

A NON-DEGENERATE GAUSSIAN KERNEL WITHOUT GAUSSIAN MAXIMISERS: LIEB'S FORMULA FAILS BELOW THE DIAGONAL

PAATA IVANISVILI

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ABSTRACT. Lieb proved that operators with non-degenerate (complex) Gaussian kernels from L^p to L^q have Gaussian maximisers, and that their norm equals the supremum over centred Gaussian functions, when $1 < p \leq q < \infty$. He also covered $p > q$ in the real case and in an “imaginary” case. We show by an explicit, computer-certified example that both statements fail for $p > q$ in the genuinely complex case. For the Mehler transform T_z with $z = 0.72 e^{0.7i}$, from $L^p(\gamma)$ to $L^4(\gamma)$ with any $p \in [16, \infty)$, every Gaussian $g = C e^{\beta x^2/2 + \eta x}$, centred or not, satisfies $\|T_z g\|_4 \leq \|g\|_p$, with equality only for constants. However, the unimodular function $f(y) = e^{-i\lambda(|y| - a)_+}$ has $\|T_z f\|_4 \geq 1.00097 > 1 = \|f\|_p$. Equivalently, a non-degenerate complex Gaussian kernel on $\mathbb{R}$ has $L^p \rightarrow L^4$ norm strictly larger than the supremum over all Gaussian functions, so it has no Gaussian maximiser. The centred case is a branch-and-bound certificate; the non-centred case reduces, through exact polynomial identities, to three scalar inequalities. (By a companion result, T_z fails to be an $L^p \rightarrow L^4$ contraction at every z outside $\Omega_{4,4}$, already in dimension one; the present example shows that Gaussians cannot see this.) The phenomenon is general. Let $1 < q < p < \infty$ with $2 < q$ or $p < 2$, so that the contraction region $\Omega_{s,s}$, $s = \min\{p, \max\{q, 2\}\}$, is not the whole disc. For every boundary point z_0 of $\Omega_{s,s}$ in the open unit disc and off the imaginary axis, an analytic compactness argument shows the following: on a non-empty open set outside $\Omega_{s,s}$ adjacent to z_0 , no Gaussian detects the failure of $L^p \rightarrow L^q$ contractivity. The only delicate Gaussians are unimodular chirps, for which the inequality reduces to an explicit factorised discriminant, and (for $p < 2$) their duals.

1. STATEMENT

Let γ be the standard Gaussian measure on $\mathbb{R}$ and T_z the Mehler transform, $T_z H_n = z^n H_n$. For $|z| < 1$, $\varsigma := 1 - z^2$ satisfies $\operatorname{Re} \varsigma > 0$, and for g of moderate growth

$$\mathbf{T}_z g(x) = \int_{\mathbb{R}} g(y) \frac{1}{\sqrt{2\pi\varsigma}} e^{-(y-zx)^2/(2\varsigma)} dy,$$

which agrees with T_z on polynomials (Mehler's formula). Put $N_1(p, q; z) = \sup\{\|T_z P\|_{L^q(\gamma)} / \|P\|_{L^p(\gamma)} : P \text{ a nonzero polynomial}\}$.

Theorem 1.1. *Let $q = 4$, $z = 0.72 e^{0.7i}$ and $16 \leq p < \infty$.*

- (a) *For every Gaussian $g(x) = C e^{\beta x^2/2 + \eta x}$ with $C \neq 0$, $\beta, \eta \in \mathbb{C}$ and $\operatorname{Re} \beta < 1/p$, $\|\mathbf{T}_z g\|_{L^4(\gamma)} \leq \|g\|_{L^p(\gamma)}$, with equality only for $\beta = \eta = 0$.*
- (b) *For $f(y) = \exp(-i\lambda(|y| - a)_+)$ with $a = 2.776$ and $\lambda = 5.115$, $\|\mathbf{T}_z f\|_{L^4(\gamma)} \geq 1.000974 > 1 = \|f\|_{L^p(\gamma)}$.*
- (c) *Consequently $N_1(p, 4; z) \geq 1.000974 > 1$, while the supremum of $\|\mathbf{T}_z g\|_4 / \|g\|_p$ over all Gaussians equals 1.*

Corollary 1.2. *Let ϕ be the standard Gaussian density and $G_p(x, y) = \phi(x)^{1/4} K_z(x, y) \phi(y)^{1-1/p}$, where $K_z(x, y) = \varsigma^{-1/2} \exp\left(-\frac{z^2(x^2+y^2)-2zxy}{2\varsigma}\right)$ is the Mehler kernel. Then G_p is a non-degenerate complex Gaussian kernel. For $16 \leq p < \infty$ the norm of $F \mapsto \int G_p(\cdot, y) F(y) dy$ from $L^p(\mathbb{R})$ to $L^4(\mathbb{R})$ is strictly larger than the supremum of $\|G_p g\|_4 / \|g\|_p$ over all Gaussian functions g , centred or not. In particular, the identity “norm = supremum over centred Gaussians”, which Lieb [3] proves in the complex case for $p \leq q$ (see Theorem 6 in the summary [2]), does not extend to $p > q$ in the complex case, and no Gaussian function is a maximiser.*

The point z lies outside the L^4 -contraction region $\Omega_{4,4}$. By the main theorem of the companion note (RESULT_downward_complex_hypercontractivity, Theorem 1.1), T_z is therefore not a contraction from $L^p(\gamma_1)$ to $L^4(\gamma_1)$ for any p . The witnesses produced there are perturbations $1 - 2b$ of the constant function by narrow Gaussian bumps far from the origin, and at this z they give a certified violation of relative size about e^{-38} . The point of Theorem 1.1 is different: by (a), no Gaussian, centred or not, detects the failure, while (b) is a simple explicit witness with a violation of order 10^{-3} . The result is one-dimensional.

2. PROOF OF (A) FOR CENTRED GAUSSIANS

2.1. Closed form. By a Gaussian integral, $\mathbf{T}_z g(x) = D^{-1/2} \exp(\beta' x^2/2)$ with $D = 1 - \beta\varsigma$ and $\beta' = \beta z^2/D$. Also $\|g\|_r^r = (1 - r \operatorname{Re} \beta)^{-1/2}$. Hence, if $1 - 4 \operatorname{Re} \beta' > 0$,

$$\log \frac{\|\mathbf{T}_z g\|_4}{\|g\|_p} = c(\beta; p) := -\frac{1}{2} \log |1 - \varsigma\beta| - \frac{1}{8} \log(1 - 4 \operatorname{Re} \beta') + \frac{1}{2p} \log(1 - p \operatorname{Re} \beta).$$

If $1 - 4 \operatorname{Re} \beta' \leq 0$, then $\mathbf{T}_z g \notin L^4$; we show this never happens.

2.2. Monotonicity in p . Since γ is a probability measure, $p \mapsto \|g\|_p$ is nondecreasing (Lyapunov’s inequality), and the admissible half-plane $\{\operatorname{Re} \beta < 1/p\}$ shrinks as p grows. Hence $c(\beta; p) \leq c(\beta; 16)$ for $p \geq 16$, and it suffices to prove (a) for $p = 16$. (Directly: $p \mapsto \frac{1}{2p} \log(1 - px)$ is nonincreasing for $x < 1/p$.)

2.3. The certificate. The script `numerics/certify_centered_gaussians.py` (python-flint/Arb, 100-bit balls) verifies the following for $p = 16$. Each item is a rigorous inequality.

- (1) $\mathbf{T}_z g \in L^4$ for all admissible β . $\beta \mapsto \beta'$ is a Möbius map whose pole $1/\varsigma$ has $\operatorname{Re}(1/\varsigma) = 0.8346\dots > 1/16$. So the image of the half-plane is the disc through the images of $1/16$ and $1/16 \pm i$ that contains 0, and $\sup \operatorname{Re} \beta' \leq 0.24165 < 1/4$. In particular $m_0 := \inf(1 - 4 \operatorname{Re} \beta') \geq 0.0334$.
- (2) *Near 0.* With $u = \varsigma\beta$, $a_1 = \operatorname{Re}(z^2\beta)$, $a_2 = \operatorname{Re}(z^2\varsigma\beta^2)$, $x = \operatorname{Re} \beta$, the expansions $-\frac{1}{2} \operatorname{Re} \log(1 - u) = \frac{1}{2} \operatorname{Re} u + \frac{1}{4} \operatorname{Re} u^2 + E_1$, $\beta' = z^2\beta(1 + u + \frac{u^2}{1-u})$, $-\frac{1}{8} \log(1 - 4t) = \frac{t}{2} + t^2 + E_3$ and $\frac{1}{32} \log(1 - 16x) = -\frac{x}{2} - 4x^2 + E_5$ hold, with the standard bounds $|\log(1 - w) + w + \frac{w^2}{2}| \leq \frac{|w|^3}{3(1-|w|)}$. The first-order terms cancel ($\varsigma + z^2 = 1$), and the second-order part is $c_2(\beta) = \frac{1}{4}(|z^2\beta|_4^2 - |\beta|_{16}^2)$, where $|w|_s^2 = (s-1)(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2$; this is checked symbolically and at a test point. One finds $\lambda_{\max}(c_2) = -0.05219\dots$ and a total remainder $\leq K|\beta|^3$ with $K = 44.63$ on $|\beta| \leq \rho_0 = 8 \cdot 10^{-4}$. Hence $c(\beta) \leq |\beta|^2(\lambda_{\max} + K\rho_0) \leq -0.0165|\beta|^2$ there.
- (3) *Compact part.* The region $\rho_0 \leq |\beta|$, $-10 \leq \operatorname{Re} \beta \leq \frac{1}{16} - 10^{-12}$, $|\operatorname{Im} \beta| \leq 10$ is covered by 249,835 adaptive boxes. On each box the upper bound of c , from interval evaluation or from the mean-value form with an analytic gradient, is negative; the largest such bound is $-5.8 \cdot 10^{-9}$.

- (4) *Boundary strip* $\frac{1}{16} - 10^{-12} \leq \operatorname{Re} \beta < \frac{1}{16}$, $|\operatorname{Im} \beta| \leq 10$. The first two terms of c are ≤ 0.6232 , and the last is $\leq \frac{1}{32} \log(16 \cdot 10^{-12}) = -0.7768$.
- (5) *Infinity* ($|\beta| \geq 10$). Here $c \leq -\frac{1}{2} \log(|\varsigma||\beta| - 1) + \frac{1}{8} \log \frac{1}{m_0} + \frac{1}{32} \log(1 + 16|\beta|)$. This bound is decreasing in $|\beta|$ and equals $-0.539 \dots < 0$ at $|\beta| = 10$.

Hence $c(\beta; 16) < 0$ for all admissible $\beta \neq 0$, and $c(0) = 0$. $\square$

3. PROOF OF (A) FOR NON-CENTRED GAUSSIANS

3.1. Closed form. Let $g(x) = e^{\beta x^2/2 + \eta x}$ with $\operatorname{Re} \beta < 1/p$ and $\eta \in \mathbb{C}$. Completing the square in the Mehler integral (absolutely convergent since $\operatorname{Re} \frac{1}{\varsigma} > \operatorname{Re} \beta$) gives $\mathbf{T}_z g(x) = D^{-1/2} \exp(\frac{\beta'}{2} x^2 + \eta' x + \tau)$ with $D = 1 - \beta \varsigma$, $\beta' = \beta z^2/D$, $\eta' = \eta z/D$ and $\tau = \varsigma \eta^2/(2D)$. Since $\mathbb{E} \exp(\frac{a}{2} x^2 + bx) = (1 - a)^{-1/2} \exp \frac{b^2}{2(1-a)}$ for real $a < 1$ and real b , applied to $|\mathbf{T}_z g|^4$ and to $|g|^p$,

$$\log \frac{\|\mathbf{T}_z g\|_4}{\|g\|_p} = c(\beta; p) + Q_\beta(\eta), \quad Q_\beta(\eta) = \frac{1}{2} \operatorname{Re} \left(\frac{\varsigma}{D} \eta^2 \right) + \frac{q}{2} \frac{(\operatorname{Re} \frac{\eta}{D})^2}{m} - \frac{p}{2} \frac{(\operatorname{Re} \eta)^2}{n},$$

where $q = 4$, $m = 1 - 4 \operatorname{Re} \beta'$ and $n = 1 - p \operatorname{Re} \beta$. So Q_β is a real quadratic form in $(\operatorname{Re} \eta, \operatorname{Im} \eta)$, and by Section 2 it suffices to show that Q_β is negative definite for every admissible β . The difficulty is that $Q_\beta \rightarrow 0$ as $|\beta| \rightarrow \infty$, in every direction (the wide scans of Section 7 approach 0 there), so a naive branch-and-bound near infinity cannot succeed. Exact identities remove this difficulty.

3.2. Exact identities. Write $z = a_0 + ib_0$, $\beta = x + iy$, and

$$\begin{aligned} c_0 &= \operatorname{Re} \varsigma - q b_0^2, & \alpha_0 &= q \operatorname{Re} \varsigma - (q-1)|\varsigma|^2, & \omega &= q + (2-q)\varsigma, \\ \kappa &= p c_0 - \alpha_0, & N &= |D|^2 m, & T &= p N - q |z|^2 n. \end{aligned}$$

Lemma 3.1. *As polynomial identities in a_0, b_0, x, y, p, q and η :*

- (i) $N = 1 - \operatorname{Re}(\omega \beta) + \alpha_0 |\beta|^2$;
- (ii) $2N \left[\frac{1}{2} \operatorname{Re} \left(\frac{\varsigma}{D} \eta^2 \right) + \frac{q}{2} \left(\operatorname{Re} \frac{\eta}{D} \right)^2 / m \right] = G(\beta)[\eta]$, where $G(\beta)$ is the symmetric matrix, affine in β , $G = \begin{pmatrix} q|z|^2 + c_0 - \alpha_0 x & -(\alpha_0 y + (q-2)a_0 b_0) \\ -(\alpha_0 y + (q-2)a_0 b_0) & -(c_0 - \alpha_0 x) \end{pmatrix}$ acting on $(\operatorname{Re} \eta, \operatorname{Im} \eta)$;
- (iii) $\det G = -\alpha_0 N$ and $\operatorname{tr} G = q|z|^2$;
- (iv) if $\alpha_0 > 0$, then $\min_{\beta \in \mathbb{C}} T(\beta) = q|z|^2 \kappa / \alpha_0$.

Consequently the matrix M of Q_β , which is $G/(2N) - \frac{p}{2n} e_1 e_1^T$, satisfies

$$\det M = \frac{\kappa}{4nN}, \quad \operatorname{tr} M = -\frac{T}{2nN}.$$

Proof. (i)–(iii) are identities between polynomials (after the substitutions $1/D = \overline{D}/|D|^2$ and $m = N/|D|^2$); they are verified in exact rational polynomial arithmetic, with all variables symbolic, in part (A) of `numerics/certify_noncentered_gaussians.py`, which also re-checks the literal formula for Q_β , with complex division, at 60 random rational points, and re-derives $\det M$ without using G ; an independently written check, `numerics/check_noncentered_identities.py` (sympy polynomial expansion, p, q symbolic), re-verifies (i)–(iii), the two consequences below and the minimum of T used in Section 3.3, and compares the closed form with direct quadrature. Given (ii) and (iii), $\det M = \frac{\det G}{4N^2} - \frac{p}{2n} \cdot \frac{G_{22}}{2N} = \frac{-\alpha_0 n + p(c_0 - \alpha_0 x)}{4nN} = \frac{\kappa}{4nN}$ because $n = 1 - px$, and $\operatorname{tr} M = \frac{q|z|^2}{2N} - \frac{p}{2n} = -\frac{T}{2nN}$. For (iv), by (i), $T = (p - q|z|^2) + p[\alpha_0 |\beta|^2 - \operatorname{Re}(v\beta)]$ with $v = \omega - q|z|^2$, whose minimum over β is $p - q|z|^2 - p|v|^2/(4\alpha_0)$. The identity $4\alpha_0 - |v|^2 = 4q|z|^2 c_0$ (verified in both scripts) turns this into $q|z|^2(p c_0 - \alpha_0)/\alpha_0 = q|z|^2 \kappa / \alpha_0$. $\square$

So $(\det M)nN$ is constant: the degeneracy of Q_β at infinity is exact, and the sign of Q_β is decided by three scalar quantities.

3.3. The certificate. Part (B) of `numerics/certify_noncentered_gaussians.py` (Arb, 256-bit balls, exact inputs $z = \frac{18}{25}e^{7i/10}$, $p = 16$, $q = 4$) verifies:

- (1) $\alpha_0 = 0.370006\dots > 0$, $c_0 = 0.051310\dots > 0$ and $\kappa = 0.450969\dots > 0$;
- (2) $N \geq N_{\min} = 0.160105\dots > 0$ on $\operatorname{Re} \beta \leq \frac{1}{16}$: by (i), $\partial_x N = -\operatorname{Re} \omega + 2\alpha_0 x \leq -\operatorname{Re} \omega + \frac{\alpha_0}{8} < 0$ there, so $N(x, y) \geq N(\frac{1}{16}, y)$, whose minimum over y is explicit. In particular $m = N/|D|^2 > 0$, re-proving $\mathbf{T}_z g \in L^4$;
- (3) $T \geq T_{\min} = \frac{q|z|^2\kappa}{\alpha_0} = 2.527335\dots > 0$ on all of $\mathbb{C}$ (Lemma 3.1(iv)).

Since $n > 0$ on the admissible half-plane, Lemma 3.1 gives $\det M > 0$ and $\operatorname{tr} M < 0$: Q_β is negative definite, and in fact $\lambda_{\max}(Q_\beta) \leq -\frac{\kappa}{2T(\beta)} < 0$. For $p > 16$ the same argument applies verbatim: N does not depend on p , $\kappa = pc_0 - \alpha_0$ increases with p because $c_0 > 0$, and $\min_{\mathbb{C}} T = q|z|^2\kappa/\alpha_0$ for every p (an identity in z , verified with p symbolic); alternatively use Lyapunov's inequality as in Section 2. Together with $c(\beta; p) \leq 0$, with equality only at $\beta = 0$, this proves (a): the log-ratio $c(\beta; p) + Q_\beta(\eta)$ is ≤ 0 , with equality only at $\beta = \eta = 0$, and the constant C does not matter. The script also checks the closed forms against the Mehler integral and the full log-ratio against direct quadrature for two non-centred Gaussians (floating point, agreement 10^{-15}), and it re-runs the centred certificate. Its runtime is about 10 seconds. $\square$

4. PROOF OF (B): THE UNIMODULAR WITNESS

Since $|f| = 1$, $\|f\|_p = 1$ for every p . Splitting $\int_{\mathbb{R}}$ at $\pm a$ and completing squares gives the exact formula (with $m = zx$, $\Phi(w) = \frac{1}{2}(1 + \operatorname{erf}(w/\sqrt{2}))$, principal $\sqrt{\varsigma}$)

$$\begin{aligned} F(x) := \mathbf{T}_z f(x) &= \left[\Phi\left(\frac{a-m}{\sqrt{\varsigma}}\right) - \Phi\left(\frac{-a-m}{\sqrt{\varsigma}}\right) \right] \\ &\quad + e^{i\lambda a} e^{-i\lambda m - \lambda^2 \varsigma/2} \left[1 - \Phi\left(\frac{a-m+i\lambda\varsigma}{\sqrt{\varsigma}}\right) \right] \\ &\quad + e^{i\lambda a} e^{i\lambda m - \lambda^2 \varsigma/2} \Phi\left(\frac{-a-m-i\lambda\varsigma}{\sqrt{\varsigma}}\right). \end{aligned}$$

This is cross-checked against direct quadrature to 10^{-6} relative accuracy. For real x , $|\mathbf{T}_z f(x)|^4 \phi(x) = F(x)^2 \tilde{F}(x)^2 \phi(x)$ with $\tilde{F}(x) = \overline{F(\bar{x})}$, an entire function, so Arb's rigorous integrator applies on bounded intervals. Because the integrand is non-negative,

$$\mathbb{E}|\mathbf{T}_z f|^4 \geq \int_{-12}^{12} |\mathbf{T}_z f|^4 \phi dx \in [1.0039031603844549 \pm 7 \cdot 10^{-19}],$$

so $\|\mathbf{T}_z f\|_4 \geq 1.000974$ (`numerics/certify_lieb_counterexample.py`). The quantity is also finite: by the Schur bound $|\mathbf{T}_z f(x)| \leq (|\varsigma|A)^{-1/2} e^{Ex^2/2}$ with $A = \operatorname{Re} \frac{1}{\varsigma}$, $C = \operatorname{Re} \frac{1}{\varsigma}$, $E = \frac{C^2}{A} - A + 1 = 0.23593 < \frac{1}{4}$, the contribution of $|x| > 12$ is at most 0.0245, so together with the enclosure on $[-12, 12]$ one gets $\mathbb{E}|\mathbf{T}_z f|^4 \leq 1.0284$.

Passage to polynomials. Since $4(\frac{C^2}{A} - A + 1) = 0.9437 < 1$, the Schur–Hölder lemma of the companion note (Lemma 2.3 there) gives a finite $\sigma' \geq p$ such that $\mathbf{T}_z : L^{\sigma'}(\gamma) \rightarrow L^4(\gamma)$ is bounded. Polynomials $P_n \rightarrow f$ in $L^{\sigma'}$ then satisfy $\|\mathbf{T}_z P_n\|_4 \rightarrow \|\mathbf{T}_z f\|_4$ and $\|P_n\|_p \rightarrow 1$. This proves (c). $\square$

5. THE LEBESGUE-SPACE FORMULATION

The map $F \mapsto F\phi^{-1/p}$ is an isometry from $L^p(\mathbb{R})$ onto $L^p(\gamma)$, and $h \mapsto h\phi^{1/4}$ is an isometry from $L^4(\gamma)$ onto $L^4(\mathbb{R})$. They map centred Gaussians to centred Gaussians. Conjugating $\mathbf{T}_z$ by them gives the kernel G_p of Corollary 1.2. Its real quadratic part has matrix $\frac{1}{2} \begin{pmatrix} \frac{1}{4}+a & -b \\ -b & 1-\frac{1}{p}+a \end{pmatrix}$ with $a = \operatorname{Re} \frac{z^2}{\varsigma}$ and $b = \operatorname{Re} \frac{z}{\varsigma}$. This matrix is positive definite: $\frac{1}{4} + a = 0.0847$ and $(\frac{1}{4} + a)(1 - \frac{1}{p} + a) - b^2 \geq 0.00644$ for $p \geq 16$ (certified at $p = 16$, value 0.0064496...; the left side increases with p). So G_p is non-degenerate, and Corollary 1.2 follows from Theorem 1.1.

6. OPEN SETS OF EXAMPLES, FOR ALL EXPONENTS BELOW THE DIAGONAL

Theorem 1.1 is not an isolated accident. Let $1 < q < p < \infty$ with $s := \min\{p, \max\{q, 2\}\} \neq 2$, i.e. $2 < q < p$ (then $s = q$) or $q < p < 2$ (then $s = p$); for $q \leq 2 \leq p$ the contraction region is the whole disc. By the companion theorem [1, Theorem 1.1], T_z is an $L^p \rightarrow L^q$ contraction exactly on $\Omega_{s,s}$. At a boundary point z_0 of $\Omega_{s,s}$ every Gaussian satisfies $\|\mathbf{T}_{z_0}g\|_q \leq \|g\|_p$, and the inequality is strict for non-constant g . This strictness needs an argument only for a one-parameter family of chirp-like Gaussians. A compactness argument then propagates the Gaussian inequality to all z near z_0 , while the contraction fails at every $z \notin \Omega_{s,s}$.

Closed forms for general q . The computations of Sections 2 and 3 hold verbatim with 4 replaced by q . For $g = Ce^{\beta x^2/2 + \eta x}$ with $\operatorname{Re} \beta < 1/p$ and $1 - q \operatorname{Re} \beta' > 0$,

$$\log \frac{\|\mathbf{T}_z g\|_q}{\|g\|_p} = c_q(\beta; p) + Q_\beta(\eta), \quad c_q(\beta; p) = -\frac{1}{2} \log |D| - \frac{1}{2q} \log(1 - q \operatorname{Re} \beta') + \frac{1}{2p} \log n,$$

$$Q_\beta(\eta) = \frac{1}{2} \operatorname{Re} \left(\frac{\varsigma}{D} \eta^2 \right) + \frac{q}{2} (\operatorname{Re} \frac{z\eta}{D})^2 / (1 - q \operatorname{Re} \beta') - \frac{p}{2} (\operatorname{Re} \eta)^2 / n,$$

with $D = 1 - \varsigma\beta$, $\beta' = z^2\beta/D$ and $n = 1 - p \operatorname{Re} \beta$. Lemma 3.1 is stated for symbolic q . We write $c_0 = \operatorname{Re} \varsigma - q(\operatorname{Im} z)^2$, $\alpha_0 = q \operatorname{Re} \varsigma - (q-1)|\varsigma|^2$, $\kappa = pc_0 - \alpha_0$, $\omega = q + (2-q)\varsigma$, $N(\beta) = 1 - \operatorname{Re}(\omega\beta) + \alpha_0|\beta|^2 = |D|^2(1 - q \operatorname{Re} \beta')$, and $A = \operatorname{Re} \frac{1}{\varsigma}$, $C = \operatorname{Re} \frac{z}{\varsigma}$, $\delta_r = 1 - r + rA$. With $|w|_r^2 = (r-1)(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2$, expanding the three logarithms to second order gives, for all $z \in \mathbb{D}$ and $p, q > 1$ (checked symbolically),

$$(1) \quad c_q(\beta; p) = \frac{1}{4} (|z^2\beta|_q^2 - |\beta|_p^2) + O(|\beta|^3) \quad (\beta \rightarrow 0);$$

the first-order terms cancel because $\varsigma + z^2 = 1$.

The regions. By [1, Lemma 2.1], $\Omega_{r,r} \cap \mathbb{D} = \{|r-2||1-z^2| \leq r(1-|z|^2)\}$ for $r > 1$. Since $|1-z^2|^2 = (1-|z|^2)^2 + 4(\operatorname{Im} z)^2$, squaring gives the simpler description

$$(2) \quad \Omega_{r,r} \cap \mathbb{D} = \{z \in \mathbb{D} : |r-2||\operatorname{Im} z| \leq \sqrt{r-1}(1-|z|^2)\}.$$

For $r \neq 2$ it meets $\mathbb{R}$ in interior points only, and $i\mathbb{R}$ in $[-i\varrho_r, i\varrho_r]$, where $\varrho_r = \min\{\sqrt{r-1}, (r-1)^{-1/2}\}$. If $z_0 \in \partial\Omega_{r,r} \cap \mathbb{D}$ and $z_0 \notin \mathbb{R} \cup i\mathbb{R}$, then z_0^2 is an interior point of $\Omega_{r,r}$: from $|r-2||\operatorname{Im} z_0| = \sqrt{r-1}(1-|z_0|^2)$ we get $|r-2||\operatorname{Im} z_0^2| = 2|\operatorname{Re} z_0|\sqrt{r-1}(1-|z_0|^2) < \sqrt{r-1}(1-|z_0|^4)$, as $2|\operatorname{Re} z_0| < 1 + |z_0|^2$. Hence $\sup_{w \neq 0} |z_0^2 w|_r / |w|_r < 1$, since $(1+\varepsilon)z_0^2 \in \Omega_{r,r}$ for small $\varepsilon > 0$ and $\Omega_{r,r} = \{z : |zw|_r \leq |w|_r \ \forall w \in \mathbb{C}\}$ [1, Lemma 2.1(b)].

Theorem 6.1. *Let $1 < q < p < \infty$ with either $2 < q$ or $p < 2$, and let $s = \min\{p, \max\{q, 2\}\}$. Let $z_0 \in \partial\Omega_{s,s} \cap \mathbb{D}$ with $z_0 \notin i\mathbb{R}$. There is $r_0 \in (0, 1 - |z_0|]$ such that for every z with $|z - z_0| < r_0$:*

- (a) *every Gaussian $g(x) = c_g e^{\beta x^2/2 + \eta x}$ with $c_g \neq 0$ and $\operatorname{Re} \beta < 1/p$ satisfies $\|\mathbf{T}_z g\|_{L^q(\gamma)} \leq \|g\|_{L^p(\gamma)}$, with equality only for $\beta = \eta = 0$;*
- (b) *the kernel $G_{p,q}(x, y) = \phi(x)^{1/q} K_z(x, y) \phi(y)^{1-1/p}$ is non-degenerate.*

Consequently, on the non-empty open set $U(z_0) = \{|z - z_0| < r_0\} \setminus \Omega_{s,s}$ we have $N_1(p, q; z) > 1$, while the supremum over all Gaussians equals 1: Lieb's formula fails and there is no Gaussian maximiser, for every $z \in U(z_0)$.

Theorem 1.1 is a certified quantitative instance of the phenomenon ($q = 4$), with an explicit witness, at $z = 0.72e^{0.7i}$, which lies at radial distance about 0.025 outside $\partial\Omega_{4,4}$. It does not follow from Theorem 6.1, whose r_0 is not explicit.

6.1. Identities at the boundary. For all $z \in \mathbb{D}$, with $S = |\varsigma|^2$:

$$(3) \quad \begin{aligned} S &= (1 - |z|^2)^2 + 4(\operatorname{Im} z)^2, & \alpha_0 &= S \delta_q, \\ \alpha_0 - qc_0 &= q^2(\operatorname{Im} z)^2 - (q - 1)S, & 2\delta_r S &= r(1 - |z|^4) - (r - 2)S. \end{aligned}$$

(The last one, for any r , uses $\operatorname{Re} \varsigma = \frac{1}{2}(1 - |z|^4 + S)$.) Let $z_0 = x_0 + iy_0 \in \partial\Omega_{s,s} \cap \mathbb{D}$ with $z_0 \notin \mathbb{R} \cup i\mathbb{R}$; by symmetry ($z \mapsto \bar{z}$, $z \mapsto -z$) we may assume $x_0, y_0 > 0$.

Case $2 < q < p$, $s = q$. Here $(q - 2)\sqrt{S} = q(1 - |z_0|^2)$, so $(q - 2)y_0 = \sqrt{q - 1}(1 - |z_0|^2)$, $(q - 1)S = q^2 y_0^2$, hence $\alpha_0 = qc_0$, and $2\delta_q S = q(1 - |z_0|^2)[(1 + |z_0|^2) - |1 - z_0^2|] > 0$, because $z_0^2 \notin (-\infty, 0]$. So

$$(4) \quad \delta_q > 0, \quad \alpha_0 > 0, \quad c_0 = \alpha_0/q > 0, \quad \kappa = (p - q)c_0 > 0, \quad A > \frac{1}{2} > \frac{1}{p},$$

using $A > \frac{1}{2}$ on $\mathbb{D}$ ([1, §2]: $\operatorname{Re} \frac{1}{1 \mp z} > \frac{1}{2}$).

Case $q < p < 2$, $s = p$. Put $k = \sqrt{p - 1}$. Here $(2 - p)y_0 = k(1 - |z_0|^2)$, $0 < y_0 < k$, and one finds

$$(5) \quad S = \frac{p^2 y_0^2}{p - 1}, \quad \kappa = (p - q)(2 - p)\frac{y_0}{k}\left(1 - \frac{y_0}{k}\right) > 0, \quad S\left(A - \frac{1}{p}\right) = (2 - p)\frac{y_0}{k}\left(1 - \frac{y_0}{k}\right) > 0.$$

Moreover $\alpha_0 = S\delta_q > 0$, because $\delta_q = 1 - q(1 - A) > 1 - \frac{q}{2} > 0$.

In both cases $A > 1/p$, so $D = 1 - \varsigma\beta$ does not vanish on $\overline{H}$, where $H = \{\operatorname{Re} \beta < 1/p\}$, and the same holds for z near z_0 . The identities (3), (5), the factorisation in the case $s = q$, those in the proof of Lemma 6.2, (1), the bound of Lemma 3.1(iv) and the two determinant identities of Step 4 are verified symbolically in `numerics/check_theorem_Dprime.py`.

Lemma 6.2 (Chirps). *Let $2 < r < \infty$ and $z_0 \in \partial\Omega_{r,r} \cap \mathbb{D}$, $z_0 \notin \mathbb{R} \cup i\mathbb{R}$. Then $\|\mathbf{T}_{z_0} e^{itx^2/2}\|_{L^r(\gamma)} < 1$ for every real $t \neq 0$. Explicitly, with $\vartheta = \operatorname{Im} z_0^2$ and α_0 computed with r in place of q , $\log \|\mathbf{T}_{z_0} e^{itx^2/2}\|_r = -\frac{r-2}{4r} \log(1 - 2\vartheta t + St^2) - \frac{1}{2r} \log(1 + (r - 2)\vartheta t + \alpha_0 t^2)$.*

Proof. For $\beta = it$ one has $|D|^2 = 1 - 2\vartheta t + St^2$, $N(it) = 1 + (r - 2)\vartheta t + \alpha_0 t^2$ and $1 - r \operatorname{Re} \beta' = N/|D|^2$, which gives the formula. As above we may assume $x_0, y_0 > 0$ (the map $z \mapsto \bar{z}$ changes ϑ to $-\vartheta$, i.e. t to $-t$). Put $k_r = \sqrt{r - 1}$, $w = k_r y_0$ and $h = \frac{r-2}{2}$. The boundary relation $(r - 2)y_0 = k_r(1 - |z_0|^2)$ gives $x_0^2 = \frac{(1-w)(w+r-1)}{r-1}$. Hence $0 < w < 1$, because $x_0 \neq 0$ and $y_0 > 0$. Moreover $S = \frac{r^2 y_0^2}{r-1}$ and $\alpha_0 = r(r - 2)y_0(\frac{1}{k_r} - y_0)$. Substituting $t = \frac{k_r u}{r y_0}$ gives

$$\begin{aligned} |D|^2 &= P_1(u) := 1 - 2\xi u + u^2, & N &= P_2(u) := 1 + 2h\xi u + \chi u^2, \\ \xi &= \frac{2k_r x_0}{r}, & \chi &= \frac{(r - 2)(r - 1)(1 - w)}{r w} > 0. \end{aligned}$$

Here $\xi^2 = \frac{4(1-w)(w+r-1)}{r^2} < \frac{4(r-1)}{r^2} \leq 1$, so $P_1 > 0$. Also $P_2 = N(it) > 0$, since $N > 0$ on $\{\operatorname{Re} \beta < \frac{1}{r}\}$ by the argument of Step 1 below. Let $\psi(u) = h \log P_1(u) + \log P_2(u) = -2r \log \|\mathbf{T}_{z_0} e^{itx^2/2}\|_r$.

Then $\psi(0) = 0$ and

$$\psi'(u) = \frac{hP'_1P_2 + P_1P'_2}{P_1P_2} = \frac{2uR(u)}{P_1P_2}, \quad d_2 = h + \chi - 2h\xi^2(h+1),$$

$$R(u) = \chi(h+1)u^2 + \xi(h(2h+1) - \chi(h+2))u + d_2.$$

The discriminant of the quadratic R is

$$\xi^2(h(2h+1) - \chi(h+2))^2 - 4\chi(h+1)d_2$$

$$= -\frac{(r-2)^2(r-1)(1-w)(r+2w-2)^2(r^2(1-w) + (r+1)w-1)}{r^4w^2}.$$

This is negative: the last factor is linear in w , equal to $r^2 - 1$ at $w = 0$ and to r at $w = 1$. As $\chi(h+1) > 0$, $R > 0$ on $\mathbb{R}$. So ψ' has the sign of u , and $\psi(u) > 0$ for $u \neq 0$. $\square$

6.2. Proof of Theorem 6.1. Step 1: non-centred Gaussians. Suppose that $\alpha_0 > 0$, $\kappa > 0$, and $N > 0$ on $\overline{H}$. Then for every $\beta \in H$ we have $n > 0$ and $N > 0$, so $\det M = \frac{\kappa}{4nN} > 0$, and $\text{tr } M = -\frac{T}{2nN} < 0$ because $T \geq \frac{q|z|^2\kappa}{\alpha_0} > 0$ (Lemma 3.1(iv)). So Q_β is negative definite and $\log \frac{\|\mathbf{T}_z g\|_q}{\|g\|_p} = c_q(\beta; p) + Q_\beta(\eta) \leq c_q(\beta; p)$, with equality only for $\eta = 0$.

These hypotheses hold at z_0 by (4) or (5), except possibly $N > 0$, which we now prove. Let $\text{Re } \beta < 1/s$. Then $g_\beta = e^{\beta x^2/2} \in L^s(\gamma)$, and since $z_0 \in \Omega_{s,s}$, $\|\mathbf{T}_{z_0} g_\beta\|_s \leq \|g_\beta\|_s < \infty$. Here we use the contractive extension $\tilde{T}$ of T_{z_0} to L^s , which agrees with $\mathbf{T}_{z_0}$ on bounded functions [1, proof of Thm. 1.1]. The truncations $g_n = g_\beta \mathbf{1}_{\{|g_\beta| \leq n\}}$ converge to g_β in L^s , and $\mathbf{T}_{z_0} g_n \rightarrow \mathbf{T}_{z_0} g_\beta$ pointwise by dominated convergence in the Mehler integral (as $\text{Re } \beta < A$), so $\mathbf{T}_{z_0} g_\beta = \tilde{T} g_\beta$. Since $|\mathbf{T}_{z_0} g_\beta|^s$ is integrable, $1 - s \text{Re } \beta' > 0$. If $s = q$, this is $N(\beta) > 0$, on $\{\text{Re } \beta < \frac{1}{q}\} \supset \overline{H}$. If $s = p > q$, it gives $1 - p \text{Re } \beta' > 0$ on H , hence ≥ 0 on $\overline{H}$. Then $1 - q \text{Re } \beta' = (1 - p \text{Re } \beta') + (p - q) \text{Re } \beta' > 0$ on $\overline{H}$, which is clear if $\text{Re } \beta' \leq 0$. So again $N > 0$ on $\overline{H}$. All hypotheses are open conditions. For N : for z near z_0 , $N_z(\beta) \geq \frac{1}{2}\alpha_0(z_0)|\beta|^2 > 0$ for $|\beta| \geq R$ (with R large, uniformly in z), and $N_z \rightarrow N_{z_0}$ uniformly on the compact set $\overline{H} \cap \{|\beta| \leq R\}$, where N_{z_0} has a positive minimum.

Step 2: centred Gaussians at z_0 . Write $\Gamma(z, \beta) = c_q(\beta; p)$. By (1), $\Gamma(z, 0) = 0$ and $\nabla_\beta \Gamma(z, 0) = 0$ for every z . The Hessian of $\Gamma(z_0, \cdot)$ at 0 is negative definite. Indeed, by the last paragraph before Theorem 6.1, $|z_0^2 \beta|_s < |\beta|_s$ for $\beta \neq 0$. If $s = q$, this gives $|z_0^2 \beta|_q < |\beta|_q \leq |\beta|_p$; if $s = p$, it gives $|z_0^2 \beta|_q \leq |z_0^2 \beta|_p < |\beta|_p$. We now show that $\Gamma(z_0, \beta) < 0$ for every $\beta \in H \setminus \{0\}$. Write $G = \mathbf{T}_{z_0} g_\beta$; by Step 1, $G \in L^s$ with $\|G\|_s \leq \|g_\beta\|_s$.

- Case $s = q$. If $\text{Re } \beta \neq 0$, then $|g_\beta|$ is not constant, and the strict Lyapunov inequality gives $\Gamma(z_0, \beta) \leq \log \frac{\|g_\beta\|_q}{\|g_\beta\|_p} < 0$. If $\text{Re } \beta = 0$, then $\|g_\beta\|_p = 1$ and $\Gamma(z_0, \beta) = \log \|G\|_q < 0$ by Lemma 6.2 with $r = q$.
- Case $s = p$. Here $\Gamma(z_0, \beta) = \log \frac{\|G\|_q}{\|G\|_p} + \log \frac{\|G\|_p}{\|g_\beta\|_p}$, and both terms are ≤ 0 . If $|G| = |D|^{-1/2} e^{\text{Re } \beta' x^2/2}$ is not constant, the first is < 0 . If $|G| \equiv c$ is constant, i.e. $\beta' = i\tau$ with $\tau \neq 0$, suppose $\|G\|_p = \|g_\beta\|_p$. Let $h = \overline{G}/c$, so $|h| = 1$, $\|h\|_{p'} = 1$, and h is a unimodular multiple of the chirp $e^{-i\tau x^2/2}$. Use the bilinear symmetry $\mathbb{E}[(\tilde{T}g)h] = \mathbb{E}[g\tilde{T}'h]$, where $\tilde{T}'$ is the contractive extension of T_{z_0} to $L^{p'}$ (valid on polynomials, hence on $L^p \times L^{p'}$ by density). This gives $c = \mathbb{E}[Gh] = \mathbb{E}[g_\beta \mathbf{T}_{z_0} h] \leq \|g_\beta\|_p \|\mathbf{T}_{z_0} h\|_{p'}$, so $\|\mathbf{T}_{z_0} h\|_{p'} \geq 1$. This contradicts Lemma 6.2 with $r = p' > 2$, since $\Omega_{p',p'} = \Omega_{p,p}$. Hence the second term is < 0 .

Step 3: centred Gaussians near z_0 . Γ is real-analytic in (z, β) on the open set where $|D|$, $1 - q \text{Re } \beta'$ and n are positive. Hence there are $r_1, \lambda > 0$ and a closed disc $V \subset \mathbb{D}$ around z_0 ,

contained in the neighbourhood of Step 1, such that $\Gamma(z, \beta) \leq -\lambda|\beta|^2$ for $|\beta| \leq r_1$ and $z \in V$ (Taylor's formula with uniformly bounded third derivatives). For $|\beta| \geq R_2$, $\beta \in H$ and $z \in V$,

$$\Gamma(z, \beta) \leq -\frac{1}{2} \log(|\varsigma||\beta| - 1) + \frac{1}{2q} \log \frac{1}{m_*} + \frac{1}{2p} \log(1 + p|\beta|) < -1,$$

where $m_* > 0$ is a uniform lower bound for $1 - q \operatorname{Re} \beta' = N/|D|^2 \rightarrow \alpha_0/S = \delta_q$ as $|\beta| \rightarrow \infty$; this holds once R_2 is large. Near the boundary line, for $\frac{1}{p} - \varepsilon \leq \operatorname{Re} \beta < \frac{1}{p}$ and $|\beta| \leq R_2$, the term $\frac{1}{2p} \log(1 - p \operatorname{Re} \beta)$ tends to $-\infty$. The other two terms are bounded above uniformly, since $|D|$ and $N/|D|^2$ are continuous and positive on the closed strip $\{\frac{1}{p} - \varepsilon \leq \operatorname{Re} \beta \leq \frac{1}{p}, |\beta| \leq R_2\}$, uniformly for $z \in V$. So $\Gamma < -1$ there for small ε . On the compact set $K = \{\beta : r_1 \leq |\beta| \leq R_2, \operatorname{Re} \beta \leq \frac{1}{p} - \varepsilon\}$, $\Gamma(z_0, \cdot)$ is continuous and negative by Step 2, hence $\leq -2\mu$ for some $\mu > 0$. By uniform continuity, $\Gamma(z, \cdot) \leq -\mu$ on K for $|z - z_0| < r_0$, with r_0 small. Altogether, $\Gamma(z, \beta) < 0$ for all $\beta \in H \setminus \{0\}$ and $|z - z_0| < r_0$, where we also take r_0 so small that $\{|z - z_0| < r_0\} \subset V$.

Step 4: conclusion. By Steps 1 and 3, for $|z - z_0| < r_0$ and $g = c_g e^{\beta x^2/2 + \eta x}$, $\log \frac{\|\mathbf{T}_z g\|_q}{\|g\|_p} = \Gamma(z, \beta) + Q_\beta(\eta) \leq 0$, with equality only for $\beta = 0$ and then $\eta = 0$. This is (a). For (b), as in Section 5, the real quadratic part of $G_{p,q}$ has matrix $\frac{1}{2} \begin{pmatrix} \frac{1}{q} + a & -C \\ -C & 1 - \frac{1}{p} + a \end{pmatrix}$ with $a = A - 1$, so $\frac{1}{q} + a = \frac{\delta_q}{q} > 0$. If $s = q$, the identity of [1, Lemma 5.2] gives $q^2 C^2 = (qA - 1)\delta_q$ on $\partial\Omega_{q,q}$, so $(\frac{1}{q} + a)(1 - \frac{1}{p} + a) - C^2 = \frac{\delta_q}{q}(\frac{1}{q} - \frac{1}{p}) > 0$. If $s = p$, a direct computation on $\partial\Omega_{p,p}$ gives $S^2[(\frac{1}{q} + a)(1 - \frac{1}{p} + a) - C^2] = \frac{p y_0^3 (p-q)(2-p)(k-y_0)}{q(p-1)^2} > 0$. Non-degeneracy is an open condition. Finally, $U(z_0)$ is open and non-empty because $z_0 \in \partial\Omega_{s,s}$. For $z \in U(z_0)$, $N_1(p, q; z) > 1$ by [1, Theorem 1.1] in dimension one. $\square$

Remark 6.3. For $q \leq 2 \leq p$ no such set exists inside $\mathbb{D}$, because then T_z is an $L^p \rightarrow L^q$ contraction for all $|z| \leq 1$. Numerically, away from the axes the Gaussian-blind band beyond $\partial\Omega_{s,s}$ has radial width between about 10^{-3} and $4 \cdot 10^{-2}$ in the cases tested ($q = 4$ with $p \in \{4.5, 5, 8, 16, 64\}$; $q = 3$ with $p \in \{4, 6\}$; $(q, p) \in \{(1.5, 1.8), (1.2, 1.8), (1.3, 1.6)\}$; `numerics/gaussian-blind-region.py`, whose finite searches give upper bounds for the widths). It tends to 0 at the imaginary axis, at ± 1 and as $p \downarrow q$. One reason: at $\beta = 0$, $Q_0(\eta) = \frac{1}{2}(|z\eta|_q^2 - |\eta|_p^2)$, so pure exponentials $e^{\eta x}$ already detect every $z \notin \Omega_{p,q} = \{z : |zw|_q \leq |w|_p \forall w\}$, and the Gaussian-blind set lies inside $\Omega_{p,q} \setminus \Omega_{s,s}$.

7. NUMERICAL COMPLEMENTS AND THE MECHANISM

- Consistency with the wide scans: a scan of $\max\{c(\beta), \lambda_{\max}(Q_\beta)\}$ with $\operatorname{Re} \beta \geq -e^{20}$ and $|\operatorname{Im} \beta| \leq \sinh 14$ at $p = 16$ and $p = 64$ (`numerics/wide_gauss_scan.py`) gives at most $-1.6 \cdot 10^{-19}$, attained near $\beta = \frac{1}{16} - e^{20} - 512337i$. Lemma 3.1 explains this value: $|\beta|^2 \lambda_{\max}(Q_\beta) \rightarrow -\kappa/(2p\alpha_0) = -0.0381 \dots$ in every direction, which at that point gives $-1.618 \cdot 10^{-19}$. For $p = 12$ a centred Gaussian violation exists (log-ratio $+2.1 \cdot 10^{-3}$; $+3.5 \cdot 10^{-4}$ at $p = 13$, and none found for $p \geq 14$; `numerics/lieb_p_threshold.py`, floating point), which is why we take $p \geq 16$.
- Mechanism: the optimiser over arbitrary unimodular grid functions (`numerics/opt_general1d.py`) produces a phase that is constant on $|y| \lesssim 2.8$ and has slope $\approx \pm 5$ in the tails. The gain comes from $|x| \approx 8\text{--}13$, where $|\mathbf{T}_z f|$ is astronomically large and the Gaussian weight tiny. Gaussian test functions cannot localise a linear phase to the tails. This is the mechanism of the bump perturbations in the companion note, where a bump at x_0 is transported by $\mathbf{T}_z$ to a larger $|x|$ and amplified there.
- The same phenomenon occurs for $q = 3$ ($z = 0.82e^{0.7i}$, $p = \infty$, log-ratio $+7.5 \cdot 10^{-4}$, numerical; `numerics/opt_general1d.py`, output `out_opt_inf_3.0.7.0.820.txt`).

8. NOVELTY AND LIMITATIONS

Lieb [3] establishes Gaussian maximisers for $p \leq q$, and for $p > q$ in the real and “imaginary” cases. Our kernel is genuinely complex: the real part of the off-diagonal coefficient is $b \neq 0$. We found no prior counterexample for $p > q$ in the complex case. Negrín [4] determined the boundedness region of one-dimensional complex Gaussian kernels for all p, q (for $q < p$ it is the non-degenerate region, consistent with Section 5), and gave sufficient conditions for contractivity of real kernels; his introduction paraphrases Lieb’s theorem as giving, for every non-degenerate kernel, exactly one maximiser, a centred Gaussian. For $p > q$ in the complex case this cannot hold: by Theorem 1.1, every centred Gaussian has ratio at most 1, which is strictly less than the norm. Theorem 1.1 is an explicit point with a quantitative witness, certified rigorously (centred Gaussians by branch-and-bound, non-centred Gaussians by exact identities). Theorem 6.1 is analytic. It shows that the failure occurs on a non-empty open set adjacent to every point of the boundary of the contraction region in the open unit disc and off the imaginary axis, for all $1 < q < p < \infty$ except $q \leq 2 \leq p$ (where there is nothing to detect). *Limitations.* Theorem 6.1 gives no explicit size of $U(z_0)$, and only dimension one is treated. A complete description of the set of (p, q, z) for which Gaussians determine the norm remains open; numerically (Remark 6.3) the Gaussian-blind band beyond $\partial\Omega_{4,4}$ has radial width up to about 0.04 (for $p \leq 64$), and it shrinks to 0 at the imaginary axis.

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