . So M is a “self-centred” quasi-arithmetic mean, and Lemma 4.1(a) gives the exact correction when the centre u is not $M[a, b]$.

5 Proof of Theorem 2

Lemma 5.1 (critical case). *Let $0 \leq u < 1$, and let $x', y', z, w \in \mathbb{R}$ satisfy $0 \leq x' \leq y' \leq u$ and $M[x', w] = M[y', z] = u$. Then*

$$M[M[x', y'], M[z, w]] \leq u.$$

Proof. Use the chart of §4 centred at u . By Lemma 4.1(b), $\rho(w) = 2\varphi - \rho(x')$ and $\rho(z) = 2\varphi - \rho(y')$. With $a := \varphi - \rho(x')$ and $b := \varphi - \rho(y')$,

$$x' = k \sinh(\varphi - a), \quad w = k \sinh(\varphi + a), \quad y' = k \sinh(\varphi - b), \quad z = k \sinh(\varphi + b), \quad 0 \leq b \leq a \leq \varphi,$$

the last inequalities because $0 \leq x' \leq y' \leq u$. If $a = b$, then $x' = y'$ and $z = w$, and the claim reads $M[x', w] \leq u$. So let $a > b$. Put $S = \frac{a+b}{2} \in (0, \varphi)$, $\Delta = \frac{a-b}{2} > 0$, $\tau = k \tanh \Delta$, and

$$A_{\mp} = \sinh S \sinh(2\varphi \mp S) \ (\geq 0), \quad D_{\mp} = 1 \mp \tau^2 A_{\mp}, \quad N_{\mp} = \cosh^2(\varphi \mp S) \mp \tau^2 A_{\mp}.$$

The pair (x', y') has chart midpoint $\varphi - S$ and chart half-difference Δ , and the pair (z, w) has $\varphi + S$ and Δ . By Lemma 4.1(a), applied with $\bar{r} = \varphi \mp S$,

$$p := M[x', y'] = \frac{k \sinh(\varphi - S)}{\sqrt{D_-}}, \quad q := M[z, w] = \frac{k \sinh(\varphi + S)}{\sqrt{D_+}},$$

with $D_{\mp} > 0$. Consequently $k^2 + p^2 = k^2 N_- / D_-$ and $k^2 + q^2 = k^2 N_+ / D_+$, using $1 + \sinh^2 = \cosh^2$; note $N_{\mp} = D_{\mp} + \sinh^2(\varphi \mp S) > 0$. By Lemma 4.1(c) and $2u = k^2 \sinh 2\varphi$, the claim $M[p, q] \leq u$ is equivalent to

$$\sinh(\varphi - S) \sqrt{N_+} + \sinh(\varphi + S) \sqrt{N_-} \leq \sinh(2\varphi) \sqrt{D_- D_+}. \quad (5.1)$$

Left side. Apply Cauchy-Schwarz with the positive weights $w_1 = \sinh(2\varphi - 2S)$ and $w_2 = \sinh(2\varphi + 2S)$:

$$\left(\sinh(\varphi - S) \sqrt{N_+} + \sinh(\varphi + S) \sqrt{N_-} \right)^2 \leq \left(\frac{\sinh^2(\varphi - S)}{w_1} + \frac{\sinh^2(\varphi + S)}{w_2} \right) (w_1 N_+ + w_2 N_-).$$

Here $\sinh^2(\varphi \mp S) / w_{1,2} = \frac{1}{2} \tanh(\varphi \mp S)$, and $\tanh(\varphi - S) + \tanh(\varphi + S) = \sinh 2\varphi / C$ with $C := \cosh(\varphi - S) \cosh(\varphi + S)$. Moreover

$$\begin{aligned} w_1 \cosh^2(\varphi + S) + w_2 \cosh^2(\varphi - S) &= 2C [\sinh(\varphi - S) \cosh(\varphi + S) + \sinh(\varphi + S) \cosh(\varphi - S)] \\ &= 2C \sinh 2\varphi, \end{aligned}$$

and, writing products as sums ($2 \sinh X \sinh Y = \cosh(X + Y) - \cosh(X - Y)$),

$$\begin{aligned} w_1 A_+ - w_2 A_- &= \sinh S [\sinh(2\varphi - 2S) \sinh(2\varphi + S) - \sinh(2\varphi + 2S) \sinh(2\varphi - S)] \\ &= \frac{1}{2} \sinh S [\cosh(4\varphi - S) - \cosh(4\varphi + S)] = -\sinh^2 S \sinh 4\varphi. \end{aligned}$$

Since $\sinh 4\varphi = 2 \sinh 2\varphi \cosh 2\varphi$, the square of the left side of (5.1) is at most

$$\frac{\sinh 2\varphi}{2C} (2C \sinh 2\varphi - \tau^2 \sinh^2 S \sinh 4\varphi) = \sinh^2 2\varphi \left(1 - \tau^2 \sinh^2 S \frac{\cosh 2\varphi}{C} \right).$$

Right side. Since $A_+ - A_- = 2 \sinh^2 S \cosh 2\varphi$ and $A_+ A_- = \sinh^2 S \sinh(2\varphi + S) \sinh(2\varphi - S)$,

$$\sinh^2 2\varphi D_- D_+ = \sinh^2 2\varphi (1 + 2\tau^2 \sinh^2 S \cosh 2\varphi - \tau^4 \sinh^2 S \sinh(2\varphi + S) \sinh(2\varphi - S)).$$

Comparison. Subtracting, the square of the right side of (5.1) exceeds the bound for the square of the left side by

$$\tau^2 \sinh^2 S \sinh^2 2\varphi \left[\cosh 2\varphi \left(2 + \frac{1}{C} \right) - \tau^2 \sinh(2\varphi + S) \sinh(2\varphi - S) \right].$$

Finally $\tau^2 = \tanh^2 \Delta / \cosh^2 \varphi \leq 1 / \cosh^2 \varphi$ and $\sinh(2\varphi + S) \sinh(2\varphi - S) = \sinh^2 2\varphi - \sinh^2 S \leq 4 \sinh^2 \varphi \cosh^2 \varphi$. So $\tau^2 \sinh(2\varphi + S) \sinh(2\varphi - S) \leq 4 \sinh^2 \varphi = 2 \cosh 2\varphi - 2 < 2 \cosh 2\varphi$, the bracket is positive, and (5.1) follows because its right side is nonnegative. $\square$

Proof of Theorem 2. Let $0 \leq x \leq y \leq z \leq w \leq 1$ and $u := \max\{M[x, w], M[y, z]\}$. All values of M below lie in $[0, 1]$ (Lemma 2.1(c)), where M is increasing in each argument (Lemma 2.1(b)).

Case $M[x, w] \geq z$. By monotonicity, then Theorem 1(b) with base z (here $x \leq z \leq w$ and $z(x + w) \leq 2$), and then Lemma 2.1(c),

$$M[M[x, y], M[z, w]] \leq M[M[x, z], M[w, z]] \leq M[M[x, w], z] \leq M[x, w] \leq u.$$

Case $M[x, w] < z$. Then $y \leq M[y, z] \leq z$, so $y \leq u \leq z$. If $u = 1$, then $z = w = 1$ and $M[M[x, y], 1] \leq 1 = u$. Let $u < 1$. Since $t \mapsto M[t, w]$ is continuous and increasing with $M[x, w] \leq u \leq M[u, w]$, there is $x' \in [x, u]$ with $M[x', w] = u$. Likewise there is $y' \in [y, u]$ with $M[y', z] = u$. Since $M[y', w] \geq M[y', z] = u = M[x', w]$, we have $x' \leq y'$. By monotonicity and Lemma 5.1, $M[M[x, y], M[z, w]] \leq M[M[x', y'], M[z, w]] \leq u$. $\square$

Remark 5.2. The reduction in [1, §4.4] starts from “ $x \leq y \leq u \leq z \leq w$ ”. The inequality $u \leq z$ can fail; for example, $x = y = z = 0 < w$ gives $u = M[0, w] > 0$. That situation is the first case above, which Theorem 1(b) handles. Lemma 5.1 does not use $w \leq 1$.

6 Proof of Theorem 3

Let $0 \leq z \leq y \leq 1$, $y - z \leq \frac{3}{4}$, and $x := M[z, y]$. If $z = y$, both sides equal y . So let $z < y$. Then $0 < x < y \leq 1$ by Lemma 2.1(c). Use the chart of §4 centred at $u = x$, so $k = \sqrt{1 - x^2}$ and $\varphi = \operatorname{artanh} x$. By Lemma 4.1(b), $\rho(z) + \rho(y) = 2\varphi$. Writing $\rho(y) - \varphi = \varphi - \rho(z) = 2\beta$ with $\beta > 0$,

$$z = k \sinh(\varphi - 2\beta), \quad y = k \sinh(\varphi + 2\beta), \quad y - z = 2k \cosh \varphi \sinh 2\beta = 2 \sinh 2\beta.$$

Thus the hypotheses read $\sinh 2\beta \leq \frac{3}{8}$ and, from $z \geq 0$, $\varphi \geq 2\beta$. Put

$$A := \frac{z+x}{2} = k \sinh(\varphi - \beta) \cosh \beta, \quad B := M[x, y].$$

The pair (x, y) has chart midpoint $\varphi + \beta$ and half-difference β , so by Lemma 4.1(a)

$$B = \frac{k \sinh(\varphi + \beta)}{\sqrt{D}}, \quad D := 1 + \tau^2 A_+, \quad \tau := \frac{\tanh \beta}{\cosh \varphi}, \quad A_+ := \sinh \beta \sinh(2\varphi + \beta),$$

and $k^2 + B^2 = k^2 N / D$ with $N := \cosh^2(\varphi + \beta) + \tau^2 A_+$. By Lemma 4.1(c) (with $\geq$), the claim $M[A, B] \geq x$ is equivalent to $A\sqrt{k^2 + B^2} + B\sqrt{k^2 + A^2} \geq 2x = k^2 \sinh 2\varphi$. After dividing by k^2 and multiplying by $\sqrt{D}$, this is

$$\sinh(\varphi - \beta) \cosh \beta \sqrt{N} + \sinh(\varphi + \beta) \sqrt{1 + \sinh^2(\varphi - \beta) \cosh^2 \beta} \geq \sinh(2\varphi) \sqrt{D}. \quad (6.1)$$

Two one-line bounds. Since $\sinh(\varphi - \beta) \geq 0$, $N \geq \cosh^2(\varphi + \beta)$ and $1 + \sinh^2(\varphi - \beta) \cosh^2 \beta \geq \cosh^2(\varphi - \beta)$, the left side of (6.1) is at least

$$\begin{aligned} & \sinh(\varphi - \beta) \cosh \beta \cosh(\varphi + \beta) + \sinh(\varphi + \beta) \cosh(\varphi - \beta) \\ &= \sinh 2\varphi + (\cosh \beta - 1) \sinh(\varphi - \beta) \cosh(\varphi + \beta). \end{aligned}$$

The right side is at most $\sinh 2\varphi (1 + \frac{1}{2}\tau^2 A_+) = \sinh 2\varphi + \tanh \varphi \tanh^2 \beta \sinh \beta \sinh(2\varphi + \beta)$. Hence (6.1) follows from

$$(\cosh \beta - 1) \sinh(\varphi - \beta) \cosh(\varphi + \beta) \geq \tanh \varphi \tanh^2 \beta \sinh \beta \sinh(2\varphi + \beta). \quad (6.2)$$

A concave cubic. Divide (6.2) by $\cosh \beta - 1 > 0$ and use $\tanh^2 \beta / (\cosh \beta - 1) = (\cosh \beta + 1) / \cosh^2 \beta$ and $2 \sinh(\varphi - \beta) \cosh(\varphi + \beta) = \sinh 2\varphi - \sinh 2\beta$. Then multiply by $2 \cosh^2 \beta (1 - t^2)$, where $t := \tanh \varphi = x$, and use $(1 - t^2) \sinh 2\varphi = 2t$, $(1 - t^2) \cosh 2\varphi = 1 + t^2$. Then (6.2) becomes $P(t) \geq 0$, where

$$P(t) = \cosh^2 \beta (2t - (1 - t^2) \sinh 2\beta) - 2t \sinh \beta (1 + \cosh \beta) (2t \cosh \beta + (1 + t^2) \sinh \beta).$$

The coefficient of t^3 is $-2 \sinh^2 \beta (1 + \cosh \beta) < 0$, and that of t^2 is $2 \sinh \beta \cosh \beta (\cosh^2 \beta - 2 \cosh \beta - 2) < 0$ (as $\cosh \beta < 1 + \sqrt{3}$). So $P'' < 0$ on $[0, \infty)$ and P is concave there. Since $t = \tanh \varphi \in [\tanh 2\beta, 1)$, it suffices to check $P(1) \geq 0$ and $P(\tanh 2\beta) \geq 0$. From $\sinh 2\beta \leq \frac{3}{8}$ we get $\sinh \beta \leq \frac{3}{16}$ (as $\cosh \beta \geq 1$), $\cosh \beta \leq \sqrt{1 + 9/256} < 1.02$, $e^\beta = \sinh \beta + \cosh \beta < 1.21$, and $5\beta \leq \frac{5}{2} \operatorname{arsinh} \frac{3}{8} \leq \frac{15}{16} < 1$.

- $P(1) = 2 \cosh^2 \beta - 4 \sinh \beta (1 + \cosh \beta) e^\beta \geq 2 - 4 \cdot \frac{3}{16} \cdot 2.02 \cdot 1.21 > 2 - 1.84 > 0$.
- $t = \tanh 2\beta$ corresponds to $\varphi = 2\beta$, i.e. $z = 0$. There

$$P(\tanh 2\beta) = 2(1 - \tanh^2 2\beta) \sinh \beta [\cosh^2 \beta \cosh 3\beta - \tanh 2\beta (1 + \cosh \beta) \sinh 5\beta],$$

and the bracket is positive because $\tanh 2\beta (1 + \cosh \beta) \sinh 5\beta \leq \frac{3}{8} \cdot 2.02 \cdot \sinh 1 < \frac{3}{8} \cdot 2.02 \cdot 1.18 < 0.9 < 1 \leq \cosh^2 \beta \cosh 3\beta$.

Hence $P \geq 0$ on $[\tanh 2\beta, 1]$, which gives (6.2), then (6.1), and so Theorem 3. $\square$

Remark 6.1 (range). The proof uses only $z \geq 0$, $y - z \leq \frac{3}{4}$ and $M[z, y] < 1$. The hypothesis $y \leq 1$ serves only to guarantee the last condition. The binding constraint is the endpoint $P(1) \geq 0$, i.e. $\cosh^2 \beta \geq 2 \sinh \beta (1 + \cosh \beta) e^\beta$. Numerically this holds for $2 \sinh 2\beta \leq 0.856$, so the same argument covers somewhat larger gaps than $\frac{3}{4}$. Numerically, I6 itself appears to hold on $[0, 1]$ for all gaps up to 1.

7 Remarks, failed routes, and numerical evidence

Ranges. The condition $x(y + z) \leq 2$ in Theorem 1 is exactly what Step 1 needs ($c \geq xK$), but it is not the true range of I3. Near the diagonal point (t, t, t) , a Taylor expansion gives

$$\begin{aligned} & M[M[t + \varepsilon y, t + \varepsilon x], M[t + \varepsilon z, t + \varepsilon x]] - M[M[t + \varepsilon y, t + \varepsilon z], t + \varepsilon x] \\ &= \frac{t(3 - t^2)}{128} (x - y)(x - z)(y - z)^2 \varepsilon^4 + O(\varepsilon^5). \end{aligned}$$

So the true threshold on the diagonal is $t = \sqrt{3}$, whereas Step 1 needs $t \leq 1$ there. Numerically, both (a) and (b) hold on the whole cube $[0, \sqrt{3}]^3$ and fail on $[0, 1.75]^3$. They also fail for some $x \leq 1$ with $y, z \leq 5$, outside $\{x(y + z) \leq 2\}$.

Routes that failed, and why.

1. *Clearing all radicals (the route of [1]).* Repeated squaring leads to polynomials of very high degree (for I4 the factor $P_1(u, s, t)$ of [1, §4.4]), whose positivity needs a computer. Trying to rationalize all square roots of I3 at once also failed: no simultaneous rational parametrization was found, because the curve involved has genus one.
2. *A symmetric separator.* The hyperboloid barycentre of P_x, P_x, P_y, P_z gives the candidate $\Psi = \text{mean} / \sqrt{1 + \text{var}}$ of (x, x, y, z) , and one could hope for $L \geq \Psi \geq R$. This is false. On $[0, 1]^3$, $L \geq \Psi$ fails for almost all samples with $x \leq y, z$, and $\Psi \geq R$ fails for all samples with $x \geq y, z$.
3. *A sign rule for the bisymmetry defect.* To fourth order at the diagonal, the defect $\Delta = M[M[a, b], M[c, d]] - M[M[a, c], M[b, d]]$ has the sign of $(a - d)(c - b)$ (for $t < \sqrt{3}$), and I3 is the case $b = d$. The global sign rule is false: about 4% of random points of $[0, 1]^4$ violate it. So I3 cannot be obtained by specializing a general four-point sign rule.
4. *Quasi-arithmetic sandwiches.* For quasi-arithmetic means the I3 difference vanishes identically, by bisymmetry. M agrees with a quasi-arithmetic mean up to third order at the diagonal, but the margin of I3 is of fourth order. Every sandwich we tried lost terms of that same order, so none closed.
5. *Three-gap triangle bound for I4.* Writing $y = x + 2 \sinh l_1$, $z = y + 2 \sinh \kappa$, $w = z + 2 \sinh l_2$ and bounding $M[x, w] \geq (x + w)/(2 \cosh(l_1 + \kappa + l_2))$ loses a term of order $\kappa^2(l_1 + l_2)$. The margin is only of order $\kappa^2(l_1 + l_2)^2$, so this bound is too lossy.
6. *I4 via I3 alone.* Theorem 1(a) at base $q = M[z, w]$ gives $M[M[x', y'], q] \leq M[M[x', q], M[y', q]]$. Numerically the right side is again at most u , but we found no direct proof; the chart argument of Lemma 5.1 made it unnecessary. The same happened with the stronger statement “adjacent $\leq$ nested”, namely $M[M[a, b], M[c, d]] \leq M[M[a, d], M[b, c]]$ for $a \leq b \leq c \leq d$. It holds numerically on $[0, \sqrt{3}]^4$, and its faces $a = b$, $c = d$ and $b = c$ are Theorem 1(a), (a), (b).
7. *I6 in additive coordinates.* With $z, y = x \cosh b \mp \sinh b$ and $T = \tanh(b/2)$, one has $(z+x)/2 = (x - T)/(1 - T^2)$ and $M[x, y] = (x + T)/W$, where $W^2 = (1 - T^2)^2 + T^2(1 + xT)^2$. The bound $W \leq (1 + W^2)/2$ leaves a two-variable polynomial of degree 14 with no transparent sign. The chart of §4 replaces this computation by a concave cubic.

Numerical evidence (not used in the proofs). All identities above were also checked exactly by computer algebra, as identities between rational functions of exponentials. Every inequality step was checked on random samples:

statement	samples	violations
Thm. 1, Steps 1–3, $[0, 1]^3$	$3 \cdot 10^6$	0
Thm. 1, Steps 1–3, $[0, 4]^3 \cap \{x(y + z) \leq 2\}$	$5 \cdot 10^5$	0
Thm. 1, Steps 1–3, $[0, 40]^3 \cap \{x(y + z) \leq 2\}$	$5 \cdot 10^3$	0
Lemma 5.1 and (5.1), critical configurations in $[0, 1]$	$1.3 \cdot 10^6$	0
Lemma 5.1 and (5.1), without $w \leq 1$	$3 \cdot 10^5$	0
Thm. 2, sorted $[0, 1]^4$	$2 \cdot 10^6$	0
Thm. 3 and each link of its proof	$3.7 \cdot 10^6$	0

The checks are reproduced by these scripts in the accompanying repository, whose README also lists the exploratory scripts:

- `explore/i3_proof_check.py` (Theorem 1);
- `explore/i4_chart_proof_check.py` (Lemmas 4.1, 5.1, Theorem 2);
- `explore/i6_chart_proof_check.py` (Theorem 3).

References

- [1] F. Nazarov, V. Vasyunin, A. Volberg, *On a Bellman function associated with the Chang–Wilson–Wolff theorem: a case study*, arXiv:2204.01336.