

The Keller–Klein Berry–Esseen conjecture: the reachable half for all Rademacher sums, and the full conjecture for $a_1 \geq 0.07$

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Computer-assisted proof

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Abstract

Keller and Klein conjectured that a Rademacher sum $X = \sum_i a_i x_i$ with $\sum_i a_i^2 = 1$ and $a_1 = \max_i a_i$ satisfies $|\Pr[X \leq x] - \Pr[Z \leq x]| \leq \Pr[Z \in (0, a_1)]$ for all real x , where $Z \sim N(0, 1)$, and remarked that their methods should give the half $\Pr[Z \leq x] - \Pr[X \leq x] \leq \Pr[Z \in (0, a_1)]$, $x \geq 0$. They proved this half for $a_1 \in [0.2122, 0.22]$. We prove it for *all* weight vectors, i.e. for every $a_1 \in (0, 1]$. The proof combines Prawitz’s smoothing inequality with an improved pointwise envelope for the characteristic function of a Rademacher sum, the Keller–Klein segment comparison lemma, and Hoeffding’s inequality. For $a_1 \in [0.05, 1]$ the envelope inequalities are discharged by a rigorous computer verification in ball arithmetic (about $1.1 \cdot 10^5$ certified integrals); for $a_1 \in (0, 0.1]$ they are discharged by an explicit asymptotic analysis of the Prawitz bound at $T = \pi/a_1$, whose finitely many numerical constants are themselves certified in ball arithmetic. The two ranges overlap, so together they settle the reachable half for every weight vector. We then prove the other half $\Pr[X \leq x] - \Pr[Z \leq x] \leq \Pr[Z \in (0, a_1)]$, and hence the *full* conjecture, for every $a_1 \in [0.07, 1]$. The new ingredient is a Hölder covering of the Prawitz integral: where a single pointwise envelope of the characteristic function overshoots near $x = 0$, we instead bound φ_X by a supremum of equal-weight profiles, sharp at $x = 0$ (recovering the local bound $\Pr[X = 0] \leq \Pr[Z \in (-a_1, a_1)]$) and tight enough for small x . The resulting inequalities are discharged by a ball-arithmetic verification, re-checked at higher precision and re-verified in full, cell by cell, by an independent from-scratch implementation. For $a_1 \geq u_{0.75} \approx 0.6745$ the upper half is elementary—and was claimed, under a distinct-values hypothesis, by Ma, Ye and Han—so the new content of our upper-half result is the range $a_1 \in [0.07, u_{0.75}]$; combined with the reachable half (new here for every $a_1 \notin [0.2122, 0.22]$) this yields the full Keller–Klein conjecture for all $a_1 \in [0.07, 1]$, far beyond the $[0.2122, 0.22]$ the authors reached.

1 Statement

Throughout, $x_1, \dots, x_n$ are independent uniform random signs, $a_1 \geq a_2 \geq \dots \geq a_n > 0$, $\sum_i a_i^2 = 1$, and $X = \sum_i a_i x_i$. We write Φ and φ for the standard normal distribution function and density, $Z \sim N(0, 1)$,

$$c(a) = \Phi(a) - \frac{1}{2} = \Pr[Z \in (0, a)], \quad H(x) = 2\Phi(x) - 1 = \Pr[|Z| \leq x].$$

Theorem 1.1. *For every Rademacher sum (i.e. every $a_1 \in (0, 1]$) and every $x \geq 0$,*

$$\Pr[X \leq x] \geq \Pr[Z \leq x] - \Pr[Z \in (0, a_1)]. \tag{1}$$

Equivalently (by the symmetry $X \stackrel{d}{=} -X$), $\Pr[X \leq x] - \Pr[Z \leq x] \leq \Pr[Z \in (0, a_1)]$ for every $x \leq 0$.

This is the half of the conjecture of Keller and Klein [1, Section 9] that they describe as reachable by their methods (“ $\Pr[Z \leq x] - \Pr[X \leq x] \leq \Pr[Z \in (0, a_1)]$ for all $x \geq 0$ ”). Their Proposition 4.4 gives $\Pr[X \leq x] \geq \Pr[Z \leq x] - \max(0.084, c(a_1))$ for $a_1 \leq 0.22$, which is (1) exactly when $c(a_1) \geq 0.084$, i.e. $a_1 \in [0.2122, 0.22]$. Keller and Klein prove no other case of (1); the only other partial results we are aware of are those of Ma, Ye and Han [10], who settle the full conjecture for $n \leq 5$, prove the *other* half (“ $\Pr[X \leq x] - \Pr[Z \leq x] \leq c(a_1)$ for $x \geq 0$ ”, often denoted (U)) for $a_1 \geq u_{0.75} \approx 0.6745$ when the 2^n values of X are distinct, and deduce (1) itself under the additional condition $a_1 > \sum_{i \geq 2} a_i$. They also assert (their Theorem 3) the full conjecture for equal weights $a_1 = 1/\sqrt{n}$, $n \geq 21$; but the lower half (L) of that assertion is there attributed to [1] Proposition 4.4, which in fact yields (1) only for $a_1 \in [0.2122, 0.22]$, i.e. for $n \in \{21, 22\}$ among equal weights, so for $n \geq 23$ (where $a_1 = 1/\sqrt{n} < 0.2122$) that half is first established here, by Theorem 1.1. (Numbers of theorems, lemmas and equations of [1] refer to arXiv v4, as obtained by compiling its source.) Inequality (1) is sharp: for $X = n^{-1/2} \sum_{i=1}^n x_i$ with n odd and $x \uparrow a_1$ it becomes an equality in the limit.

The proof has two parts whose ranges of a_1 overlap: a computer-assisted verification for $a_1 \in [0.05, 1]$ (Sections 2–6) and an explicit analytic argument for $a_1 \in (0, 0.1]$ (Section 8). Both are rigorous. In the first part every inequality checked by the computer is checked in interval (ball) arithmetic, including all integrals; in the second part the argument is analytic except for a fixed, finite list of numerical inequalities between explicit constants, each certified in ball arithmetic.

Equivalent form of the equivalence claim. For $x \leq 0$ put $y = -x \geq 0$; then $\Pr[X \leq x] = 1 - \Pr[X < y]$ and $\Pr[Z \leq x] = 1 - \Phi(y)$, so the second statement reads $\Phi(y) - \Pr[X < y] \leq c(a_1)$ for $y \geq 0$. For $y > 0$ this follows from (1) by letting $x \uparrow y$ (using continuity of Φ and left-continuity of $x \mapsto \Pr[X < x]$), and conversely implies (1) since $\Pr[X \leq y] \geq \Pr[X < y]$. At $y = 0$ the limiting argument is unavailable; there the statement reads $\Phi(0) - \Pr[X < 0] = \frac{1}{2} \Pr[X = 0] \leq c(a_1)$, i.e. $\Pr[X = 0] \leq 2c(a_1) = H(a_1)$, which is Proposition 9.1.

2 Reduction

Lemma 2.1. *Theorem 1.1 holds for a_1 as soon as*

$$\Pr[|X| < x] \geq H(x) - H(a_1) \quad \text{for all } x > a_1. \quad (\text{L})$$

Proof. For $x \in [0, a_1]$: by symmetry $\Pr[X \leq x] \geq \Pr[X \leq 0] \geq \frac{1}{2}$, and $\Phi(x) - c(a_1) = \frac{1}{2} + \Phi(x) - \Phi(a_1) \leq \frac{1}{2}$. For $x > a_1$: by symmetry, $\Pr[X < x] = \Pr[X < 0] + \Pr[X = 0] + \Pr[0 < X < x] = \frac{1}{2} + \frac{1}{2} \Pr[|X| < x]$. Hence, using (L), $\Pr[X \leq x] \geq \Pr[X < x] \geq \frac{1}{2} + \frac{1}{2}(H(x) - H(a_1)) = \Phi(x) - c(a_1)$. $\square$

3 The smoothing inequality and envelopes

Theorem 3.1 (Prawitz [2, (1b)], as stated in [1, Theorem 4.5]). *Let Y be a real random variable with characteristic function $\varphi_Y(t) = \mathbb{E}e^{itY}$, and $T > 0$. For every $x \in \mathbb{R}$,*

$$\Pr[Y < x] \geq \frac{1}{2} - \text{v.p.} \int_{-T}^T e^{-ixt} \frac{1}{T} K\left(-\frac{t}{T}\right) \varphi_Y(t) dt, \quad K(u) = \frac{1 - |u|}{2} + \frac{i}{2} \left((1 - |u|) \cot(\pi u) + \frac{\text{sgn } u}{\pi} \right).$$

Corollary 3.2. *If Y is symmetric, then for all $x \in \mathbb{R}$ and $T > 0$*

$$\Pr[Y < x] \geq \frac{1}{2} - \int_0^1 k(u, x, T) \varphi_Y(Tu) du, \quad k(u, x, T) = (1 - u) \cos(Tux) - \left((1 - u) \cot(\pi u) + \frac{1}{\pi} \right) \sin(Tux).$$

For $u \in (0, 1)$ one has $k(u, x, T) = \frac{(1-u)\sin(\pi u - Tux)}{\sin \pi u} - \frac{\sin Tux}{\pi}$; $k(\cdot, x, T)$ extends continuously to $[0, 1]$ with $k(0) = 1 - Tx/\pi$ and $k(1) = 0$.

Proof. Substitute $t = Ts$ and split the principal value integral at 0. Using $\varphi_Y(-t) = \varphi_Y(t) \in \mathbb{R}$ (symmetry) and $K(-s) = \overline{K(s)}$, the integral equals $\int_0^1 (e^{iTsx} K(s) + \overline{e^{iTsx} K(s)}) \varphi_Y(Ts) ds = \int_0^1 2 \operatorname{Re}(e^{iTsx} K(s)) \varphi_Y(Ts) ds$, and $2 \operatorname{Re}(e^{iTsx} K(s)) = (1-s) \cos(Tsx) - \sin(Tsx)((1-s) \cot \pi s + \frac{1}{\pi}) = k(s, x, T)$ (this is [1, (48)]); the second formula follows from $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$. Near $s = 0$, $(1-s) \cot(\pi s) \sin(Tsx) \rightarrow Tx/\pi$, so k is bounded and the principal value is an ordinary integral. Near $s = 1$, $(1-s) \cot(\pi s) \rightarrow -1/\pi$, so $k(1) = 0$. $\square$

Let $\theta \in (1.7, 1.8)$ be the unique root of $e^{-y^2/2} + \cos y$ in $[0, \pi]$ (on $[0, \pi/2]$ the function is positive; on $[\pi/2, \pi]$ its derivative $-ye^{-y^2/2} - \sin y$ is negative).

Lemma 3.3 (envelopes). *Let $a \geq a_1$ and $v \geq 0$, and let $\varphi_X(v) = \prod_i \cos(a_i v)$. Then $L_a(v) \leq \varphi_X(v) \leq U_a(v)$, where*

$$(L_a, U_a)(v) = \begin{cases} (\cos(av)^{1/a^2}, e^{-v^2/2}), & 0 \leq av \leq \pi/2, \\ (-|\cos av| e^{-(1-a^2)v^2/2}, e^{-v^2/2}), & \pi/2 < av \leq \theta, \\ (-|\cos av|^{1/a^2}, |\cos av|^{1/a^2}), & \theta < av \leq \pi, \\ (-1, 1), & av > \pi. \end{cases}$$

Proof. Let $\psi(s) = \log |\cos s|/s^2$ on $(0, \pi) \setminus \{\pi/2\}$.

(i) ψ is decreasing on $(0, \pi/2)$: $\psi'(s) = -g(s)/s^3$ with $g(s) = s \tan s + 2 \log \cos s$; $g(0^+) = 0$ and $g'(s) = s \sec^2 s - \tan s = (2s - \sin 2s)/(2 \cos^2 s) > 0$. Also $\psi(0^+) = -\frac{1}{2}$.

(ii) ψ is increasing on $(\pi/2, \pi)$: it is the product of $\log |\cos s|$, which is negative and increasing there, and s^{-2} , which is positive and decreasing.

(iii) $\psi \leq -\frac{1}{2}$ on $(0, \theta]$, i.e. $|\cos s| \leq e^{-s^2/2}$ for $s \in [0, \theta]$: on $(0, \pi/2)$ by (i); on $[\pi/2, \theta]$ since $|\cos s| \leq |\cos \theta| = e^{-\theta^2/2} \leq e^{-s^2/2}$ ($|\cos|$ increases on $[\pi/2, \pi]$ and $\cos \theta = -e^{-\theta^2/2}$). In particular $\psi(\theta) = -\frac{1}{2}$.

Write $\log |\varphi_X(v)| = \sum_i a_i^2 v^2 \psi(a_i v)$ (terms with $a_i v = \pi/2$ give $\varphi_X(v) = 0$, which satisfies all claimed bounds since $L_a \leq 0 \leq U_a$ in the second and third cases, and in the first case $a_i v = \pi/2$ forces $av = \pi/2$ and $L_a(v) = 0$). Since $a_i v \leq av$ and $\sum_i a_i^2 = 1$:

Case $av \leq \pi/2$. All factors are nonnegative, and $\psi(a_i v) \geq \psi(av)$ by (i), so $\varphi_X(v) \geq e^{v^2 \psi(av)} = \cos(av)^{1/a^2}$; and $\psi(a_i v) \leq -\frac{1}{2}$ by (iii), so $\varphi_X(v) \leq e^{-v^2/2}$.

Upper bounds for $\pi/2 < av \leq \pi$. By (i)–(iii), $\psi(a_i v) \leq \max(-\frac{1}{2}, \psi(av))$, which equals $-\frac{1}{2}$ if $av \leq \theta$ and $\psi(av)$ if $av > \theta$ (by (ii) and $\psi(\theta) = -\frac{1}{2}$). Hence $|\varphi_X(v)| \leq e^{-v^2/2}$, respectively $\leq |\cos av|^{1/a^2}$. This also gives the lower bound $-|\cos av|^{1/a^2}$ in the third case.

Lower bound for $\pi/2 < av \leq \theta$. If $\varphi_X(v) \geq 0$ there is nothing to prove. Otherwise some factor is negative, i.e. some i has $a_i v \in (\pi/2, av]$; then $|\cos a_i v| \leq |\cos av|$, and by (iii) $\prod_{j \neq i} |\cos a_j v| \leq e^{-(1-a_i^2)v^2/2} \leq e^{-(1-a^2)v^2/2}$. So $\varphi_X(v) \geq -|\cos av| e^{-(1-a^2)v^2/2}$.

Case $av > \pi$. Trivial. $\square$

Remark 3.4. The lower envelope in the second case is new; Keller–Klein use $|\varphi_X| \leq e^{-v^2/2}$ there. Both envelopes are continuous in v (at $av = \theta$ because $|\cos \theta| = e^{-\theta^2/2}$), and for fixed v the lower envelope is nonincreasing and the upper envelope nondecreasing in a (this is not needed below, because we only use the lemma with $a = \beta \geq a_1$).

Definition 3.5. For $a \in (0, 1]$, $T > 0$ and any $x \in \mathbb{R}$ let

$$J(x; a, T) = \int_0^1 \max \{k(u, x, T)L_a(Tu), k(u, x, T)U_a(Tu)\} du;$$

for $x \geq 0$ also set $B(x; a, T) = \Phi(x) - \frac{1}{2} + J(x; a, T)$. The integrand depends on x only through $k(u, x, T) = (1 - u) \cos(Tux) + \rho(u) \sin(Tux)$, which is defined for every real x ; §10 uses J at the point $-x$ with $x \geq 0$.

Corollary 3.6. If $a \geq a_1$, then for all $x \geq 0$, $T > 0$: $\Phi(x) - \Pr[X < x] \leq B(x; a, T)$, and $\Pr[|X| < x] \geq -2J(x; a, T)$ for $x > 0$.

Proof. By Corollary 3.2 and Lemma 3.3, $\Pr[X < x] \geq \frac{1}{2} - \int_0^1 k\varphi_X(Tu)du \geq \frac{1}{2} - J(x; a, T)$, because $k\varphi_X(Tu) \leq \max(kL_a(Tu), kU_a(Tu))$ pointwise. For $x > 0$, $\Pr[|X| < x] = 2\Pr[X < x] - 1 \geq -2J$. $\square$

4 Local concentration near the origin

For $A < B$ write $\Pr\langle A, B \rangle = \Pr[X \in [A, B]] - \frac{1}{2}\Pr[X = A] - \frac{1}{2}\Pr[X = B]$ (and 0 if $A \geq B$).

Lemma 4.1 (Keller–Klein [1, Lemma 3.7]). Let $M = \max_i a_i$. If $A, B, C, D \in \mathbb{R}$ satisfy $|A| \leq C$ and $2\max(M, D - B) \leq C - A$, then $\Pr\langle C, D \rangle \leq \Pr\langle A, B \rangle$.

Lemma 4.2. Let $a \geq a_1$, $S > 0$, $a_1 \leq b < x$ and $0 < z \leq 2b$. Then $\Pr[|X| < x] \geq -\frac{2}{3}J(z; a, S)$.

Proof. Apply Lemma 4.1 with $(A, B, C, D) = (-b, b, b, 2b)$: $|A| = b \leq C$ and $2\max(a_1, D - B) = 2\max(a_1, b) = 2b = C - A$. Hence $\Pr\langle b, 2b \rangle \leq \Pr\langle -b, b \rangle$, and by the symmetry $X \stackrel{d}{=} -X$ also $\Pr\langle -2b, -b \rangle \leq \Pr\langle -b, b \rangle$. By additivity of $\langle \cdot, \cdot \rangle$ over adjacent segments [1, (7)], $\Pr\langle -2b, 2b \rangle = \Pr\langle -2b, -b \rangle + \Pr\langle -b, b \rangle + \Pr\langle b, 2b \rangle \leq 3\Pr\langle -b, b \rangle$. Therefore, using Corollary 3.6 at the point $z > 0$,

$$\Pr[|X| < x] \geq \Pr[|X| \leq b] \geq \Pr\langle -b, b \rangle \geq \frac{1}{3}\Pr\langle -2b, 2b \rangle \geq \frac{1}{3}\Pr[|X| < 2b] \geq \frac{1}{3}\Pr[|X| < z] \geq -\frac{2}{3}J(z; a, S). \quad \square$$

Lemma 4.3 (Hoeffding). If $e^{-x^2/2} \leq c(a_1)$, then (L) holds at x .

Proof. $\Pr[|X| \geq x] \leq 2e^{-x^2/2}$ (Hoeffding), so $\Pr[|X| < x] \geq 1 - 2c(a_1) = 1 - H(a_1) \geq H(x) - H(a_1)$. $\square$

5 A finite certificate for a block of values of a_1

Proposition 5.1. Let $0 < \alpha < \beta \leq 1$. Suppose there are numbers $\alpha = y_0 < y_1 < \dots < y_M$, $S_0, \dots, S_{M-1} > 0$ and, if $y_M < x_e$, numbers $y_M = x_0 < x_1 < \dots < x_N = x_e$, $T_0, \dots, T_{N-1} > 0$, such that

$$e^{-x_e^2/2} \leq c(\alpha), \tag{E}$$

$$H(y_{j+1}) \leq H(\alpha) - \frac{2}{3}J(2y_j; \beta, S_j) \quad (0 \leq j < M), \tag{C}$$

$$\Phi(x_{i+1}) \leq \Phi(\alpha) - J(x_i; \beta, T_i) \quad (0 \leq i < N), \tag{P}$$

where in the case $y_M \geq x_e$ there is no grid (x_i) and $N = 0$. Then (L) holds for every Rademacher sum with $a_1 \in [\alpha, \beta]$, and hence so does Theorem 1.1.

Proof. Fix X with $a_1 \in [\alpha, \beta]$, so $a = \beta \geq a_1$ is admissible in Corollary 3.6 and Lemma 4.2; note $H(a_1) \geq H(\alpha)$ and $c(a_1) \geq c(\alpha)$. Let $x > a_1$.

Case $x \leq \min(y_M, x_e)$. Choose j with $x \in (y_j, y_{j+1}]$ (possible since $x > a_1 \geq \alpha = y_0$) and put $b = \max(a_1, y_j)$, so $a_1 \leq b < x$ and $0 < 2y_j \leq 2b$. By Lemma 4.2 with $z = 2y_j$, by (C), and since $x \leq y_{j+1}$ and $a_1 \geq \alpha$, $\Pr[|X| < x] \geq -\frac{2}{3}J(2y_j; \beta, S_j) \geq H(y_{j+1}) - H(\alpha) \geq H(x) - H(a_1)$.

Case $y_M < x \leq x_e$. Choose i with $x \in [x_i, x_{i+1}]$. Since $\Pr[X < \cdot]$ is nondecreasing and $\Phi(x) \leq \Phi(x_{i+1})$, Corollary 3.6 and (P) give

$$\Phi(x) - \Pr[X < x] \leq \Phi(x_{i+1}) - \Pr[X < x_i] \leq \Phi(x_{i+1}) - \frac{1}{2} + J(x_i; \beta, T_i) \leq c(\alpha) \leq c(a_1),$$

which is (L) at x (recall $\Pr[X < x] = \frac{1}{2} + \frac{1}{2}\Pr[|X| < x]$ and $\Phi(x) = \frac{1}{2} + \frac{1}{2}H(x)$).

Case $x \geq x_e$ (or $x \geq y_M \geq x_e$). By (E), $e^{-x^2/2} \leq e^{-x_e^2/2} \leq c(\alpha) \leq c(a_1)$; apply Lemma 4.3. $\square$

Remark 5.2. The three cases cover every $x > a_1$: if $y_M \geq x_e$ the first and third cases do; otherwise all three are needed. Only finitely many values of J enter, all with the single parameter $a = \beta$; this is what makes a finite certificate possible for a whole interval of values of a_1 .

6 The computer verification for $a_1 \in [0.05, 1]$

Proposition 6.1. *There is an explicit list of 362 blocks $[\alpha, \beta]$ covering $[0.05, 1]$, each with parameters satisfying (E), (C), (P) of Proposition 5.1. All these inequalities have been verified in ball arithmetic.*

By Lemma 2.1 and Propositions 5.1 and 6.1, Theorem 1.1 holds for every $a_1 \in [0.05, 1]$. Section 8 proves it for every $a_1 \in (0, 0.1]$; the two ranges cover all of $(0, 1]$.

6.1 How the inequalities are verified

All parameters $(\alpha, \beta, y_j, S_j, x_i, T_i, x_e)$ are IEEE double numbers, i.e. exact dyadic rationals. Every quantity is evaluated in the ball arithmetic library Arb/FLINT [6, 7] (through `python-flint` 0.9.0), which returns a ball guaranteed to contain the exact value; an inequality $u \leq v$ is accepted only if the upper end of the ball for u is at most the lower end of the ball for v . Φ is computed from the Arb error function.

Upper bounds for $J(x; a, T)$. The integrand is written region by region (the regions refer to $av = aTu$): in regions 1–2, $\max(kL, kU) = \frac{1}{2}[k(U+L) + |k|(U-L)]$ (valid since $U \geq L$); in region 3 it equals $|k|U$, and in region 4 it equals $|k|$. To avoid removable singularities, k is evaluated as

$$k = \frac{(1-u)w \operatorname{sinc}(wu)}{\pi \operatorname{sinc}(\pi u)} - \frac{\sin(Tux)}{\pi} \quad (0 \leq u \leq \tfrac{1}{2}), \quad k = \frac{\sin(wu)}{\pi \operatorname{sinc}(\pi(1-u))} - \frac{\sin(Tux)}{\pi} \quad (\tfrac{1}{2} \leq u \leq 1),$$

with $w = \pi - Tx$ and $\operatorname{sinc} z = \sin z/z$; both formulas are analytic on the respective intervals. The region boundaries $u_1 = \pi/(2aT)$, $u_2 = \theta/(aT)$, $u_3 = \pi/(aT)$ are enclosed in intervals $[l_m, h_m]$ with double endpoints (for θ we use an enclosure $\theta \in [\theta_-, \theta_+]$ obtained by bisection with certified signs of $e^{-y^2/2} + \cos y$). On $[0, l_1]$, $[h_1, l_2]$, $[h_2, l_3]$, $[h_3, 1]$ (intersected with $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$) the integral is enclosed by Arb's rigorous integrator `acb_calc_integrate` [7] (Petras' adaptive Gauss–Legendre scheme with error bounds from complex-analytic estimates). That integrator is rigorous provided the integrand reports non-holomorphic points: we use `acb_real_abs` for $|k|$ and `acb_log` for $\cos(av)^{1/a^2} = \exp(a^{-2} \log \cos av)$ and $|\cos av|^{1/a^2} = \exp(a^{-2} \log(-\cos av))$ with the analyticity

flag set; all other functions involved are entire or meromorphic (poles automatically produce non-finite balls). On the slivers $[l_m, h_m]$ we use the trivial envelope $L = -1$, $U = 1$ (always valid), so the integrand is $|k|$, bounded by $\sup_{[l_m, h_m]} |k| \cdot (h_m - l_m)$ evaluated in interval arithmetic. The sum of all pieces is a ball whose upper end is an upper bound for $J(x; a, T)$.

Choice of parameters. For a block $[\alpha, \beta]$ the parameters are produced greedily (in floating point; this part needs no rigour): y_{j+1} is the largest double with (C) certified, where S_j approximately minimises $J(2y_j; \beta, \cdot)$; the C -grid stops once $y_M \geq 1.75\alpha$ (or $y_M \geq x_e$); then x_{i+1} is the largest double with (P) certified, with T_i approximately minimising $J(x_i; \beta, \cdot)$; $x_e = \sqrt{2 \log(1/c(\alpha))}(1 + 10^{-9})$. The blocks are $\alpha_{k+1} = \alpha_k + \min(0.02, 0.6\alpha_k^3)$, $\alpha_0 = 0.05$, the last block ending at 1. Typically $y_M \approx 1.82\alpha$ for small α , and the minimal slack $\Phi(\alpha) - \Phi(x_{i+1}) - J(x_i; \beta, T_i)$ along the P -grid is about $0.13\alpha^3$ for α near 0.05 (attained near $x = 2\alpha$).

Results. All 362 blocks passed; in total 110 950 certified integrals (each one an upper bound for one value $J(\cdot; \beta, \cdot)$). The full certificate (all blocks with all grids and all S_j, T_i) is the file `rigorous/results/L_005_1_cert.jsonl.gz`; the summary is `rigorous/results/L_005_1_summary.txt`. The independent checker `rigorous/check_cert.py` re-verifies (E), (C), (P) for every stored block from the certificate alone, at a different working precision (96 bits instead of 64), and checks that the blocks cover $[0.05, 1]$; its output is “checked 362 blocks at prec=96: 0 failures; coverage of $[0.05, 1.0]$: COMPLETE” (`rigorous/results/L_005_1_check96.txt`).

range of α	blocks	$\beta - \alpha$	integrals per block	time per block (1 core)
$[0.05, 0.1)$	252	$7.5 \cdot 10^{-5} - 6.0 \cdot 10^{-4}$	181–625	3–13 s
$[0.1, 0.3)$	75	$6.1 \cdot 10^{-4} - 0.0146$	25–179	0.5–3 s
$[0.3, 1]$	35	0.0169–0.02	1–23	≤ 0.5 s

Reproduction.

```
cd rigorous
python3 run_range.py 0.05 1.0 0.6 0.02 8 L_005_1      # generate + verify (about 5 min, 8 cores)
python3 check_cert.py results/L_005_1_cert.jsonl.gz 0.05 1.0 96 8  # independent re-check
```

7 Sanity checks

- For random weight vectors ($n \leq 16$) and equal weights ($n \leq 151$), the exact value of $\Phi(x) - \Pr[X < x]$ never exceeded $B(x; a_1, T)$ (`numerics/break_bound.py`); for equal weights at lattice points and $T = \pi/a_1$ the gap is as small as $2 \cdot 10^{-6}$, showing the envelopes are essentially sharp.
- The certified enclosures of J (radius $\approx 10^{-12}$) agree with an independent floating-point composite Gauss–Legendre quadrature; the discrepancy shrinks under refinement of the floating-point rule (to 10^{-9} – 10^{-11} with 3000 subintervals for strongly oscillating cases, $x \approx 2$ – 3 , $T \approx 10$ – 50).

8 The range $a_1 \in (0, 0.1]$: an analytic proof

The computer verification of Section 6 becomes expensive as $a_1 \rightarrow 0$, because the margin in (P) near $x = 2a_1$ is of order a_1^3 , forcing blocks of width $O(a_1^3)$. We therefore treat small a_1 analytically.

Throughout this section $a := a_1 \in (0, 0.1]$ is fixed, $T := \pi/a$, $\xi := x/a$, and

$$F(x) := B(x; a, \frac{\pi}{a}) - c(a) = \Phi(x) - \frac{1}{2} + J(x; a, \frac{\pi}{a}) - c(a).$$

By Corollary 3.6 (with the admissible choice $a \geq a_1$, here $a = a_1$), $\Phi(x) - \Pr[X < x] \leq c(a) + F(x)$ for all $x \geq 0$; so, exactly as in the proof of Proposition 5.1 (case $y_M < x \leq x_e$), $F(x) \leq 0$ implies (L) at x for every $x > a$. The goal of this section is:

Theorem 8.1 ((L) for small a_1). *If $a_1 \in (0, 0.1]$ then (L) holds for every $x > a_1$; hence so does Theorem 1.1.*

Combined with Section 6 (which gives Theorem 1.1 for $a_1 \in [0.05, 1]$) this proves Theorem 1.1 for all $a_1 \in (0, 1]$, since $(0, 0.1] \cup [0.05, 1] = (0, 1]$.

The scheme mirrors Proposition 5.1 but with $\beta = a_1$ and no grids: a C -range $x \in (a, 1.8a]$ handled by local concentration, a P -range $x \in [1.8a, x_e]$ handled by $F \leq 0$, and a Hoeffding tail $x \geq x_e$, where

$$x_e = x_e(a) := \sqrt{2 \log(1/c(a))}.$$

The heart of the matter is an explicit upper bound F^+ for F on the P -range, and the proof that $F^+ \leq 0$ there.

8.1 Notation and the bound F^+

Write $\varphi_0 = \varphi(0) = 1/\sqrt{2\pi}$, so $\int_0^\infty \cos(sx)e^{-s^2/2}ds = \sqrt{\pi/2}e^{-x^2/2} = \pi\varphi(x)$. Define

$$D(x) = \int_0^\infty \sin(sx)e^{-s^2/2}ds, \quad M(x) = \int_0^\infty s \cos(sx)e^{-s^2/2}ds, \quad N(x) = \int_0^\infty s^2 \sin(sx)e^{-s^2/2}ds.$$

Integrating by parts, $M(x) = 1 - xD(x)$ and $N(x) = D(x) + xM(x)$; also $D(x) = \sqrt{2} \operatorname{daw}(x/\sqrt{2})$, where daw is Dawson's function, so $D > 0$ and D, M, N are entire and computable in ball arithmetic. We use the moments $m_j = \int_0^\infty s^j e^{-s^2/2}ds$: $m_1 = 1$, $m_3 = 2$, $m_5 = 8$, $m_7 = 48$, and $m_{2k} = \sqrt{\pi/2}(2k-1)!!$, so $m_2 = \sqrt{\pi/2}$, $m_4 = 3\sqrt{\pi/2}$, $m_6 = 15\sqrt{\pi/2}$, $m_8 = 105\sqrt{\pi/2}$; note $\sqrt{\pi/2} = \pi\varphi_0$.

Fix the constants

$$K_g = 0.0323, \quad C_\rho = 2.09, \quad \xi_0 = \frac{1}{2} - \frac{C_\rho}{\pi^2}, \quad c_R = 0.01415,$$

and set

$$D_1^+(\xi) = \frac{\varphi_0}{4}(\xi-1)a^3 + \frac{2}{3\pi^2}a^4 + 15\varphi_0 K_g(\xi-1)a^5 + \frac{48K_g}{\pi^2}a^6 + \left(\frac{\xi^2}{2} + \frac{\xi}{3}\right)\left(\frac{5}{4}\varphi_0 a^5 + 105\varphi_0 K_g a^7\right), \quad (2)$$

$$F^+(x) = a\varphi(x) - c(a) + \frac{a^2}{\pi^2}xD(x) + \frac{a^2}{3}x\varphi(x) - \frac{a^3}{3\pi^2}N(x) + D_1^+(\xi) + \frac{\varphi_0^2}{2}(\xi - \xi_0)^2 a^4 + \frac{2C_\rho}{\pi^4}(\xi-1)a^4 + c_R a^4. \quad (3)$$

8.2 Four elementary constants

Lemma 8.2. *Let $h(u) = (1-u)\cot(\pi u) + \frac{1}{\pi} - \frac{1}{\pi u} = (1-u)(\cot \pi u - \frac{1}{\pi u})$ and $q(z) = \cot z - \frac{1}{z} + \frac{z}{3}$. Then*

$$(i) \quad |h(u)| \leq \frac{\pi u}{3} \text{ for } u \in [0, 1];$$

$$(ii) \quad (1-u)|q(\pi u)| \leq \frac{\pi^3}{45} u^3 \text{ for } u \in (0, 1);$$

$$(iii) \quad g(y) := -\log \cos y - \frac{y^2}{2} \text{ satisfies } 0 \leq g(y) \leq \frac{y^4}{12} + K_g y^6 \text{ for } y \in [0, 1];$$

$$(iv) \quad \rho(t) := \frac{\pi t}{\sin \pi t} \text{ satisfies } \rho(t) - 1 \leq C_\rho t^2 \text{ for } t \in [0, 1 - \theta/\pi].$$

Proof. From the partial-fraction expansion $\pi \cot \pi u = \frac{1}{u} - 2u \sum_{n \geq 1} \frac{1}{n^2 - u^2}$ we get, for $u \in (0, 1)$,

$$\cot \pi u - \frac{1}{\pi u} = -\frac{2u}{\pi} \sum_{n \geq 1} \frac{1}{n^2 - u^2}, \quad q(\pi u) = -\frac{2u^3}{\pi} \sum_{n \geq 1} \frac{1}{n^2(n^2 - u^2)},$$

the second because $\frac{\pi u}{3} = \frac{2u}{\pi} \sum_{n \geq 1} n^{-2}$ (as $\sum n^{-2} = \pi^2/6$) and $\frac{1}{n^2} - \frac{1}{n^2 - u^2} = -\frac{u^2}{n^2(n^2 - u^2)}$. All summands are positive on $(0, 1)$. Put $t_n(u) = \frac{1-u}{n^2 - u^2}$; then $t'_n(u) = -\frac{n^2 - 1 + (1-u)^2}{(n^2 - u^2)^2} \leq 0$ on $[0, 1]$ (the numerator is ≥ 0 since $n \geq 1$), so each t_n is nonincreasing and $t_n(u) \leq t_n(0) = n^{-2}$. Hence

$$|h(u)| = \frac{2u}{\pi}(1-u) \sum_{n \geq 1} \frac{1}{n^2 - u^2} = \frac{2u}{\pi} \sum_{n \geq 1} t_n(u) \leq \frac{2u}{\pi} \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi u}{3},$$

which is (i); and $(1-u)|q(\pi u)| = \frac{2u^3}{\pi} \sum_{n \geq 1} \frac{t_n(u)}{n^2} \leq \frac{2u^3}{\pi} \sum_{n \geq 1} \frac{1}{n^4} = \frac{2u^3}{\pi} \cdot \frac{\pi^4}{90} = \frac{\pi^3}{45} u^3$, which is (ii).

For (iii), $g'(y) = \tan y - y$, and $\tan$ has nonnegative Taylor coefficients on $[0, \pi/2)$ [11, 4.3.67]; thus $g(y) = \int_0^y (\tan t - t) dt = \sum_{k \geq 2} c_k y^{2k}$ with $c_k \geq 0$ and $c_2 = \frac{1}{12}$, so $g \geq 0$ and $(g(y) - \frac{y^4}{12})/y^6 = \sum_{k \geq 3} c_k y^{2k-6}$ is nondecreasing on $(0, \pi/2)$. Hence for $y \in [0, 1]$, $g(y) - \frac{y^4}{12} \leq (g(1) - \frac{1}{12})y^6$, and $g(1) - \frac{1}{12} = -\log \cos 1 - \frac{1}{2} - \frac{1}{12} \leq K_g$ (Lemma 8.9(K1)). For (iv), $\frac{y}{\sin y} = y \csc y$ has nonnegative Taylor coefficients on $[0, \pi)$ [11, 4.3.68], so $(\frac{y}{\sin y} - 1)/y^2$ is nondecreasing; with $y = \pi t$, $(\rho(t) - 1)/t^2 = \pi^2(\frac{y}{\sin y} - 1)/y^2$ is nondecreasing in t and hence on $[0, 1 - \theta/\pi]$ is at most its value at $t = 1 - \theta/\pi$ (i.e. $y = \pi - \theta$), which is $\leq C_\rho$ (Lemma 8.9(K2)). $\square$

8.3 The decomposition of J and the closed form of the Gaussian part

With $T = \pi/a$ we have $aTu = \pi u$, so for $v = Tu$ the envelope regions of Lemma 3.3 are $u \in [0, \frac{1}{2}]$ (region 1), $(\frac{1}{2}, \frac{\theta}{\pi}]$ (region 2), $(\frac{\theta}{\pi}, 1]$ (region 3); region 4 is empty. Using $\max(kL, kU) = kU + k^-(U - L)$ with $k^- = \max(-k, 0)$ in regions 1–2, and $\max(kL, kU) = |k|U = |k| |\cos \pi u|^{1/a^2}$ in region 3, and writing $E(x) = \int_0^1 k(u, x, T) e^{-v^2/2} du$,

$$J(x; a, \frac{\pi}{a}) = E(x) + D_1 + D_2 + A_1 + A_2, \tag{4}$$

where (all integrands use $v = \pi u/a$)

$$D_1 = \int_0^{1/2} k^- W_1 du, \quad D_2 = \int_{1/2}^{\theta/\pi} k^- W_2 du, \quad A_1 = \int_{\theta/\pi}^1 |k| |\cos \pi u|^{1/a^2} du, \quad A_2 = -\int_{\theta/\pi}^1 k e^{-v^2/2} du,$$

with $W_1 = e^{-v^2/2} - \cos(\pi u)^{1/a^2} \geq 0$ (region 1) and $W_2 = e^{-v^2/2} + |\cos \pi u| e^{-(1-a^2)v^2/2} \geq 0$ (region 2). Thus $F(x) = [\Phi(x) - \frac{1}{2} + E(x)] + D_1 + D_2 + A_1 + A_2 - c(a)$.

Lemma 8.3 (closed form of E). *For $a \in (0, 0.1]$ and $x \geq 0$,*

$$\Phi(x) - \frac{1}{2} + E(x) = a\varphi(x) - \frac{a^2}{\pi^2} M(x) + \frac{a^2}{3} x\varphi(x) - \frac{a^3}{3\pi^2} N(x) - R_q + \varepsilon_1,$$

with $|R_q| \leq \frac{2a^4}{45\pi}$ and $|\varepsilon_1| \leq 5e^{-\pi^2/(2a^2)}$.

Proof. Substitute $s = Tu = \pi u/a$ (so $v = s$, $du = \frac{a}{\pi}ds$, and $u \in [0, 1] \mapsto s \in [0, \pi/a]$). Using $\pi u = as$ and the identity $(1 - u) \cot(\pi u) + \frac{1}{\pi} = \frac{1}{as} + h(u)$ (which is the definition of h),

$$k(u, x, T) = \left(1 - \frac{as}{\pi}\right) \cos(sx) - \left(\frac{1}{as} + h(u)\right) \sin(sx), \quad E(x) = \frac{a}{\pi} \int_0^{\pi/a} k e^{-s^2/2} ds.$$

Every integrand below *except* the R_q -term is a polynomial of degree ≤ 2 in s times one of $\cos(sx), \sin(sx), \text{sinc}(sx)$ times $e^{-s^2/2}$ — hence entire in s — so replacing its integral over $[0, \frac{\pi}{a}]$ by the integral over $[0, \infty)$ changes E by a total of $-\varepsilon_1$ with $|\varepsilon_1| \leq 5e^{-\pi^2/(2a^2)}$ (a crude bound on $\frac{a}{\pi} \int_{\pi/a}^{\infty} (1 + s + s^2) e^{-s^2/2} ds$). The remaining R_q -term is kept on $[0, \frac{\pi}{a}]$, where its integrand is regular — the pole of $\cot$ at $s = \frac{\pi}{a}$ being cancelled by the factor $1 - \frac{as}{\pi}$ — and Lemma 8.2(ii) applies there (below). On $[0, \infty)$:

$$\frac{a}{\pi} \int_0^{\infty} \cos(sx) e^{-s^2/2} ds = a\varphi(x), \quad -\frac{a^2}{\pi^2} \int_0^{\infty} s \cos(sx) e^{-s^2/2} ds = -\frac{a^2}{\pi^2} M(x), \quad -\frac{a}{\pi} \cdot \frac{1}{a} \int_0^{\infty} \frac{\sin(sx)}{s} e^{-s^2/2} ds = -(\Phi(x) -$$

the last because $\frac{d}{dx} \int_0^{\infty} \frac{\sin sx}{s} e^{-s^2/2} ds = \int_0^{\infty} \cos(sx) e^{-s^2/2} ds = \pi\varphi(x)$ and the integral vanishes at $x = 0$. For the h -term, write $h(u) = (1 - \frac{as}{\pi})(\cot(as) - \frac{1}{as}) = -\frac{as}{3} + \frac{a^2 s^2}{3\pi} + (1 - \frac{as}{\pi})q(as)$. Since $\int_0^{\infty} s \sin(sx) e^{-s^2/2} ds = \pi x\varphi(x)$ and $\int_0^{\infty} s^2 \sin(sx) e^{-s^2/2} ds = N(x)$,

$$-\frac{a}{\pi} \int_0^{\infty} h(u) \sin(sx) e^{-s^2/2} ds = \frac{a^2}{3} x\varphi(x) - \frac{a^3}{3\pi^2} N(x) - R_q, \quad R_q = \frac{a}{\pi} \int_0^{\pi/a} (1 - \frac{as}{\pi}) q(as) \sin(sx) e^{-s^2/2} ds$$

(the first two, polynomial, parts of $-h$ are entire and were extended to $[0, \infty)$, contributing to ε_1 ; the q -part is the one kept on $[0, \frac{\pi}{a}]$). By Lemma 8.2(ii) with $u = as/\pi \in (0, 1)$ on $[0, \frac{\pi}{a}]$, $(1 - \frac{as}{\pi})|q(as)| \leq \frac{\pi^3}{45} (\frac{as}{\pi})^3 = \frac{a^3 s^3}{45}$, so $|R_q| \leq \frac{a}{\pi} \cdot \frac{a^3}{45} \int_0^{\pi/a} s^3 e^{-s^2/2} ds \leq \frac{a}{\pi} \cdot \frac{a^3}{45} \int_0^{\infty} s^3 e^{-s^2/2} ds = \frac{a^4}{45\pi} m_3 = \frac{2a^4}{45\pi}$. Summing the four contributions and cancelling $-(\Phi(x) - \frac{1}{2})$ against $\Phi(x) - \frac{1}{2}$ gives the claim. $\square$

8.4 The envelope-loss and aliasing bounds

Lemma 8.4 (envelope loss). *For $a \in (0, 0.1]$ and $1 \leq \xi \leq x_e(a)/a$ (i.e. $a \leq x \leq x_e(a)$), $D_1 \leq D_1^+(\xi) + \varepsilon_2$ with $0 \leq \varepsilon_2 \leq 10^{-10}a^4$.*

Proof. On region 1, $v = \pi u/a$ and $\cos(\pi u)^{1/a^2} = e^{-v^2/2} e^{-g(as)/a^2}$ with $s = \pi u/a$ and g as in Lemma 8.2(iii) (valid since $\pi u \leq \pi/2 \leq \theta$, so $\cos \pi u \in [0, 1]$). Hence by $1 - e^{-z} \leq z$,

$$W_1 = e^{-s^2/2} (1 - e^{-g(as)/a^2}) \leq e^{-s^2/2} \frac{g(as)}{a^2} \leq e^{-s^2/2} \left(\frac{a^2 s^4}{12} + K_g a^4 s^6 \right) \quad (s \leq 1/a),$$

using $g(y) \leq \frac{y^4}{12} + K_g y^6$ with $y = as \leq 1$. For $k^- = \max(-k, 0)$: from $k = (1 - u) \cos(sx) - (\frac{1}{as} + h(u)) \sin(sx)$ and $\frac{\sin(sx)}{as} = \xi \text{sinc}(sx)$ (as $x = \xi a$),

$$-k = \xi \text{sinc}(sx) - (1 - u) \cos(sx) + h(u) \sin(sx) \leq \xi + \left(-1 + u + \frac{(sx)^2}{2} \right) + \frac{\pi u}{3} sx,$$

using $\text{sinc} \leq 1$, $-(1 - u) \cos(sx) \leq -(1 - u) + (1 - \cos sx) \leq -1 + u + \frac{(sx)^2}{2}$, $h(u) \sin(sx) \leq |h(u)| sx \leq \frac{\pi u}{3} sx$ (Lemma 8.2(i)). With $u = as/\pi$ and $sx = \xi as$ this is

$$k^- \leq (-k)^+ \leq \kappa(s) := (\xi - 1) + \frac{as}{\pi} + a^2 s^2 \left(\frac{\xi^2}{2} + \frac{\xi}{3} \right).$$

On $s \in [1/a, \pi/(2a)]$ we have $0 \leq W_1 \leq e^{-s^2/2}$ ($\cos(\pi u)^{1/a^2} \geq 0$ as $u \leq \frac{1}{2}$), and we bound $\int k^- W_1 du$ there by $\varepsilon_2 \leq \frac{a}{\pi} \int_{1/a}^{\infty} \kappa(s) e^{-s^2/2} ds$. In $\kappa(s)$ the coefficient $a^2 \xi^2$ satisfies $a^2 \xi^2 = x^2 \leq x_e^2 = 2 \log \frac{1}{c(a)}$, so $\kappa(s) \leq \frac{x_e}{a} + \frac{as}{\pi} + 2s^2 \log \frac{1}{c(a)}$, a fixed polynomial in s with prefactors that are at most $\text{poly}(1/a)$ (as $x_e/a, \log \frac{1}{c(a)} = O(a^{-1})$). Since $\int_{1/a}^{\infty} s^j e^{-s^2/2} ds \leq \frac{2}{a} e^{-1/(2a^2)}$ and, by Lemma 8.9(X2), $e^{-1/(2a^2)} \leq 2 \cdot 10^{-12} a^{10}$ for $a \leq 0.1$, the a^{10} absorbs every such prefactor and leaves $\varepsilon_2 \leq 10^{-10} a^4$. On $s \in [0, 1/a]$, extend to $[0, \infty)$ (integrand ≥ 0) and integrate $\frac{a}{\pi} \kappa(s) e^{-s^2/2} (\frac{a^2 s^4}{12} + K_g a^4 s^6)$ term by term with the moments $m_4, \dots, m_8$; using $\frac{a}{\pi} \cdot \frac{a^2}{12} m_4 = \frac{\varphi_0}{4} a^3$, $\frac{a}{\pi} \cdot a^4 K_g m_6 = 15 \varphi_0 K_g a^5$, $\frac{a}{\pi} \cdot \frac{a}{\pi} \cdot \frac{a^2}{12} m_5 = \frac{2}{3\pi^2} a^4$, $\frac{a}{\pi} \cdot \frac{a}{\pi} a^4 K_g m_7 = \frac{48 K_g}{\pi^2} a^6$, $\frac{a}{\pi} \cdot \frac{a^4}{12} m_6 = \frac{5}{4} \varphi_0 a^5$, and $\frac{a}{\pi} \cdot a^6 K_g m_8 = 105 \varphi_0 K_g a^7$, one gets exactly $D_1^+(\xi)$ of (2). $\square$

Lemma 8.5 (aliasing). *For $a \in (0, 0.1]$ and $\xi \geq 1$,*

$$A_1 \leq \frac{a^2}{\pi^2} |\cos \pi \xi| + \frac{\varphi_0}{\pi} (\xi - \xi_0) |\sin \pi \xi| a^3 + \frac{2C_\rho}{\pi^4} (\xi - 1) a^4.$$

Proof. Put $t = 1 - u \in [0, 1 - \theta/\pi]$ and $\beta = \pi \xi = Tx$. Then $|\cos \pi u|^{1/a^2} = (\cos \pi t)^{1/a^2} \leq e^{-\pi^2 t^2/(2a^2)}$, because $\pi t \leq \pi - \theta < \theta$ and $\cos y \leq e^{-y^2/2}$ on $[0, \theta]$ (Lemma 3.3). From $k(u, x, T) = \frac{(1-u) \sin(\pi u - Tux)}{\sin \pi u} - \frac{\sin Tux}{\pi}$ (Corollary 3.2), with $\rho(t) = \pi t / \sin \pi t$, $P = (\pi - \beta)(1 - t)$ and $Q = \beta(1 - t)$,

$$k(1 - t) = \frac{1}{\pi} [\rho(t) \sin P - \sin Q] = \frac{1}{\pi} \left[(\rho(t) - 1) \sin P + 2 \sin \frac{\pi t}{2} \sin \frac{(\pi - 2\beta)(1 - t)}{2} \right].$$

Now $|\sin P| \leq |\sin \beta| + \pi(\xi - 1) |\cos \beta| t$ and $|\sin \frac{(\pi - 2\beta)(1 - t)}{2}| \leq |\cos \beta| + \frac{\pi(2\xi - 1)}{2} |\sin \beta| t$ (expand $P = (\pi - \beta) - (\pi - \beta)t$ and $\frac{(\pi - 2\beta)(1 - t)}{2} = (\frac{\pi}{2} - \beta) - \frac{(\pi - 2\beta)t}{2}$, use $\sin(\pi - \beta) = \sin \beta$, $\cos(\pi - \beta) = -\cos \beta$, $|\sin \theta| \leq |\theta|$, and $\xi \geq 1$). With $\rho(t) - 1 \leq C_\rho t^2$ (Lemma 8.2(iv)) and $\sin \frac{\pi t}{2} \leq \frac{\pi t}{2}$,

$$|k(1 - t)| \leq t |\cos \beta| + \left(\frac{C_\rho}{\pi} + \frac{\pi(2\xi - 1)}{2} \right) |\sin \beta| t^2 + C_\rho (\xi - 1) |\cos \beta| t^3.$$

Bounding $A_1 \leq \int_0^\infty |k(1 - t)| e^{-\pi^2 t^2/(2a^2)} dt$ and substituting $t = as/\pi$ (so $\int_0^\infty t^j e^{-\pi^2 t^2/(2a^2)} dt = (a/\pi)^{j+1} m_j$) gives, with $m_1 = 1$, $m_2 = \sqrt{\pi/2} = \pi \varphi_0$, $m_3 = 2$,

$$A_1 \leq \frac{a^2}{\pi^2} |\cos \beta| + \left(\frac{C_\rho}{\pi} + \frac{\pi(2\xi - 1)}{2} \right) \frac{\varphi_0 a^3}{\pi^2} |\sin \beta| + \frac{2C_\rho}{\pi^4} (\xi - 1) |\cos \beta| a^4.$$

Finally $(\frac{C_\rho}{\pi} + \frac{\pi(2\xi - 1)}{2}) \frac{\varphi_0 a^3}{\pi^2} = \frac{\varphi_0 a^3}{\pi} (\xi - \frac{1}{2} + \frac{C_\rho}{\pi^2}) = \frac{\varphi_0}{\pi} (\xi - \xi_0) a^3$, and $|\cos \beta| \leq 1$ in the last term. $\square$

Lemma 8.6 (oscillation). *For $Q > 0$, $K \geq 0$ and $\beta \in \mathbb{R}$, $-Q(1 - |\cos \beta|) + K|\sin \beta| \leq \frac{K^2}{2Q}$.*

Proof. Let $c = |\cos \beta| \in [0, 1]$, $w = \sqrt{1 - c}$. Then $|\sin \beta| = \sqrt{(1 - c)(1 + c)} \leq \sqrt{2} w$, so the left side is $\leq -Qw^2 + \sqrt{2} Kw \leq \max_{w \geq 0} (-Qw^2 + \sqrt{2} Kw) = K^2/(2Q)$. $\square$

8.5 $F \leq F^+$ and $F^+ \leq 0$

Proposition 8.7. *For $a \in (0, 0.1]$ and every x with $a \leq x \leq x_e(a)$ (i.e. $1 \leq \xi \leq x_e(a)/a$), $F(x) \leq F^+(x)$.*

Proof. Combine $F = [\Phi - \frac{1}{2} + E] + D_1 + D_2 + A_1 + A_2 - c(a)$ with Lemmas 8.3, 8.4, 8.5. The terms $\frac{a^2}{\pi^2} |\cos \pi \xi|$ (from A_1) and $-\frac{a^2}{\pi^2} M(x)$ (from E) combine, using $M = 1 - xD$, into

$$-\frac{a^2}{\pi^2} M(x) + \frac{a^2}{\pi^2} |\cos \pi \xi| + \frac{\varphi_0}{\pi} (\xi - \xi_0) |\sin \pi \xi| a^3 = \frac{a^2}{\pi^2} xD(x) - \frac{a^2}{\pi^2} (1 - |\cos \pi \xi|) + K |\sin \pi \xi|,$$

with $K = \frac{\varphi_0}{\pi} (\xi - \xi_0) a^3 \geq 0$ (as $\xi \geq 1 > \xi_0$). By Lemma 8.6 with $Q = a^2/\pi^2$ and $\beta = \pi \xi$, the last two terms are $\leq \frac{K^2}{2Q} = \frac{\varphi_0^2}{2} (\xi - \xi_0)^2 a^4$. The remaining pieces of A_1 ($\frac{2C_p}{\pi^4} (\xi - 1) a^4$), of E ($a\varphi(x) + \frac{a^2}{3} x\varphi(x) - \frac{a^3}{3\pi^2} N(x)$), of D_1 (D_1^+), and $-c(a)$ reproduce the named terms of F^+ in (3); and the leftover errors obey $|R_q| + |\varepsilon_1| + D_2 + |A_2| + \varepsilon_2 \leq c_R a^4$: indeed $D_2, |A_2|, |\varepsilon_1| \leq \text{poly}(1/a) e^{-c/a^2}$ with $c \geq (1-a^2)\pi^2/8 \geq \frac{1}{2}$ for $a \leq 0.1$ (the smallest exponent is that of D_2 , whose slowest term decays like $e^{-(1-a^2)v^2/2}$ with $v \geq \pi/(2a)$; for $A_2, c \geq \theta^2/2$; for $\varepsilon_1, c \geq \pi^2/2$), so (as in the bound on ε_2) the a^{10} of $e^{-1/(2a^2)} \leq 2 \cdot 10^{-12} a^{10}$ absorbs the $\text{poly}(1/a)$ and each is below $10^{-10} a^4$ by Lemma 8.9(X2), and with $\varepsilon_2 \leq 10^{-10} a^4$ and $|R_q| \leq \frac{2a^4}{45\pi}$ we have $|R_q| + |\varepsilon_1| + D_2 + |A_2| + \varepsilon_2 \leq (\frac{2}{45\pi} + 4 \cdot 10^{-10}) a^4 \leq c_R a^4$ (Lemma 8.9(K3)). $\square$

For the next proposition write $P(\xi, a) := F^+(x)/a^3$ (with $x = \xi a$) under the elementary replacements $x\varphi(x) \leq \varphi_0 x$, $xD(x) \leq x^2$ ($D(x) \leq x$ since $\sin sx \leq sx$), $N(x) \geq 0$ for $x \leq 1$ (as $N = x + D(1 - x^2) \geq x \geq 0$ there), and $a\varphi(x) - c(a) \leq \varphi_0 a^3 (\frac{1}{6} - \frac{\xi^2}{2}) + \varphi_0 \frac{\xi^4}{8} a^5$ (from $\varphi(x) \leq \varphi_0(1 - \frac{x^2}{2} + \frac{x^4}{8})$ and $c(a) \geq \varphi_0(a - \frac{a^3}{6})$). Collecting powers of a ,

$$P(\xi, a) = -\frac{\varphi_0}{12} (\xi - 1)(6\xi - 1) + \left[\frac{\varphi_0^2}{2} (\xi - \xi_0)^2 + \frac{2C_p}{\pi^4} (\xi - 1) + \frac{\xi^2}{\pi^2} + \frac{2}{3\pi^2} + c_R \right] a + (\dots) a^2 + \dots, \quad (5)$$

all coefficients of positive powers of a being ≥ 0 for $\xi \geq 1$; the exact expression is the function `Pfun` verified in `rigorous/small_a_check.py`. In particular $P(\xi, \cdot)$ is nondecreasing in a on $[0, 0.1]$ for every $\xi \geq 1$.

Proposition 8.8. *For $a \in (0, 0.1]$ and $1.8a \leq x \leq x_e(a)$, $F^+(x) \leq 0$.*

Proof. *Small- x range* $x \leq 1$ (so $\xi \geq 1.8$, $a \leq 1/\xi$). Here $F^+(x) \leq a^3 P(\xi, a)$ and, since $P(\xi, \cdot)$ is nondecreasing, $P(\xi, a) \leq P(\xi, a^*(\xi))$ with $a^*(\xi) = \min(0.1, 1/\xi)$. For $\xi \in [1.8, 10]$, $a^* = 0.1$ and $P(\xi, 0.1) \leq -0.15$ (Lemma 8.9(P1)); for $\xi \geq 10$, $a^* = 1/\xi$ and $\eta^2 P(1/\eta, \eta) \leq -0.1$ for $\eta = 1/\xi \in (0, 0.1]$ (Lemma 8.9(P2)), so $P(\xi, 1/\xi) < 0$. In both cases $F^+(x) \leq a^3 P(\xi, a) < 0$.

Large- x range $1 \leq x \leq x_e(a)$. Here $\varphi(x) \leq \varphi(1)$, and $c(a)/a \geq \varphi_0(1 - \frac{a^2}{6})$ from $c(a) \geq \varphi_0(a - \frac{a^3}{6})$, so $-c(a)/a \leq -\varphi_0 + \varphi_0 \frac{a^2}{6}$. (We must not use $c(a)/a \geq c(0.1)/0.1$: this gives only $F^+/a \leq \varphi_0 - c(0.1)/0.1$, which is > 0 since $c(a) < a\varphi_0$ always.) From (3), $F^+(x)/a = \varphi(x) - c(a)/a + \Sigma' \leq \varphi(1) - \varphi_0 + \Sigma$, where $\Sigma := \varphi_0 \frac{a^2}{6} + \Sigma'$ adds the correction $\varphi_0 a^2/6 \leq \varphi_0 \cdot 0.1^2/6$ to the sum Σ' of the remaining terms of (3) divided by a , each bounded using $x \leq x_e(a)$: the quantities $ax^2 \leq 2a \log(1/c(a)) \leq 2 \cdot 0.1 \log(1/c(0.1)) =: L$ (as $a \log \frac{1}{c(a)}$ is increasing on $(0, 0.1]$; see after Lemma 8.9), $ax \leq \sqrt{aL} \leq \sqrt{0.1L}$, $xD(x) \leq 2$, $x\varphi(x) \leq \varphi(1)$, $|N(x)| \leq \sqrt{\pi/2}$. Lemma 8.9(Q) certifies $\Sigma \leq \varphi_0 - \varphi(1)$ (in fact $\Sigma \leq 0.1247 < 0.1570 = \varphi_0 - \varphi(1)$), whence $F^+(x)/a \leq \varphi(1) - \varphi_0 + \Sigma \leq 0$. $\square$

8.6 Proof of Theorem 8.1

Proof. Fix $a = a_1 \in (0, 0.1]$ and $x > a$.

P-range $x \in [1.8a, x_e]$. By Proposition 8.7 and Proposition 8.8, $F(x) \leq F^+(x) \leq 0$, which gives (L) at x as noted at the start of the section.

Hoeffding tail $x \geq x_e$. Then $e^{-x^2/2} \leq e^{-x_e^2/2} = c(a)$, so Lemma 4.3 gives (L).

C-range $x \in (a, 1.8a]$. Apply Lemma 4.2 with $S = \pi/a$, $a = a_1$ and $b \uparrow x$: since $t \mapsto \Pr[|X| < t]$ is left-continuous and $J(z; a, S)$ is continuous in z , letting $b \uparrow x$ (so $z = 2b \uparrow 2x$) yields $\Pr[|X| < x] \geq \frac{1}{3} \Pr[|X| < 2x]$. Now $2x \in (2a, 3.6a] \subset [1.8a, x_e]$ (as $3.6a \leq 0.36 < x_e(a)$ for $a \leq 0.1$), so by the *P-range*, $F(2x) \leq 0$, i.e. $\Phi(2x) - \Pr[X < 2x] \leq c(a)$; hence $\Pr[|X| < 2x] = 2\Pr[X < 2x] - 1 \geq H(2x) - H(a) - 2F(2x) \geq H(2x) - H(a)$. Therefore

$$\Pr[|X| < x] \geq \frac{1}{3}(H(2x) - H(a)) \geq H(x) - H(a)$$

provided $H(2x) + 2H(a) - 3H(x) \geq 0$. Using $2\varphi_0(y - \frac{y^3}{6}) \leq H(y) \leq 2\varphi_0 y$,

$$H(2x) + 2H(a) - 3H(x) \geq 2\varphi_0\left(2a - x - \frac{4x^3 + a^3}{3}\right),$$

which at the worst point $x = 1.8a$ equals $2\varphi_0(0.2a - 8.109a^3) > 0$ for $a \leq 0.1$ (Lemma 8.9(H)); the bracket is decreasing in x , so it is > 0 on all of $(a, 1.8a]$. $\square$

8.7 The certified numerical facts

Lemma 8.9. *The following hold and are verified in ball arithmetic by `rigorous/small_a_check.py` (output “ALL CHECKS PASSED”), at working precision 128 bits, with $\theta \in [\theta_-, \theta_+]$ the enclosure of Section 6:*

$$(K1) \quad -\log \cos 1 - \frac{1}{2} - \frac{1}{12} \leq 0.0323; \quad (K2) \quad \pi^2\left(\frac{y_1}{\sin y_1} - 1\right)/y_1^2 \leq 2.09 \text{ for } y_1 = \pi - \theta_-; \quad (K3) \quad \frac{2}{45\pi} + 4 \cdot 10^{-10} \leq 0.01415.$$

$$(P1) \quad P(\xi, 0.1) \leq -0.15 \text{ for } \xi \in [1.8, 10] \text{ (checked on 8200 subintervals);}$$

$$(P2) \quad \eta^2 P(1/\eta, \eta) \leq -0.1 \text{ for } \eta \in [0, 0.1] \text{ (checked on 2000 subintervals);}$$

$$(Q) \quad \text{the term sum } \Sigma \text{ of Proposition 8.8 satisfies } \Sigma \leq \varphi_0 - \varphi(1) \text{ (0.1246 < 0.1570);} \quad (X1) \quad x_e(0.1) = \sqrt{2 \log(1/c(0.1))} = 2.539 \text{ and } c(0.1) = 0.03983;$$

$$(H) \quad 2a - x - \frac{4x^3 + a^3}{3} > 0 \text{ at } x = 1.8a, a = 0.1; \quad (X2) \quad e^{-1/(2a^2)}/a^{10} \leq 2 \cdot 10^{-12} \text{ at } a = 0.1.$$

(P1) and (P2) are evaluated on the stated subintervals with ξ (resp. η) enclosed by an interval; since $P(\xi, a)$ is a fixed polynomial in ξ, η, a with the explicit coefficients of (5), this is a finite ball computation. Two monotonicity facts used in the proofs are elementary and extend the single-point checks (X1), (X2) to the whole range $(0, 0.1]$: first, $a \log \frac{1}{c(a)}$ is increasing there, since $\frac{d}{da} a \log \frac{1}{c(a)} = \log \frac{1}{c(a)} - \frac{a\varphi(a)}{c(a)} > 0$ (because $\log \frac{1}{c(a)} \geq \log \frac{1}{c(0.1)} > 3 > 1 > \frac{a\varphi(a)}{c(a)}$, the last since $c(a) = \int_0^a \varphi > a\varphi(a)$), so $2a \log \frac{1}{c(a)} \leq 2 \cdot 0.1 \log \frac{1}{c(0.1)} = 0.6446$ and $x_e(a) \geq x_e(0.1) > 0.36$; second, $-\frac{1}{2a^2} - 10 \log a$ is increasing on $(0, 0.1]$, so by (X2) $e^{-1/(2a^2)} \leq 2 \cdot 10^{-12} a^{10}$ for all $a \in (0, 0.1]$. This completes the analytic proof for $a_1 \leq 0.1$.

9 A complement: the upper half at $x = 0$

The other half of the conjecture is (U): $\Pr[|X| \leq x] \leq H(x) + H(a_1)$ for $x \geq 0$. At $x = 0$ it holds in general:

Proposition 9.1. *For every Rademacher sum, $\Pr[X = 0] \leq H(a_1) = 2c(a_1)$.*

Proof. (a) Since X takes finitely many values, $\Pr[X = 0] = \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \varphi_X(t) dt \leq \limsup_R \frac{1}{2R} \int_{-R}^R \prod_i |\cos a_i t|$. With $p_i = a_i^{-2}$ (so $\sum_i 1/p_i = 1$), Hölder's inequality gives $\frac{1}{2R} \int_{-R}^R \prod_i |\cos a_i t| dt \leq \prod_i \left(\frac{1}{2R} \int_{-R}^R |\cos a_i t|^{p_i} dt \right)^{1/p_i}$, and $\frac{1}{2R} \int_{-R}^R |\cos a_i t|^{p_i} dt \rightarrow \frac{1}{\pi} I(p_i)$, $I(p) = \int_0^\pi |\cos s|^p ds$. Since I is nonincreasing and $p_i \geq p_1 = a_1^{-2}$, $\Pr[X = 0] \leq \frac{1}{\pi} I(a_1^{-2})$.

(b) $\frac{1}{\pi} I(p) = \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi} \Gamma(\frac{p}{2}+1)}$. By the inequality $\Gamma(y+1)/\Gamma(y+\frac{1}{2}) > \sqrt{y+\frac{1}{4}}$ for $y > 0$ (Watson; the case $s = \frac{1}{2}$ of Kershaw's lower bound [9]) with $y = p/2$, $\frac{1}{\pi} I(p) < \sqrt{2/\pi} (p + \frac{1}{2})^{-1/2}$. On the other hand $H(h) = \sqrt{2/\pi} \int_0^h e^{-t^2/2} dt \geq \sqrt{2/\pi} (h - h^3/6)$. With $h = p^{-1/2}$ and $u = 1/p \in (0, 1]$ it suffices that $(1 + \frac{u}{2})^{-1/2} \leq 1 - \frac{u}{6}$, i.e. $(1 - \frac{u}{6})^2 (1 + \frac{u}{2}) \geq 1$, i.e. $u(\frac{1}{6} - \frac{5u}{36} + \frac{u^2}{72}) \geq 0$; the bracket is decreasing on $[0, 1]$ with value $\frac{3}{72} > 0$ at $u = 1$. Hence $\frac{1}{\pi} I(a_1^{-2}) \leq H(a_1)$. $\square$

Part (b) of this argument was supplied by an independent audit; equality is approached by even equal weights, $X = n^{-1/2} \sum_{i \leq n} x_i$, n even, $n \rightarrow \infty$.

10 The upper half (U) for $x > 0$

The other half of the conjecture is (U): $\Pr[X \leq x] - \Phi(x) \leq c(a_1)$ for $x \geq 0$. Since, for $x \geq 0$, $\Pr[X \leq x] = \Pr[X < 0] + \Pr[0 \leq X \leq x] = \frac{1}{2}(1 - \Pr[X = 0]) + (\frac{1}{2} \Pr[X = 0] + \frac{1}{2} \Pr[|X| \leq x]) = \frac{1}{2} + \frac{1}{2} \Pr[|X| \leq x]$ (by symmetry), and $\Phi(x) - \frac{1}{2} = \frac{1}{2} H(x)$, (U) is *equivalent* to

$$\Pr[|X| \leq x] \leq H(x) + H(a_1) \quad (x \geq 0). \quad (U)$$

At $x = 0$ this is Proposition 9.1, valid for every weight vector. In this section we prove (U) for all $x \geq 0$ once $a_1 \in [\alpha_*, 1]$, where the admissible lower end α_* is as small as the computer verification of §10.4 reaches.

Near $x = 0$ the pointwise envelope of Lemma 3.3 does not suffice (at $x = 0$ the bound $2J(0; a_1, \pi/a_1)$ already overshoots $H(a_1)$ at third order in a_1). The remedy is a second, sharper bound of the Prawitz integral by a Hölder/covering argument whose supremum form is sharp at $x = 0$ (Lemma 10.3 recovers the local bound of Proposition 9.1) and which we use for small x ; the envelope bound takes over for larger x .

10.1 The Prawitz upper bound and the Hölder covering

Lemma 10.1. *For every Rademacher sum, every $x \geq 0$ and $T > 0$,*

$$\Pr[|X| \leq x] \leq 2 \int_0^1 k(u, -x, T) \varphi_X(Tu) du. \quad (U\text{-Praw})$$

Proof. Corollary 3.2 at the point $-x$ gives $\Pr[X < -x] \geq \frac{1}{2} - \int_0^1 k(u, -x, T) \varphi_X(Tu) du$. By symmetry $\Pr[X < -x] = \Pr[X > x] = 1 - \Pr[X \leq x]$, so $\Pr[X \leq x] \leq \frac{1}{2} + \int_0^1 k(u, -x, T) \varphi_X(Tu) du$; now multiply by 2 and subtract 1, using $\Pr[|X| \leq x] = 2\Pr[X \leq x] - 1$. $\square$

Write $k(u, -x, T) = (1 - u) \cos(Tux) + \rho(u) \sin(Tux)$ with $\rho(u) = (1 - u) \cot(\pi u) + \frac{1}{\pi}$.

Lemma 10.2. *$\rho(u) \geq 0$ for $u \in (0, 1)$. Consequently, if $Tx \leq \pi/2$ then $k(u, -x, T) \geq 0$ on $[0, 1]$.*

Proof. For $u \in (0, \frac{1}{2}]$, $\cot(\pi u) \geq 0$, so $\rho(u) > 0$. For $u \in (\frac{1}{2}, 1)$ put $w = 1 - u \in (0, \frac{1}{2})$; then $\cot(\pi u) = -\cot(\pi w)$, so $\rho = \frac{1}{\pi} - w \cot(\pi w)$. Let $g(w) = \frac{1}{\pi} - w \cot(\pi w)$; then $g(0^+) = 0$ and $g'(w) = -\cot(\pi w) + \pi w \csc^2(\pi w) = \csc^2(\pi w) [\pi w - \frac{1}{2} \sin(2\pi w)] > 0$ on $(0, \frac{1}{2})$ because $2\pi w > \sin(2\pi w)$. Hence $g > 0$, i.e. $\rho > 0$. The last claim: $Tux \leq Tx \leq \pi/2$ gives $\cos(Tux) \geq 0$ and $\sin(Tux) \geq 0$, and $1 - u, \rho(u) \geq 0$. $\square$

Lemma 10.3 (Hölder covering). *Let $x \geq 0$, $T > 0$ with $Tx \leq \pi/2$. With $h(a, u) = |\cos(aTu)|^{1/a^2}$ and $G(a; x, T) = \int_0^1 k(u, -x, T) h(a, u) du$,*

$$\Pr[|X| \leq x] \leq 2 \sup_{0 < a \leq a_1} G(a; x, T).$$

Proof. By Lemma 10.2, $k := k(u, -x, T) \geq 0$, and $\varphi_X(Tu) = \prod_i \cos(a_i Tu) \leq \prod_i |\cos(a_i Tu)|$, so (U-Praw) gives $\Pr[|X| \leq x] \leq 2 \int_0^1 k \prod_i |\cos(a_i Tu)| du$. With $p_i = a_i^{-2}$ (so $\sum_i p_i^{-1} = \sum_i a_i^2 = 1$), the generalized Hölder inequality for the nonnegative measure $k du$ gives

$$\int_0^1 k \prod_i |\cos(a_i Tu)| du \leq \prod_i \left(\int_0^1 k |\cos(a_i Tu)|^{p_i} du \right)^{1/p_i} = \prod_i G(a_i; x, T)^{a_i^2} \leq \left(\sup_{0 < a \leq a_1} G(a; x, T) \right)^{\sum_i a_i^2},$$

using $0 \leq G(a_i; x, T) \leq \sup_{a \leq a_1} G$ and $\sum_i a_i^2 = 1$. $\square$

Remark 10.4 (attribution). Applying Hölder's inequality with exponents $p_i = a_i^{-2}$ to a product $\prod_i |\cos(a_i Tu)|$ (equivalently to $\prod_i |\text{sinc}|$), so as to reduce a bound over all weight vectors to a one-parameter supremum over equal-weight profiles, is the classical device of Haagerup [4] (sharp Khintchine constants) and Ball [3] (cube slicing); see also Nazarov–Podkorytov [5]. What is new here is its combination with the Prawitz kernel, which requires $k \geq 0$ and hence the restriction $Tx \leq \pi/2$ underlying Lemma 10.3.

10.2 A computable bound for the supremum

Fix $\beta \in (0, 1]$ and $T = \pi/\beta$. Recall $\theta \in (1.7, 1.8)$ from Lemma 3.3, and set $a_c = \theta\beta/\pi$ and $\psi(s) = \log |\cos s|/s^2$ (so $h(a, u) = \exp((Tu)^2 \psi(aTu))$).

Lemma 10.5. *Let $0 \leq x$ with $Tx \leq \pi/2$ and $a_1 \leq \beta$. For any partition $a_c = e_0 < e_1 < \dots < e_K = \beta$,*

$$\sup_{0 < a \leq \beta} G(a; x, T) \leq \max \left(G^g, \max_{0 \leq j < K} P_j \right), \quad G^g = \int_0^1 k(u, -x, T) e^{-(Tu)^2/2} du,$$

$$P_j = \int_0^1 k(u, -x, T) \max(h(e_j, u), h(e_{j+1}, u)) du.$$

Proof. Throughout $k := k(u, -x, T) \geq 0$ (Lemma 10.2). Split $(0, \beta] = (0, a_c] \cup \bigcup_j [e_j, e_{j+1}]$.

$a \leq a_c$. Then $aTu \leq aT \leq \theta$ for all $u \in [0, 1]$ (since $T = \pi/\beta$ and $a \leq \theta\beta/\pi$), so by Lemma 3.3(iii) $|\cos(aTu)| \leq e^{-(aTu)^2/2}$, whence $h(a, u) = |\cos(aTu)|^{1/a^2} \leq e^{-(Tu)^2/2}$ and $G(a; x, T) \leq G^g$.

$a \in [e_j, e_{j+1}]$. Fix u and let $s = aTu \in (0, \pi]$ (as $a \leq \beta$, $T = \pi/\beta$, $u \leq 1$). By Lemma 3.3(i),(ii), ψ decreases on $(0, \pi/2)$ and increases on $(\pi/2, \pi)$; hence $a \mapsto h(a, u) = \exp((Tu)^2 \psi(aTu))$ decreases while $aTu < \pi/2$ and increases while $aTu > \pi/2$ (with $h = 0$ at $aTu = \pi/2$). So $a \mapsto h(a, u)$ has a single interior minimum, and $\sup_{a \in [e_j, e_{j+1}]} h(a, u) = \max(h(e_j, u), h(e_{j+1}, u))$. Integrating against $k \geq 0$, $G(a; x, T) \leq P_j$ for all $a \in [e_j, e_{j+1}]$. Taking the supremum over the pieces proves the claim. $\square$

Write $S(x; \beta) = \max(G^g, \max_j P_j)$ for the right-hand side (it depends on the chosen partition; the verification fixes one). Lemmas 10.3 and 10.5 give, for $a_1 \leq \beta$ and $Tx \leq \pi/2$,

$$\Pr[|X| \leq x] \leq 2 S(x; \beta). \tag{S}$$

At $x = 0$, $k(u, 0, T) = 1 - u$, and for the equal-weight profile $a = \beta$ the integrand $|\cos \pi u|^{1/\beta^2}$ is symmetric about $u = \frac{1}{2}$, so

$$2G(\beta; 0, T) = 2 \int_0^1 (1-u) |\cos \pi u|^{1/\beta^2} du = \int_0^1 |\cos \pi u|^{1/\beta^2} du = \frac{1}{\pi} I(\beta^{-2}), \quad I(p) := \int_0^\pi |\cos s|^p ds.$$

Hence $2 \sup_{0 < a \leq \beta} G(a; 0, T) \geq \frac{1}{\pi} I(\beta^{-2})$, the Hölder bound on $\Pr[X = 0]$ of Proposition 9.1; numerically the supremum is attained at $a = \beta$, so this holds with equality. The *computable* bound $2S(0; \beta)$ of (S) is *not* equal to $\frac{1}{\pi} I(\beta^{-2})$: replacing $\sup_a G$ by a finite piece cover makes $2S(0; \beta)$ exceed it by a positive overshoot, from $\approx 5 \cdot 10^{-6}$ at $\beta = 0.15$ to $\approx 6 \cdot 10^{-4}$ at $\beta = 1$ (these are for $K = 10$; the stages below 0.15 use $K = 16$, which cuts the $x = 0$ overshoot to $\approx 3\text{--}6 \cdot 10^{-7}$ for $\beta \in [0.07, 0.15]$). The certificate never relies on exactness at $x = 0$: it bounds $\Pr[|X| \leq x]$ there by the first zone-A cell (U-A), not by $S(0; \beta)$.

10.3 The envelope bound and the block certificate

Lemma 10.6. *For $a_1 \leq \beta$ and all $x \geq 0$, $T > 0$: $\Pr[|X| \leq x] \leq 2J(-x; \beta, T)$.*

Proof. By (U-Praw) and Lemma 3.3 with $a = \beta \geq a_1$, $k(u, -x, T)\varphi_X(Tu) \leq \max(k(u, -x, T)L_\beta(Tu), k(u, -x, T)U_\beta(Tu))$ pointwise, so $\Pr[|X| \leq x] \leq 2 \int_0^1 \max(kL_\beta, kU_\beta) du = 2J(-x; \beta, T)$ (the definition of J at the point $-x$). $\square$

Proposition 10.7. *Let $0 < \alpha < \beta \leq 1$, $T_A = \pi/\beta$, $x_A = \alpha/2$. Suppose there are a grid $0 = x_0^A < x_1^A < \dots < x_M^A = x_A$, an integer $N \geq 0$, and—when $N \geq 1$ —a grid $x_A = x_0^B < x_1^B < \dots < x_N^B = x_e$ with parameters $T_0, \dots, T_{N-1} > 0$; when $N = 0$ put $x_e := x_A$. Suppose*

$$H(x_e) \geq 1 - H(\alpha), \tag{U-C}$$

$$2S(x_{i+1}^A; \beta) \leq H(x_i^A) + H(\alpha) \quad (0 \leq i < M), \tag{U-A}$$

$$2J(-x_{i+1}^B; \beta, T_i) \leq H(x_i^B) + H(\alpha) \quad (0 \leq i < N), \tag{U-B}$$

where in (S) the partition for $S(\cdot; \beta)$ uses $T = T_A = \pi/\beta$. (When $N = 0$ the grid (x_i^B) and (U-B) are empty, and $x_e = x_A$; zones A and C alone cover all $x \geq 0$.) Then (U) holds for every Rademacher sum with $a_1 \in [\alpha, \beta]$.

Proof. Fix X with $a_1 \in [\alpha, \beta]$; then $H(a_1) \geq H(\alpha)$, $\beta \geq a_1$ is admissible, and $x \mapsto \Pr[|X| \leq x]$ is nondecreasing. For $x \in [0, x_A]$ we have $x \leq x_A = \alpha/2 \leq a_1/2$ and $T_A x = \pi x/\beta \leq \pi x_A/\beta = \pi\alpha/(2\beta) \leq \pi/2$, so (S) applies on the whole of zone A.

Zone A, $x \in [0, x_A]$. Pick i with $x \in [x_i^A, x_{i+1}^A]$. Then by (S) and (U-A), $\Pr[|X| \leq x] \leq \Pr[|X| \leq x_{i+1}^A] \leq 2S(x_{i+1}^A; \beta) \leq H(x_i^A) + H(\alpha) \leq H(x) + H(a_1)$.

Zone B, $x \in [x_A, x_e]$ (only when $N \geq 1$). Pick i with $x \in [x_i^B, x_{i+1}^B]$. By Lemma 10.6 and (U-B), $\Pr[|X| \leq x] \leq \Pr[|X| \leq x_{i+1}^B] \leq 2J(-x_{i+1}^B; \beta, T_i) \leq H(x_i^B) + H(\alpha) \leq H(x) + H(a_1)$. When $N = 0$ we have $x_e = x_A$ and this zone is empty.

Zone C, $x \geq x_e$. $\Pr[|X| \leq x] \leq 1 \leq H(x_e) + H(\alpha) \leq H(x) + H(a_1)$, using (U-C). (When $N = 0$, $x_e = x_A$, so zone A covers $[0, x_e]$ and zone C covers $[x_e, \infty)$.) $\square$

10.4 The computer verification

Proposition 10.8. *There is an explicit list of blocks $[\alpha, \beta]$ covering $[0.07, 1]$, each with grids and parameters satisfying (U-C), (U-A), (U-B) of Proposition 10.7. All these inequalities have been verified in ball arithmetic.*

The verification follows §6 verbatim for the quantities J , and adds the covering bound $S(x; \beta)$. For each block, $T_A = \pi/\beta$ and the partition for S consists of a_c —the largest double $\leq \theta\beta/\pi$, so $a_c \leq \theta\beta/\pi$ as Lemma 10.5 requires (θ enclosed as in §6)—followed by K nodes $e_j = \beta - (\beta - a_c)(\frac{K-j}{K})^2$, with $e_0 = a_c$ and $e_K = \beta$ set exactly and the interior nodes concentrated near β (this placement cuts the piece-cover overshoot by about an order of magnitude at fixed K , because the supremum sits at $a = \beta$ for small x). Each P_j and G^g is a sum of Arb integrals over the sub-intervals of $[0, 1]$ cut at $u = \frac{1}{2}$ (kernel form) and at the ball-enclosed zeros $z_b := \pi/(2e_{j+1}T_A) \leq z_a := \pi/(2e_jT_A)$ of $\cos(e_{j+1}T_A u)$, $\cos(e_jT_A u)$, with $h(e, u) = \exp(p \log |\cos(eT_A u)|)$ and the Hölder exponent $p = e^{-2}$ formed as an exact ball $1/\text{arb}(e)^2$ (no floating-point rounding of the exponent), analyticity flag on, correct cos sign per sub-interval. By the single-minimum property of $a \mapsto h(a, u)$ (proof of Lemma 10.5), $\max(h(e_j, \cdot), h(e_{j+1}, \cdot))$ equals the single smooth factor $h(e_j, \cdot)$ on $[0, z_b]$ (there $e_jT_A u, e_{j+1}T_A u < \pi/2$) and $h(e_{j+1}, \cdot)$ on $[z_a, 1]$; only on the narrow band $[z_b, z_a]$, where the two cosines straddle $\pi/2$, is the maximum taken via $\max(p, q) = \frac{1}{2}(p + q + |p - q|)$ (`acb_real_abs`). This single-factor split removes the spurious subdivision `acb_integral` performs near $u = 0$ and near the cusps; it was validated against both the all-`real_abs` evaluation and an independent high-resolution float quadrature (90 cases, β from 0.07 to 0.9: the fast integrand’s upper end never undershoots the true value and is no looser than the `real_abs` form). The thin zero-slivers, and – at small β – any sub-interval on which `acb_integral` returns a non-finite enclosure (there the Hölder exponents e_j^{-2} run from ≈ 204 at $e = \beta$ up to ≈ 637 at $e = a_c$ when $\beta \approx 0.07$, and from ≈ 156 at $e = \beta$ to ≈ 488 at $e = a_c$ when $\beta \approx 0.08$, making $|\cos|^{1/e^2}$ too stiff near a cosine zero for the integrator), are bounded rigorously by $\sup |k| \cdot \sup \max(h(e_j, \cdot), h(e_{j+1}, \cdot)) \cdot (\text{width})$, valid since $k \geq 0$ and $0 \leq h \leq 1$; where this fallback is used the band straddles $\pi/2$, so $\max h \sim (\cos \text{ near } \pi/2)^{1/e^2} \lesssim 10^{-60}$ and the over-estimate is utterly negligible. The parameters are chosen greedily in floating point (a fast non-rigorous proxy for S predicts each step; the step is then certified in ball arithmetic, with geometric back-off on the rare miss), and only then are (U-A),(U-B),(U-C) certified. $x_e = \sqrt{2} \text{erf}^{-1}(1 - H(\alpha))$ rounded up so that (U-C) holds.

Results. All 2705 blocks passed. The re-checker `rigorous/check_cert_U.py` re-verifies (U-C),(U-A),(U-B) for every stored block from the certificate alone at working precision 96 bits, with $T = \pi/\beta$ as a ball and the Hölder exponents e_j^{-2} as exact balls, and checks the cover of $[0.07, 1]$ by exact *rational* arithmetic (block endpoints compared as `Fractions`, no tolerance); production (at 64 bits, with a float T) is only a heuristic that *finds* the grids and is not itself the certificate. Its output is

```
checked 2705 blocks at prec=96 (fast): 0 failures; coverage of [0.07,1.0] over 2705
blocks: COMPLETE
min rigorous slack: zone A +8.295e-06, zone B +1.193e-05
```

the slack being $\min_i (H(x_i^A) + H(\alpha) - 2S(x_{i+1}^A; \beta))$ in zone A and the analogous zone-B quantity, over all 1,168,413 certified *cells* (728,137 in zone A, 440,276 in zone B; each zone-A cell is itself a sum of $K + 1$ Arb integrals and each zone-B cell a sum of region integrals, so the number of integral evaluations is several times larger). In units of 10^{-6} , the by-stage minimum (zone A, zone B) margins are (8.30, 11.96) on $[0.07, 0.08]$ (521 blocks), (10.00, 11.97) on $[0.08, 0.10]$ (611), (9.69, 11.96) on $[0.10, 0.15]$ (1119) and (16.22, 11.96) on $[0.15, 1]$ (454), so the whole-range minimum sits in the bottom stage. The certificate—every block with both grids, the T_i , x_e and the piece count K —is `rigorous/results/U.007_1.cert.jsonl.gz`. The cover is audited as *exact rationals* by `rigorous/exact_coverage_U.py`, which reads each stored endpoint as the exact dyadic value of its double and the range ends and stage joins as exact decimals, and confirms that the tiling of $[7/100, 1]$ is gapless (no tolerance) and that $a_1 = 1/10$ —the 100 equal weights, the

tightest vector on the whole range—together with every stage join $\{0.07, 0.08, 0.10, 0.11, 0.15, 1\}$ lies in a block certified at its left endpoint $\alpha \leq a_1$. A self-describing copy with the partition nodes $a_c = e_0 < \dots < e_K = \beta$ stored in each record is `U_007_1_cert_nodes.jsonl.gz` (`add_nodes_U.py`; `verify_nodes_U.py` re-derives the nodes, checks $a_c \leq \theta\beta/\pi$ in ball arithmetic, and confirms every load-bearing field is identical to the verified certificate, so the re-check and the independent verifications below transfer to it). This re-check shares its integrand code with production, so it is an independent *run* (stored data, higher precision), not independent code; the genuine independent checks are three. (i) The fast region-decomposition integrand of each P_j was validated against an independent high-resolution float quadrature of $\int k \max(h, \cdot)$ (`prawitz_U_fast.py`, 90 cases with β from 0.07 to 0.9: the fast upper end never undershoots the true value, with slack $\geq +1.0 \cdot 10^{-12}$). (ii) A random sample of 1000 individual zone-A cells was re-certified at 96 bits with the obviously-correct all-`real_abs` covering bound (`safe_cross_check_U.py`: 0 failures, min safe-method margin $+3.67 \cdot 10^{-6}$), which is no tighter than the fast bound on any cell. (iii) The entire certificate (all 2705 blocks, all 1,168,413 cells) was independently re-verified by a from-scratch implementation sharing no integrator, kernel, piece decomposition or parameter code with ours (its own Clenshaw–Curtis quadrature with Bernstein-ellipse error bounds and exact-ball exponents): 0 failures, with rigorous minimum margins $+8.295 \cdot 10^{-6}$ (zone A) and $+1.196 \cdot 10^{-5}$ (zone B), matching ours. This verifier is that of the independent referee, whose report additionally certifies the $[0.15, 1]$ subrange in detail (margins $+1.62 \cdot 10^{-5}/+1.20 \cdot 10^{-5}$, and no counterexample in an extensive brute-force and simulated-annealing search); we ran the same code on the full $[0.07, 1]$ certificate.

Reproduction. The certificate is built in a_1 -stages (the per-block buffer fraction grows as a_1 shrinks, since the local-concentration margin $H(\alpha) - \frac{1}{\pi}I(\alpha^{-2}) = \Theta(\alpha^3)$ shrinks), then concatenated:

```
cd rigorous
python3 run_range_U.py 0.15 1.0 results/U_015_1_cert.jsonl.gz 10 8 0.45
python3 run_range_U.py 0.11 0.15 results/U_011_015_cert.jsonl.gz 16 8 0.65
python3 run_range_U.py 0.10 0.11 results/U_010_011_cert.jsonl.gz 16 8 0.78 1e-3 800
python3 run_range_U.py 0.08 0.10 results/U_008_010_cert.jsonl.gz 16 8 0.45 1e-5 2500
python3 run_range_U.py 0.07 0.08 results/U_007_008_cert.jsonl.gz 16 8 0.45 1e-5 2500
python3 concat_certs.py results/U_007_1_raw_cert.jsonl.gz results/U_007_008_cert.jsonl.gz \
    results/U_008_010_cert.jsonl.gz results/U_010_011_cert.jsonl.gz results/U_011_015_cert.jsonl.gz
python3 fix_bottom_block.py results/U_007_1_raw_cert.jsonl.gz results/U_007_1_cert.jsonl.gz 1e-18
python3 check_cert_U.py results/U_007_1_cert.jsonl.gz 0.07 1.0 96 8 # independent re-check
```

The `fix_bottom_block` step lowers the first block’s left endpoint by one unit in the last place, from the double `fl(0.07)` nearest 0.07 (which exceeds 0.07 by $\approx 6.7 \cdot 10^{-18}$) to the double just below it, which is < 0.07 ; the block re-certifies with an essentially unchanged margin, and the cover of real $a_1 \in [0.07, 1]$ is then exact. (For the earlier $[0.15, 1]$ cover this was automatic, as `fl(0.15) < 0.15`.)

Theorem 10.9 (the upper half, and the full conjecture, on $[0.07, 1]$). *For every Rademacher sum with $a_1 \in [0.07, 1]$ and every $x \geq 0$, $\Pr[|X| \leq x] \leq H(x) + H(a_1)$; equivalently $\Pr[X \leq x] - \Phi(x) \leq c(a_1)$. Together with Theorem 1.1, this proves the full Keller–Klein conjecture for $a_1 \in [0.07, 1]$:*

$$|\Pr[X \leq x] - \Phi(x)| \leq c(a_1) \quad \text{for all } x \in \mathbb{R}.$$

Proof. The first statement is (U), which holds for $a_1 \in [0.07, 1]$ by Propositions 10.7 and 10.8; its equivalence to $\Pr[X \leq x] - \Phi(x) \leq c(a_1)$ ($x \geq 0$) is the reduction established at the start of this section (around (U)). For the full conjecture fix $a_1 \in [0.07, 1]$. For $x \geq 0$: Theorem 1.1 gives

$\Phi(x) - \Pr[X \leq x] \leq c(a_1)$ and (U) gives $\Pr[X \leq x] - \Phi(x) \leq c(a_1)$, so $|\Pr[X \leq x] - \Phi(x)| \leq c(a_1)$. For $x < 0$ put $y = -x > 0$; by symmetry $\Pr[X \leq x] = 1 - \Pr[X < y]$ and $\Phi(x) = 1 - \Phi(y)$, so $\Pr[X \leq x] - \Phi(x) = \Phi(y) - \Pr[X < y]$. We bound $\Phi(y) - \Pr[X < y]$ above and below. Above: letting $x' \uparrow y$ in Theorem 1.1 ($\Pr[X \leq x'] \geq \Phi(x') - c(a_1)$) gives $\Pr[X < y] \geq \Phi(y) - c(a_1)$, i.e. $\Phi(y) - \Pr[X < y] \leq c(a_1)$. Below: $\Pr[X < y] \leq \Pr[X \leq y] \leq \Phi(y) + c(a_1)$ by (U) (in the form $\Pr[X \leq y] - \Phi(y) \leq c(a_1)$), so $\Phi(y) - \Pr[X < y] \geq -c(a_1)$. Hence $|\Phi(y) - \Pr[X < y]| \leq c(a_1)$. $\square$

Remark 10.10 (the range $a_1 \geq u_{0.75}$ is elementary). For $a_1 \geq u_{0.75} := \Phi^{-1}(\frac{3}{4}) \approx 0.6745$, equivalently $H(a_1) \geq \frac{1}{2}$, the upper half (U) has a one-line proof for every weight vector. Write $Y = \sum_{i \geq 2} a_i x_i$, so $X = a_1 x_1 + Y$ with Y symmetric and independent of x_1 , and $\Pr[|X| \leq x] = \Pr[|a_1 + Y| \leq x]$. For $0 \leq x < a_1$, $|a_1 + Y| \leq x$ forces $Y \leq x - a_1 < 0$, so $\Pr[|X| \leq x] \leq \Pr[Y < 0] \leq \frac{1}{2} \leq H(a_1) \leq H(x) + H(a_1)$; for $x \geq a_1$, $H(x) + H(a_1) \geq 2H(a_1) \geq 1 \geq \Pr[|X| \leq x]$. (Ma–Ye–Han [10] prove (U) on this range under the additional hypothesis that the 2^n values of X be distinct; the argument above needs no such hypothesis.) Thus the content of Theorem 10.9 beyond this elementary regime is the range $a_1 \in [0.07, u_{0.75})$; on $[u_{0.75}, 1]$ the certificate serves as an independent cross-check.

11 What remains

Theorem 1.1 settles the reachable half (L) for all $a_1 \in (0, 1]$. The small- a_1 argument of Section 8 makes rigorous, and uniform in x , the heuristic margin $\frac{\varphi(0)}{12}(m-1)(6m-1)a_1^3 + O(a_1^4)$ at $x = ma_1$ ($T = \pi/a_1$): this is exactly the coefficient $-\frac{\varphi_0}{12}(\xi-1)(6\xi-1)$ of a^3 in $P(\xi, a)$ (eq. (5)), which drives Proposition 8.8.

The full conjecture needs, in addition, the upper half (U): $\Pr[|X| \leq x] \leq H(x) + H(a_1)$ for $x \geq 0$. Section 10 proves (U), and hence the full conjecture (Theorem 10.9), for all $a_1 \in [0.07, 1]$. The one obstruction we had to get past is that a *single* pointwise envelope for φ_X does not suffice near $x = 0$: already at $x = 0$ the pointwise-envelope upper bound $2J(0; a_1, \pi/a_1)$ overshoots $c(a_1)$ at third order (the extremal equal-weight vector cannot simultaneously saturate the Gaussian envelope near $v = 0$ and the lattice envelope near $v = \pi/a_1$). The remedy is not a pointwise bound but a *covering*: Hölder’s inequality (Lemma 10.3) replaces φ_X by $\sup_{a \leq a_1} \int k |\cos(aTu)|^{1/a^2} du$, whose supremum form is sharp at $x = 0$ (Lemma 10.3 recovers Proposition 9.1) and tight enough for small x ; the envelope takes over for larger x . For $a_1 \geq u_{0.75} \approx 0.6745$, (U) is in fact elementary (Remark 10.10): Ma, Ye and Han [10] prove (U) on this range under a distinct-values hypothesis, but no such hypothesis is needed. The new content of Theorem 10.9 is thus the range $a_1 \in [0.07, u_{0.75})$, on which we are aware of no prior proof of (U).

What remains is (U) for $a_1 \in (0, 0.07)$. The covering certificate of §10.4 extends there in principle: the per-block width is set by the local-concentration margin $H(\alpha) - \frac{1}{\pi}I(\alpha^{-2}) = \Theta(\alpha^3)$, so the block *density* grows like $1/\alpha^3$ and covering $[\alpha_0, 1]$ costs $\Theta(\alpha_0^{-2})$ blocks. Each fixed floor $\alpha_0 > 0$ is thus plausibly within reach of more computation (we do not prove this), but no finite certificate can cover the whole interval $(0, \alpha_0)$; reaching $a_1 \rightarrow 0$ needs a new idea, not merely more compute. A cleaner finish would be a small- a_1 analytic upper bound mirroring Section 8: the same third-order expansion that drives (L) there controls the overshoot of the covering bound, and making it rigorous and uniform in x would close (U) for all $a_1 \in (0, 1]$ and thus the entire conjecture.

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