A FULL SOLUTION OF THE KELLER–KLEIN CONJECTURE: A SIMPLIFIED PROOF AND A COMPLETE LEAN FORMALIZATION

PAATA IVANISVILI

Abstract. Let $X = \sum_{j=1}^{n} a_j \varepsilon_j$, where $\varepsilon_1, \dots, \varepsilon_n$ are independent uniform random signs and $\sum_j a_j^2 = 1$, and let Z be a standard Gaussian random variable. Keller and Klein conjectured that $|\mathbb{P}(X \le x) - \mathbb{P}(Z \le x)| \le \mathbb{P}(0 < Z < a_1)$ for every real x, where $a_1 = \max_j |a_j|$. We prove this, for $\mathbb{P}(X \le x)$ and for $\mathbb{P}(X < x)$, and we give a complete formalization of the proof in Lean 4 with Mathlib. The final theorem depends only on Lean's three standard axioms and uses no `sorry` and no `native_decide`; an independent replay of all its dependencies in a fresh kernel environment, with lean4checker, found no error.

The proof combines four ingredients: Prawitz's smoothing inequality at the frequency π/a_1, with explicit envelopes of the characteristic function; an induction on the number of terms that peels off the largest coefficient; an analytic treatment of the window $|x| \le a_1/2$, valid for every $a_1 < 0.675$ and resting on a Wallis-type inequality; and analytic arguments for small a_1. For the last ingredient we use D. D. Goyal's proof of the upper half for $a_1 \le 0.07$, and we give a new proof of the lower half for $a_1 \le 0.1$. What remains for the computer is a single integral inequality on a two-parameter region, for $0.07 \le a_1 \le 0.55$, together with three inequalities between elementary functions of two variables. These are checked by the Lean kernel itself, using fixed-point ball arithmetic and `decide +kernel`: 1703 boxes with 141 320 cell bounds, and 105 boxes for the elementary inequalities. For comparison, the earlier computer-assisted proof of the lower half and of the range $a_1 \ge 0.07$ reported about 1.17 million certified cells and 110 950 certified integrals, computed outside any proof assistant.

Contents

1. Introduction 2
1.1. Earlier work 3
1.2. What is new 3
1.3. Organization 4
2. Notation and the reduction to two one-sided bounds 5
3. Prawitz's inequality and the envelope integral 5
3.1. Prawitz's function 5
3.2. Envelopes of the characteristic function 6
3.3. The envelope integral 7
4. Peeling the largest coefficient 8
4.1. Hoeffding's bound 8
4.2. Peeling identities 8
4.3. The induction 8
5. The window near the origin 9
5.1. Reduction to a single profile 10
5.2. An affine majorant of the kernel 10
5.3. Interpolation in ξ 11
5.4. A Wallis inequality 12

Date: 9 November 2026.

5.5. The diagonal 12
5.6. The ray 13
6. Small coefficients 13
6.1. Goyal's decomposition 14
6.2. The lower half for $a \le \frac{1}{10}$ 16
6.3. The lower half near the diagonal 17
6.4. The upper half for $a \le 0.07$ 17
7. The master decomposition 19
8. Certified numerics in the Lean kernel 21
8.1. Ball arithmetic 21
8.2. Elementary functions 21
8.3. The cell bound 21
8.4. The box check 22
8.5. Covers 22
8.6. The peeling inequalities 23
8.7. Finding the certificates 23
9. The formalization 23
9.1. Structure 23
9.2. Trusted base 24
9.3. Building 24
9.4. Independent replay 24
10. Approaches that did not work 26
Appendix A. Elementary bounds for the kernel 27
Appendix B. The Lean project 28
Acknowledgments 32
Disclosure 32
References 32

1. Introduction

Let $\varepsilon_1, \dots, \varepsilon_n$ be independent random signs, $\mathbb{P}(\varepsilon_j = 1) = \mathbb{P}(\varepsilon_j = -1) = \frac{1}{2}$, let $a_1, \dots, a_n$ be real numbers with $\sum_j a_j^2 = 1$, and put

$$X = \sum_{j=1}^{n} a_j \varepsilon_j.$$

Let Z be a standard Gaussian random variable, $\Phi(x) = \mathbb{P}(Z \le x)$, $\varphi = \Phi'$, and $c(t) = \Phi(t) - \frac{1}{2} = \mathbb{P}(0 < Z < t)$ for $t \ge 0$. In their proof of Tomaszewski's conjecture, Keller and Klein [8, Section 9] asked whether, with $a_1 = \max_j |a_j|$,

$$\sup_{x \in \mathbb{R}} \left|\mathbb{P}(X \le x) - \Phi(x)\right| \le c(a_1). \tag{1.1}$$

Since $c(a_1) \le a_1/\sqrt{2\pi}$, this is a Berry–Esseen inequality for Rademacher sums with the constant $1/\sqrt{2\pi} \approx 0.399$ in front of $\max_j |a_j|$. It is sharp: if $a_j = n^{-1/2}$ for all j and n is odd, then $\mathbb{P}(X \le x) = \frac{1}{2}$ for $0 \le x < a_1$, so $\Phi(x) - \mathbb{P}(X \le x) \to c(a_1)$ as $x \uparrow a_1$ [6, Section 1].

Theorem 1.1. *Let $n \ge 1$, let $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_j a_j^2 = 1$, and let $a \ge \max_j |a_j|$. For every $x \in \mathbb{R}$,*

$$\left|\mathbb{P}(X \le x) - \Phi(x)\right| \le \Phi(a) - \tfrac{1}{2} \qquad \text{and} \qquad \left|\mathbb{P}(X < x) - \Phi(x)\right| \le \Phi(a) - \tfrac{1}{2}.$$

In particular (1.1) holds, and it also holds with $\mathbb{P}(X < x)$ in place of $\mathbb{P}(X \le x)$.

Theorem 1.1 is proved in Lean 4 [13] with Mathlib [12] as the theorem `KellerKlein.keller_klein`:

```
theorem keller_klein {n : ℕ} (coeff : Fin n → ℝ) (hvar : ∑ i, coeff i ^ 2 = 1) {a : ℝ}
    (hca : ∀ i, |coeff i| ≤ a) (x : ℝ) :
    |rademacherCDF coeff x - cdf (gaussianReal 0 1) x| ≤ cdf (gaussianReal 0 1) a - 1 / 2 ∧
      |rademacherStrictCDF coeff x - cdf (gaussianReal 0 1) x| ≤
        cdf (gaussianReal 0 1) a - 1 / 2
```

Here `cdf (gaussianReal 0 1)` is Mathlib's distribution function of the standard Gaussian law. The Rademacher side is defined by counting sign vectors:

```
abbrev BoolCube (n : ℕ) := Fin n → Bool
def rademacherSign (b : Bool) : ℝ := if b then 1 else -1
def rademacherSum {n : ℕ} (c : Fin n → ℝ) (ω : BoolCube n) : ℝ :=
  ∑ i, c i * rademacherSign (ω i)
def cubeProbability {n : ℕ} (p : BoolCube n → Prop) : ℝ := by
  classical
  exact ((Finset.univ.filter p).card : ℝ) / (2 : ℝ) ^ n
def rademacherCDF {n : ℕ} (c : Fin n → ℝ) (x : ℝ) : ℝ :=
  cubeProbability (fun ω => rademacherSum c ω ≤ x)
def rademacherStrictCDF {n : ℕ} (c : Fin n → ℝ) (x : ℝ) : ℝ :=
  cubeProbability (fun ω => rademacherSum c ω < x)
```

For $n = 0$ the hypothesis $\sum_i a_i^2 = 1$ fails, so the formal statement holds vacuously. The variant `keller_klein_max` takes $a = |a_{i_0}|$ for an index i_0 of maximal modulus. The command `#print axioms KellerKlein.keller_klein` reports `[propext, Classical.choice, Quot.sound]`.

1.1. **Earlier work.** Keller and Klein [8, Section 9] describe the half $\Phi(x) - \mathbb{P}(X \le x) \le c(a_1)$ for $x \ge 0$ as within reach of their methods. Their Proposition 4.4 gives it when $a_1 \in [0.2122, 0.22]$ [6, Section 1]. Ma, Ye and Han [11] settle the conjecture for $n \le 5$, and they prove the other half for $a_1 \ge \Phi^{-1}(\frac{3}{4}) \approx 0.6745$ when the 2^n values of X are distinct.

The present paper builds on two recent works.

- The computer-assisted manuscript [6] proves the lower half for all a_1 (Theorem 1.1 there) and the upper half, hence the full conjecture, for $a_1 \ge 0.07$ (Theorem 10.9 there). The lower half uses an analytic argument for $a_1 \le 0.1$, with finitely many numerical constants certified in ball arithmetic, and a certificate of 362 blocks and 110 950 certified integrals on $[0.05, 1]$. Near the diagonal, $x \le 1.8a_1$, it uses the segment comparison lemma of Keller and Klein [8, Lemma 3.7]. The upper half uses 2705 blocks and 1 168 413 certified cells on $[0.07, 1]$: 728 137 for the window near the origin and 440 276 elsewhere. The certificates were computed with the ball arithmetic library Arb [7]; neither part is formalized.
- D. D. Goyal [3] proves the upper half for $a_1 \le 0.07$ analytically and formalizes the proof completely in Lean 4. That formalization also derives the full conjecture from the results of [6], which enter as hypotheses.

1.2. **What is new.**

(a) *A simpler proof.* The conjecture is reduced to seven explicit inputs (Theorem 4.5): one numerical constant, the window near the origin, and five inequalities between explicit functions of two real variables. Each input is proved analytically or checked by the computer, on a much smaller region than in [6]. In particular:
   - The window $|x| \le a_1/2$ of the upper half is handled analytically for every $a_1 < 0.675$ (Section 5). In [6] it accounts for 728 137 of the certified cells.

- An induction on n that removes the largest coefficient (*peeling*, Section 4) settles both halves for large a_1 and the lower half near the diagonal. It replaces the segment comparison lemma.
- The lower half for $a_1 \le 0.1$ has a new analytic proof, based on Goyal's decomposition of the Prawitz integral (Section 6).
- The remaining computer-assisted inequality is organized around a *master decomposition* (Section 7). It removes the Gaussian main term in closed form and leaves an integrand without cancellation, of the size of the true margin.

(b) *A complete formalization.* Every step, including all numerical verification, is checked by the Lean kernel. The certificates are Boolean programs over natural numbers, evaluated by kernel reduction (`decide +kernel`). The trusted base is therefore the Lean kernel, together with its built-in arithmetic on natural numbers, and the three standard axioms. The compiler and native code are not involved.

range	earlier proofs [6, 3]	this paper
upper half, $a_1 \le 0.07$	analytic [3]	analytic, following [3] (Section 6.4)
upper half, $\lvert x\rvert \le a_1/2$, $0.07 \le a_1 < 0.675$	728 137 certified cells [6]	analytic
upper half, $\lvert x\rvert \ge a_1/2$, $0.07 \le a_1 \le 0.5$	certified cells (440 276 on $[0.07, 1]$) [6]	790 boxes, 65 736 cell bounds
upper half, $0.5 \le a_1 < 0.675$	included in the line above	peeling, 54 boxes
upper half, $a_1 \ge 0.675$	elementary [6, Remark 10.10]	elementary
lower half, $a_1 \le 0.1$, $x \ge 1.5a_1$	analytic, with certified constants [6]	analytic
lower half, $0.1 \le a_1 \le 0.55$, $x \ge 1.5a_1$	362 blocks, 110 950 integrals on $[0.05, 1]$ [6]	913 boxes, 75 584 cell bounds
lower half, near the diagonal	segment comparison [8, Lemma 3.7]	peeling: analytic for $a_1 \le 0.3$, 31 boxes above
lower half, $a_1 \ge 0.55$	included above	peeling, 20 boxes
formalization	upper half for $a_1 \le 0.07$ [3]	complete

TABLE 1. What is analytic and what is computer-assisted. The counts for [6] are those reported there; its integrals are enclosed by Arb's rigorous integrator. A cell bound here is one interval enclosure of an explicit integrand on a subinterval of $[0, 1]$, uniform over a box of parameters (Section 8). The units are not the same, but the 141 320 cell bounds used here are about one eighth of the certified cells reported in [6] for the upper half alone. All of the computation here runs inside the proof assistant.

1.3. **Organization.** Section 2 reduces Theorem 1.1 to two one-sided statements (U) and (L) about $|X|$. Section 3 recalls Prawitz's inequality and the envelopes of the characteristic function, and reduces both halves to one inequality $J(a, y) \le c(a)+c(y)$. Section 4 contains the peeling identities and the induction (Theorem 4.5). The analytic pieces are proved in Section 5 (the window near the origin) and Section 6 (small coefficients). Section 7 contains the master decomposition behind the computer-assisted pieces, and Section 8 the certified numerics. Section 9 describes the formalization; Table 4 there maps the results of this paper to Lean declarations. Section 10 lists approaches that did not work. The results taken from [3] are also proved here, along the lines of

[3]: Prawitz's inequality and the envelopes in Section 3, the decomposition of the Prawitz integral in Section 6, Goyal's theorem (Theorem 4.6) in Section 6.4, and elementary bounds for the kernel in Appendix A.

2. Notation and the reduction to two one-sided bounds

Throughout, $\varphi_0 = 1/\sqrt{2\pi}$, $g(t) = e^{-t^2/2}$ and $\varphi = \varphi_0 g$. We write

$$c(t) = \int_0^t \varphi = \Phi(t) - \tfrac{1}{2}, \qquad H(t) = 2c(t) \qquad (t \in \mathbb{R}),$$

so that c is odd and increasing, and $H(t) = \mathbb{P}(|Z| \le t)$ for $t \ge 0$. A coefficient vector (a_j) is *normalized* if $\sum_j a_j^2 = 1$, and a number a is a *level* for it if $a \ge \max_j |a_j|$. Then $X = \sum_j a_j \varepsilon_j$ is symmetric, $\mathbb{E}X^2 = 1$, and every level satisfies $a > 0$. In Lean, c, H, φ_0, g and φ are `c`, `H`, `phi0`, `gaussian` and `phi`, and $\mathbb{P}(|X| \le x)$ and $\mathbb{P}(|X| < x)$ are `rademacherConcentration` and `rademacherOpenConcentration`.

Definition 2.1. Let X be a normalized Rademacher sum and a a level. We say that

$(\mathrm{U})_a$ holds if $\mathbb{P}(|X| \le x) \le H(a) + H(x)$ for every $x \ge 0$, and
$(\mathrm{L})_a$ holds if $\mathbb{P}(|X| < x) \ge H(x) - H(a)$ for every $x > a$.

For $0 < x \le a$ the inequality in $(\mathrm{L})_a$ holds trivially, since its right-hand side is ≤ 0. Since H is increasing, $(\mathrm{U})_a$ implies $(\mathrm{U})_{a'}$ and $(\mathrm{L})_a$ implies $(\mathrm{L})_{a'}$ for every $a' \ge a$ (`upperHalf_mono`, `lowerHalf_mono`). It therefore suffices to prove both statements at the level $a = \max_j |a_j|$.

Proposition 2.2 (`cdf_bounds_of_halves`)**.** *If $(\mathrm{U})_a$ and $(\mathrm{L})_a$ hold for X, then $|\mathbb{P}(X \le x) - \Phi(x)| \le c(a)$ and $|\mathbb{P}(X < x) - \Phi(x)| \le c(a)$ for every $x \in \mathbb{R}$.*

Proof. By symmetry, $\mathbb{P}(X \le x) = \frac{1}{2}(1 + \mathbb{P}(|X| \le x))$ for $x \ge 0$ and $\mathbb{P}(X < x) = \frac{1}{2}(1 + \mathbb{P}(|X| < x))$ for $x > 0$, while $\Phi(x) = \frac{1}{2}(1 + H(x))$.

The case $x > 0$. By $(\mathrm{U})_a$, $\mathbb{P}(X \le x) - \Phi(x) = \frac{1}{2}(\mathbb{P}(|X| \le x) - H(x)) \le \frac{1}{2}H(a) = c(a)$. If $x > a$, then $(\mathrm{L})_a$ gives $\Phi(x) - \mathbb{P}(X < x) = \frac{1}{2}(H(x) - \mathbb{P}(|X| < x)) \le c(a)$. If $0 < x \le a$, the same difference is at most $\frac{1}{2}H(x) \le c(a)$. Since $\mathbb{P}(X < x) \le \mathbb{P}(X \le x)$, both $\mathbb{P}(X < x) - \Phi(x)$ and $\mathbb{P}(X \le x) - \Phi(x)$ lie in $[-c(a), c(a)]$.

The case $x = 0$. By $(\mathrm{U})_a$ at $x = 0$, $\mathbb{P}(X = 0) \le H(a)$. Hence $\mathbb{P}(X \le 0) - \frac{1}{2} = \frac{1}{2}\mathbb{P}(X = 0) \in [0, c(a)]$ and $\mathbb{P}(X < 0) - \frac{1}{2} = -\frac{1}{2}\mathbb{P}(X = 0) \in [-c(a), 0]$.

The case $x < 0$. By symmetry, $\mathbb{P}(X \le x) - \Phi(x) = \Phi(-x) - \mathbb{P}(X < -x)$ and $\mathbb{P}(X < x) - \Phi(x) = \Phi(-x) - \mathbb{P}(X \le -x)$, so this case follows from the case $-x > 0$. □

3. Prawitz's inequality and the envelope integral

3.1. Prawitz's function.

For $0 < u < 1$ let

$$\rho(u) = (1-u)\cot(\pi u) + \frac{1}{\pi}, \qquad \Psi(z) = \frac{1}{2} + \int_0^1 \left[(1-u)\cos(uz) + \rho(u)\sin(uz)\right] \mathrm{d}u \quad (z \in \mathbb{R}).$$

The integral converges, because $\rho(u) \sim 1/(\pi u)$ as $u \to 0$. Writing $\delta(y) = 1 - y\cot y$, one has $\rho(u) = \delta(\pi(1-u))/\pi \ge 0$, since $y \cot y \le 1$ on $(0, \pi)$.

Lemma 3.1 (Prawitz [15, (1b)]; see also [8, Theorem 4.5], [6, Theorem 3.1])**.** $\Psi(z) \ge \mathbf{1}_{\{z \ge 0\}}$ *for every $z \in \mathbb{R}$.*

Proof. We give the recurrence argument formalized in [3], which credits it to the public CoolRmal/BerryEsseen project [2]. Put $s(z) = 2\int_0^1 \rho(u)\sin(uz)\,\mathrm{d}u$ and $\operatorname{sinc} w = (\sin w)/w$. Since $\int_0^1 (1-u)\cos(wu)\,\mathrm{d}u = (1-\cos w)/w^2$ for $w \ne 0$, we have $2\int_0^1 (1-u)\cos(uz)\,\mathrm{d}u = \operatorname{sinc}^2(z/2)$, so

$$\Psi(z) = \tfrac{1}{2}\left(1 + s(z) + \operatorname{sinc}^2(z/2)\right).$$

The function s is odd, and $\Psi(0)=1$. Let $A(x)=s(2\pi x)$ and $S(x)=\sin^2(\pi x)/\pi^2$; note $S(x+1)=S(x)$.

(i) *The recurrence.* Let $x\ge 0$, $\alpha=2\pi x$ and $\beta=\alpha+2\pi$. Since $\sin\beta u-\sin\alpha u=2\sin(\pi u)\cos(\pi(2x+1)u)$ and $\rho(u)\sin(\pi u)=(1-u)\cos(\pi u)+\frac{1}{\pi}\sin(\pi u)$, two product-to-sum formulas give

$$A(x+1)-A(x)=2\int_0^1(1-u)(\cos\beta u+\cos\alpha u)\,\mathrm{d}u+\frac{2}{\pi}\int_0^1(\sin\beta u-\sin\alpha u)\,\mathrm{d}u.$$

At $x=0$ the right-hand side equals 1. For $x>0$, evaluating the integrals and using $\cos\beta=\cos\alpha$,

$$A(x+1)-A(x)=2(1-\cos\alpha)\Big(\frac{1}{\alpha^2}+\frac{1}{\beta^2}-\frac{2}{\alpha\beta}\Big)=\frac{S(x)}{x^2(x+1)^2}\ge 0.$$

(ii) Hence $A(1)=A(0)+1=1$, and since $S(n)=0$ for every integer n, $A(n)=1$ for every integer $n\ge 1$.

(iii) *The limit.* For $x>0$ and $y\ge 0$, $A(x+y)-A(y)=4\int_0^1\rho(u)\sin(\pi xu)\cos\big(2\pi(y+\frac{x}{2})u\big)\,\mathrm{d}u$. The function $\rho(u)\sin(\pi xu)$ is bounded by x on $(0,1)$, because $0\le\rho(u)\le 1/(\pi u)$ (Lemma A.1). By the Riemann–Lebesgue lemma, $A(x+n)-A(n)\to 0$, and so $A(x+n)\to 1$, as $n\to\infty$ through the integers.

(iv) Let $x>0$. By (i), $n\mapsto A(x+n)$ is nondecreasing, so $A(x)\le 1$ by (iii). Also by (i),

$$A(x+1)+\frac{S(x+1)}{(x+1)^2}=A(x)+S(x)\Big(\frac{1}{x^2(x+1)^2}+\frac{1}{(x+1)^2}\Big)\le A(x)+\frac{S(x)}{x^2},$$

because $\frac{1}{x^2}-\frac{1}{(x+1)^2}-\frac{1}{x^2(x+1)^2}=\frac{2x}{x^2(x+1)^2}\ge 0$. So $n\mapsto A(x+n)+S(x)/(x+n)^2$ is nonincreasing, and it tends to 1; hence $A(x)+S(x)/x^2\ge 1$.

(v) For $z>0$ let $x=z/(2\pi)$. Then $A(x)=s(z)$ and $S(x)/x^2=\operatorname{sinc}^2(z/2)$, so (iv) gives $s(z)+\operatorname{sinc}^2(z/2)\ge 1$, that is, $\Psi(z)\ge 1$. For $z<0$, (iv) gives $s(z)=-A(-z/(2\pi))\ge -1$, so $\Psi(z)\ge\frac{1}{2}\operatorname{sinc}^2(z/2)\ge 0$. □

In Lean this is `prawitz_indicator_majorant`, from [3], proved as above in `PrawitzRecurrence.lean`, `PrawitzOscillation.lean` (the Riemann–Lebesgue step), `PrawitzLimit.lean` and `PrawitzMajorant.lean`. The definition of s, the integrability of $\rho(u)\sin(uz)$ and the triangular integral are adapted from the CausalSmith library [16].

Lemma 3.2. *For $T>0$ and $y,z\in\mathbb{R}$,*

$$\Psi(T(y-z))+\Psi(T(y+z))-1=2\int_0^1\big[(1-u)\cos(Tuy)+\rho(u)\sin(Tuy)\big]\cos(Tuz)\,\mathrm{d}u. \tag{3.1}$$

Denote the right-hand side by $P_T(y,z)$. Then $\mathbf{1}_{\{|z|\le x\}}\le P_T(x,z)$ for $x\ge 0$, and $\mathbf{1}_{\{|z|<x\}}\ge -P_T(-x,z)$ for $x>0$.

Proof. The identity follows from $\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta$ and $\sin(\alpha-\beta)+\sin(\alpha+\beta)=2\sin\alpha\cos\beta$. For $x\ge 0$ we have $\mathbf{1}_{\{|z|\le x\}}=\mathbf{1}_{\{x-z\ge 0\}}+\mathbf{1}_{\{x+z\ge 0\}}-1$, and Lemma 3.1 gives the first inequality. For $x>0$ the events $\{z\ge x\}$ and $\{-z\ge x\}$ are disjoint, so $\mathbf{1}_{\{|z|<x\}}=1-\mathbf{1}_{\{z-x\ge 0\}}-\mathbf{1}_{\{-z-x\ge 0\}}\ge 1-\Psi(T(z-x))-\Psi(T(-z-x))=-P_T(-x,z)$. □

3.2. **Envelopes of the characteristic function.** The characteristic function of X is $\varphi_X(t)=\mathbb{E}\cos(tX)=\prod_j\cos(a_jt)$. On $(\pi/2,\pi)$ the function $|\cos y|=\sin(y-\frac{\pi}{2})$ increases from 0 to 1, while $e^{-y^2/2}$ decreases. So the equation $|\cos y|=e^{-y^2/2}$ has a unique root θ in $(\pi/2,\pi)$, and $1.7<\theta<1.8$ (`crossoverRoot_bounds`) because

$$|\cos 1.7|<\tfrac{1}{5}<e^{-3/2}<e^{-1.7^2/2}\qquad\text{and}\qquad|\cos 1.8|>\tfrac{1}{5}>e^{-1.62}=e^{-1.8^2/2}.$$

Indeed, $|\cos 1.7| = \sin(1.7 - \frac{\pi}{2}) \le 1.7 - \frac{\pi}{2} < 0.13$. With $w = 1.8 - \frac{\pi}{2} \in (0.229, 0.23)$, $|\cos 1.8| = \sin w \ge w - \frac{w^3}{6} > 0.22$. Also $e^{3/2} < 2.72 \cdot 1.65 < 5$, since $1.65^2 > 2.72 > e$, and $e^{1.62} > \sum_{k=0}^{5} \frac{1.62^k}{k!} > 5$. In Lean, θ is `crossoverRoot`, defined by choice from a proved existence statement, not by an axiom. For $a > 0$ and $t \ge 0$ put

$$L_a(t) = \begin{cases} \cos(at)^{1/a^2}, & at \le \pi/2, \\ -|\cos at|\, e^{-(1-a^2)t^2/2}, & \pi/2 < at \le \theta, \\ -|\cos at|^{1/a^2}, & \theta < at \le \pi, \\ -1, & at > \pi, \end{cases} \qquad U_a(t) = \begin{cases} e^{-t^2/2}, & at \le \theta, \\ |\cos at|^{1/a^2}, & \theta < at \le \pi, \\ 1, & at > \pi. \end{cases}$$

Lemma 3.3 ([6, Lemma 3.3]; `cosProduct_enclosed` in [3])**.** *If (a_j) is normalized and a is a level, then $L_a(t) \le \varphi_X(t) \le U_a(t)$ for all $t \ge 0$.*

Proof. We follow [3] (`Envelopes.lean`). Since cos is even, we may assume $0 \le a_j \le a$. Fix $t \ge 0$ and put $y_j = a_j t$, so that $0 \le y_j \le at$ and $\sum_j y_j^2 = t^2$. Let $F(y) = \log|\cos y|/y^2$ for $y \in (0, \pi] \setminus \{\frac{\pi}{2}\}$.

(i) On $(0, \frac{\pi}{2})$ we have $F'(y) = -k(y)/y^3$ with $k(y) = y \tan y + 2 \log\cos y$. Since $k(0) = 0$ and $k'(y) = (2y - \sin 2y)/(2\cos^2 y) \ge 0$, F is nonincreasing on $(0, \frac{\pi}{2})$. Since $(-\log\cos y)' = \tan y \ge y$, also $\log\cos y \le -\frac{y^2}{2}$, that is, $F \le -\frac{1}{2}$ on $(0, \frac{\pi}{2})$, and $\cos y \le e^{-y^2/2}$ on $[0, \frac{\pi}{2}]$.

(ii) If $\frac{\pi}{2} < x \le y \le \pi$, then $\log|\cos x| \le \log|\cos y| \le 0$, so $F(x) \le \log|\cos y|/x^2 \le F(y)$. Thus F is nondecreasing on $(\frac{\pi}{2}, \pi]$. By the definition of θ, $F(\theta) = -\frac{1}{2}$.

(iii) If $\frac{\pi}{2} < y \le \theta$, then $|\cos y| \le |\cos\theta| = e^{-\theta^2/2} \le e^{-y^2/2}$. With (i), $|\cos y| \le e^{-y^2/2}$ for $0 \le y \le \theta$.

The upper bound. If $at \le \theta$, then (iii) applies to every y_j, and $|\varphi_X(t)| \le \prod_j e^{-y_j^2/2} = e^{-t^2/2}$. Let $\theta < at \le \pi$, so that $F(at) \ge F(\theta) = -\frac{1}{2}$ by (ii). If some $\cos y_j$ vanishes, then $\varphi_X(t) = 0$. Otherwise $\log|\cos y_j| \le y_j^2 F(at)$ for every j: if $y_j < \frac{\pi}{2}$, by (i), since $\log\cos y_j \le -\frac{y_j^2}{2}$; if $\frac{\pi}{2} < y_j \le at$, by (ii). Summing over j gives $\log|\varphi_X(t)| \le t^2 F(at) = a^{-2}\log|\cos at|$, that is, $|\varphi_X(t)| \le |\cos at|^{1/a^2}$. For $at > \pi$ we use $|\varphi_X| \le 1$. This proves $\varphi_X \le U_a$, and also $\varphi_X \ge -U_a = L_a$ where $at > \theta$.

The lower bound for $at \le \theta$. If $at < \frac{\pi}{2}$, every factor $\cos y_j$ is positive, and by (i) $\log\cos y_j = y_j^2 F(y_j) \ge y_j^2 F(at)$ when $y_j > 0$. Summing gives $\log\varphi_X(t) \ge t^2 F(at)$, that is, $\varphi_X(t) \ge \cos(at)^{1/a^2}$. If $at = \frac{\pi}{2}$, then $\varphi_X(t) \ge 0 = L_a(t)$. Let $\frac{\pi}{2} < at \le \theta$. Since $L_a(t) \le 0$, we may assume $\varphi_X(t) < 0$. Then some factor is negative, say $\cos y_i < 0$, so $\frac{\pi}{2} < y_i \le at$ and $|\cos y_i| \le |\cos at|$. By (iii) the other factors contribute at most $\exp(-\frac{1}{2}\sum_{j \ne i} y_j^2) = e^{-(1-a_i^2)t^2/2} \le e^{-(1-a^2)t^2/2}$. Hence $|\varphi_X(t)| \le |\cos at|\, e^{-(1-a^2)t^2/2} = -L_a(t)$. □

3.3. **The envelope integral.** For $\xi \in \mathbb{R}$ and $0 < u < 1$ let

$$\kappa_\xi(u) = (1-u)\cos(\pi\xi u) + \rho(u)\sin(\pi\xi u)$$

(`upperKernel`), and for $a > 0$ and $y \in \mathbb{R}$ define

$$J(a, y) = \int_0^1 \max\Big\{\kappa_{y/a}(u)\, L_a\big(\tfrac{\pi u}{a}\big),\ \kappa_{y/a}(u)\, U_a\big(\tfrac{\pi u}{a}\big)\Big\}\, \mathrm{d}u$$

(`JUpper a y`). With $T = \pi/a$ the kernel in (3.1) is $\kappa_{y/a}(u)$, because $Tuy = \pi u\,(y/a)$.

Proposition 3.4 (cf. [6, Corollary 3.6 and Lemma 10.6])**.** *Let (a_j) be normalized with level $a \le 1$. Then*

$$\mathbb{P}(|X| \le x) \le 2J(a, x) \quad (x \ge 0), \qquad \mathbb{P}(|X| < x) \ge -2J(a, -x) \quad (x > 0).$$

Consequently $(\mathrm{U})_a$ holds at x if $J(a, x) \le c(a) + c(x)$, and $(\mathrm{L})_a$ holds at x if $J(a, -x) \le c(a) - c(x)$.

Proof. Put $z = X$ in Lemma 3.2 with $T = \pi/a$ and take expectations; the sum over sign vectors is finite. Since $\mathbb{E}\cos(TuX) = \varphi_X(Tu)$, we get $\mathbb{P}(|X| \le x) \le 2\int_0^1 \kappa_{x/a}(u)\varphi_X(\pi u/a)\,\mathrm{d}u$ and $\mathbb{P}(|X| < x) \ge -2\int_0^1 \kappa_{-x/a}(u)\varphi_X(\pi u/a)\,\mathrm{d}u$. By Lemma 3.3, $\kappa\varphi_X \le \max(\kappa L_a, \kappa U_a)$ pointwise: use U_a where $\kappa \ge 0$ and L_a where $\kappa < 0$. The consequences follow because $H = 2c$ and c is odd. □

Remark 3.5. In [6] the sine term of the kernel has the opposite sign, so $J(a, y)$ here is $J(-y; a, \pi/a)$ in the notation of [6]. In [6] the frequency T is a free parameter. Following [3], we take $T = \pi/a$ for both halves, so that every estimate of J is proved once, for real y, and is used at $y = x$ for (U) and at $y = -x$ for (L). The upper bound is `upperHalf_at_of_JUpper`, based on [3]; the lower bound is `lowerHalf_at_of_JUpper` in `Full/LowerPrawitz.lean`.

4. Peeling the largest coefficient

4.1. Hoeffding's bound.

Lemma 4.1 (Hoeffding [5]; `hoeffding_open`, `lowerHalf_at_of_hoeffding`)**.** *For every normalized X and $t \ge 0$, $\mathbb{P}(|X| \ge t) \le 2e^{-t^2/2}$. Consequently $(\mathrm{L})_a$ holds at every x with $e^{-x^2/2} \le c(a)$.*

Proof. $\mathbb{E}e^{sX} = \prod_j \cosh(sa_j) \le e^{s^2/2}$, so $\mathbb{P}(X \ge t) \le e^{-t^2/2}$ by Markov's inequality with $s = t$. By symmetry, $\mathbb{P}(|X| < x) \ge 1 - 2e^{-x^2/2} \ge 1 - H(a) \ge H(x) - H(a)$, since $H(x) \le 1$. □

4.2. Peeling identities.

Lemma 4.2 (`peel_closed_ge`, `peel_closed_lt`, `peel_closed_lt_le_half`, `peel_open_gt`)**.** *Let $X = a\varepsilon + Y$, where $a > 0$, the sign ε is independent of Y, and Y is symmetric.*

(a) If $x \ge a$, then $\mathbb{P}(|X| \le x) = \frac{1}{2}\big(\mathbb{P}(|Y| \le x + a) + \mathbb{P}(|Y| \le x - a)\big)$.
(b) If $0 \le x < a$, then $\mathbb{P}(|X| \le x) = \frac{1}{2}\big(\mathbb{P}(|Y| \le a + x) - \mathbb{P}(|Y| < a - x)\big) \le \frac{1}{2}$.
(c) If $x > a$, then $\mathbb{P}(|X| < x) = \frac{1}{2}\big(\mathbb{P}(|Y| < x + a) + \mathbb{P}(|Y| < x - a)\big)$.

Proof. Conditioning on ε and using the symmetry of Y, $\mathbb{P}(|X| \le x) = \mathbb{P}(Y \in I)$ with $I = [-x - a, x - a]$, and also $\mathbb{P}(Y \in I) = \mathbb{P}(Y \in -I)$. If $x \ge a$, then $I \cup (-I) = [-x - a, x + a]$ and $I \cap (-I) = [-(x - a), x - a]$, which gives (a). If $0 \le x < a$, then I and $-I$ are disjoint, $I \cup (-I) = [-a - x, a + x] \setminus (-(a - x), a - x)$, and $2\mathbb{P}(Y \in I) = \mathbb{P}(Y \in I \cup (-I)) \le 1$; this gives (b). Part (c) is (a) with open intervals. □

4.3. The induction.

For $0 < a < 1$ let $\sigma = \sigma(a) = \sqrt{1 - a^2}$ and $\ell = \ell(a) = \min(a/\sigma, 1)$, and for $t \ge 0$ put

$$\mathrm{lo}(a, t) = \max\Big\{0,\ H\big(\tfrac{t}{\sigma}\big) - H(\ell),\ 1 - 2e^{-t^2/(2\sigma^2)}\Big\}, \qquad \mathrm{up}(a, t) = \min\Big\{1,\ H(\ell) + H\big(\tfrac{t}{\sigma}\big)\Big\}.$$

Lemma 4.3 (`peel_remainder_bounds`)**.** *Let $(a_j)_{j \le n+1}$ be normalized, let $a = |a_i|$ be a largest coefficient with $a < 1$, and let $Y = X - a_i\varepsilon_i$. If (U) and (L) hold at every level for every normalized sum with n terms, then for all $t \ge 0$*

$$\mathrm{lo}(a, t) \le \mathbb{P}(|Y| < t), \qquad \mathbb{P}(|Y| \le t) \le \mathrm{up}(a, t).$$

Proof. Y/σ is a normalized sum with n terms. Each of its coefficients is at most a/σ in modulus, and at most 1 because the squares sum to 1; so ℓ is a level for it. Apply $(\mathrm{U})_\ell$ and $(\mathrm{L})_\ell$ at t/σ. When $t/\sigma \le \ell$ the bound $H(t/\sigma) - H(\ell)$ is nonpositive, and the third term of lo is Lemma 4.1. □

Definition 4.4. For $0 < a < 1$ and $x > a$ let $\mathrm{PeelL}(a, x)$ be the inequality

$$H(x) - H(a) \le \tfrac{1}{2}\big(\mathrm{lo}(a, x + a) + \mathrm{lo}(a, x - a)\big),$$

and for $x \geq 0$ let $\mathrm{PeelU}(a, x)$ be the inequality

$$\begin{cases} \frac{1}{2}\big(\mathrm{up}(a, x+a) + \mathrm{up}(a, x-a)\big) \leq H(a) + H(x) & \text{if } x \geq a, \\ \frac{1}{2}\big(\mathrm{up}(a, a+x) - \mathrm{lo}(a, a-x)\big) \leq H(a) + H(x) & \text{if } x < a. \end{cases}$$

The following theorem is the structure `Pieces` together with `halves_of_pieces` and `keller_klein_of_pieces`.

Theorem 4.5 (the induction)**.** *Suppose that the following statements hold.*

(P1) $c(0.675) \geq \frac{1}{4}$.
(P2) *For every normalized* (a_j) *with level* $a < 0.675$ *and every* $0 \leq x \leq a/2$: $\mathbb{P}(|X| \leq x) \leq H(a) + H(x)$.
(P3) $J(a, x) \leq c(a) + c(x)$ *whenever* $0.07 \leq a \leq \frac{1}{2}$, $x \geq a/2$ *and* $H(x) + H(a) < 1$.
(P4) $J(a, -x) \leq c(a) - c(x)$ *whenever* $0 < a \leq 0.55$, $x \geq 3a/2$ *and* $c(a) < e^{-x^2/2}$.
(P5) $\mathrm{PeelL}(a, x)$ *whenever* $0.55 \leq a < 1$, $x > a$ *and* $c(a) < e^{-x^2/2}$.
(P6) $\mathrm{PeelL}(a, x)$ *whenever* $0 < a \leq 0.55$ *and* $a < x \leq 3a/2$.
(P7) $\mathrm{PeelU}(a, x)$ *whenever* $\frac{1}{2} \leq a < 0.675$, $x \geq a/2$ *and* $H(x) + H(a) < 1$.

Then, for every normalized X *with largest coefficient* $A = \max_j |a_j|$, $(\mathrm{U})_A$ *and* $(\mathrm{L})_A$ *hold, and therefore Theorem 1.1 holds.*

The proof also uses Goyal's theorem.

Theorem 4.6 (Goyal [3, Theorem 1.1]; `small_coefficient_upper_concentration`)**.** *If* (a_j) *is normalized with level* $0 < a \leq 0.07$, *then* $\mathbb{P}(|X| \leq x) \leq H(a) + H(x)$ *for every* $x \geq 0$.

Theorem 4.6 is proved in Section 6.4, following [3].

Proof of Theorem 4.5. By induction on the number n of terms; for $n = 0$ there is no normalized vector. Let $A = |a_i|$ be a largest coefficient, so $0 < A \leq 1$. By (P1), $H(0.675) \geq \frac{1}{2}$.

If $A = 1$, all other coefficients vanish and $X = \pm\varepsilon_i$. For $x < 1$, $\mathbb{P}(|X| \leq x) = 0$; for $x \geq 1$, $H(A) + H(x) \geq 2H(0.675) \geq 1$. For $x > 1$, $\mathbb{P}(|X| < x) = 1 \geq H(x) - H(A)$.

Let $A < 1$. By the induction hypothesis, Lemma 4.3 applies. Table 2 lists the cases. In the cases "peeling", the bound follows from Lemma 4.2 and Lemma 4.3. For example, if $A \leq x$ in the upper half, then $\mathbb{P}(|X| \leq x) \leq \frac{1}{2}(\mathrm{up}(A, x+A) + \mathrm{up}(A, x-A)) \leq H(A) + H(x)$ by (P7). By monotonicity in the level and Proposition 2.2, Theorem 1.1 follows. □

The inputs are proved as follows: (P1) by a certified evaluation of $c(0.675) \approx 0.2502$ (Section 8); (P2) analytically (Section 5); (P3) by the certified check of Sections 7 and 8; (P4) analytically for $a \leq 0.1$ (Theorem 6.6) and by the certified check for $0.1 \leq a \leq 0.55$; (P5) and (P7) by certified checks of elementary inequalities; and (P6) analytically for $a \leq 0.3$ (Proposition 6.8) and by a certified check for $0.3 \leq a \leq 0.55$.

Remark 4.7. Peeling loses about $a^3\varphi(a)$, from the rescaling of Y by σ. This is the size of the true margin, so peeling alone cannot settle small or moderate a; but it settles large a and the lower half near the diagonal. In [6] the near-diagonal range was handled by the segment comparison lemma [8, Lemma 3.7] through [6, Lemma 4.2], which loses a factor $\frac{3}{2}$ in a local concentration bound.

5. The window near the origin

This section proves (P2); in Lean it is `zoneA_piece` in `Full/ZoneAGeneral.lean`. No computation is involved.

Theorem 5.1. *Let* (a_j) *be normalized with level* $a < 0.675$, *and let* $0 \leq x \leq a/2$. *Then* $\mathbb{P}(|X| \leq x) \leq H(a) + H(x)$.

half	case	argument
(U)	$A \ge 0.675$, $x < A$	$\mathbb{P}(\lvert X\rvert \le x) \le \frac12 \le H(A)$ (Lemma 4.2(b), (P1))
(U)	$A \ge 0.675$, $x \ge A$	$H(A) + H(x) \ge 2H(0.675) \ge 1$
(U)	$A < 0.675$, $x \le A/2$	(P2)
(U)	$x \ge A/2$, $H(x) + H(A) \ge 1$	trivial
(U)	$A \le 0.07$	Theorem 4.6
(U)	$0.07 < A \le \frac12$, $x \ge A/2$, $H(x) + H(A) < 1$	Proposition 3.4 and (P3)
(U)	$\frac12 < A < 0.675$, $x \ge A/2$, $H(x) + H(A) < 1$	peeling and (P7)
(L)	$e^{-x^2/2} \le c(A)$	Lemma 4.1
(L)	$A \ge 0.55$, $c(A) < e^{-x^2/2}$	peeling and (P5)
(L)	$A < 0.55$, $A < x \le 3A/2$	peeling and (P6)
(L)	$A < 0.55$, $x > 3A/2$, $c(A) < e^{-x^2/2}$	Proposition 3.4 and (P4)

TABLE 2. The case analysis of `halves_of_pieces` for a normalized sum with largest coefficient $A < 1$.

5.1. **Reduction to a single profile.** For $b > 0$ let $W_b(t) = |\cos(bt)|^{1/b^2}$ (`cosineProfile`), and for $0 < b \le a$ and $0 \le \xi \le \frac12$ let

$$Z_b(a,\xi) = \int_0^1 \kappa_\xi(u)\, W_b\!\left(\tfrac{\pi u}{a}\right) \mathrm{d}u \qquad (\texttt{ZoneA.holderProfile a b } \xi).$$

For $0 \le \xi \le \frac12$ and $0 < u < 1$ we have $0 \le \pi\xi u \le \pi/2$, hence $\kappa_\xi(u) \ge 0$ (`upperKernel_nonneg_closed`). By the weighted arithmetic–geometric mean inequality with weights a_j^2,

$$|\varphi_X(t)| = \prod_j \left(W_{|a_j|}(t)\right)^{a_j^2} \le \sum_j a_j^2\, W_{|a_j|}(t),$$

where coefficients $a_j = 0$ contribute the factor 1 and the weight 0. Applying Hölder's inequality to a product of cosines in this way, to reduce all weight vectors to one-parameter profiles, goes back to Haagerup [4] and Ball [1] (see also [14]). Its combination with the Prawitz kernel, which requires $\kappa \ge 0$, is from [6, Remark 10.4] and [3]. In Lean it is `integral_cosProduct_le_uniform_profile` from [3]. With the first step of the proof of Proposition 3.4 it gives

$$\mathbb{P}(|X| \le x) \le 2\int_0^1 \kappa_\xi(u)\, |\varphi_X(\pi u/a)|\, \mathrm{d}u \le 2 \max_{j:\, a_j \ne 0} Z_{|a_j|}(a,\xi), \qquad \xi = x/a.$$

Hence Theorem 5.1 follows from the next statement (`holderProfile_le_general`).

Theorem 5.2. *For $0 < b \le a < 0.675$ and $0 \le \xi \le \frac12$, $Z_b(a,\xi) \le c(a) + c(a\xi)$.*

5.2. **An affine majorant of the kernel.** Recall $\delta(y) = 1 - y\cot y$, and let

$$q(u) = \pi u\rho(u) = 1 - (1-u)\delta(\pi u) = u\,\delta(\pi(1-u)), \qquad p(u) = 1 - \tfrac52 u^2 + u^3 + \tfrac12 u^4.$$

The two expressions for q follow from $\pi u\cot(\pi u) = 1 - \delta(\pi u)$ and $\rho(u) = \delta(\pi(1-u))/\pi$. In particular $0 \le q \le 1$.

Lemma 5.3 (`cotangent_upper_quartic`; `cotangent_remainder_bound` from [3])**.** *For $0 < y \le \pi/2$, $\frac{y^2}{3} + \frac{y^4}{45} \le \delta(y) \le \frac{y^2}{3} + \frac{y^4}{32}$.*

Proof. The lower bound. Let $f(y) = (1-\frac{y^2}{3}-\frac{y^4}{45})\sin y - y\cos y$, $g(y) = (1-\frac{4y^2}{15})\sin y - (y+\frac{y^3}{15})\cos y$ and $m(y) = (7+y^2)\sin y - 7y\cos y$. Then $f(0) = g(0) = m(0) = 0$, and

$$f' = \tfrac{y}{3}\, g, \qquad g' = \tfrac{y}{15}\, m, \qquad m' = 9y\sin y + y^2\cos y \geq 0 \quad \text{on } [0,\pi/2].$$

Hence $m, g, f \geq 0$ on $[0,\pi/2]$, and dividing $f \geq 0$ by $\sin y > 0$ gives the claim.

The upper bound (following [3], `cotangent_lower_product`). Let $P = 1 - \frac{y^2}{3} - \frac{y^4}{32}$. It suffices to show $y\cos y \geq P\sin y$. If $P \leq 0$, this holds because $y\cos y \geq 0$. If $P > 0$, the Taylor bounds $\cos y \geq 1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720}$ and $\sin y \leq y - \frac{y^3}{6} + \frac{y^5}{120}$ give

$$y\cos y - P\sin y \geq y^5\Big(\frac{13}{1440} - \frac{11y^2}{2880} + \frac{y^4}{3840}\Big) \geq \frac{17}{15360}\, y^5 \geq 0.$$

For the middle inequality, note that the quadratic in y^2 decreases for $y^2 \leq \frac{22}{3}$, that $y^2 \leq \pi^2/4 < \frac{5}{2}$, and that its value at $y^2 = \frac{5}{2}$ is $\frac{17}{15360}$. □

Lemma 5.4 (`pi_mul_rho_le_poly`)**.** *$q(u) \leq p(u)$ for $0 < u < 1$.*

Proof. Let $s = \pi^2 \in (9.869, 9.87)$. If $u \leq \frac{1}{2}$, the lower bound of Lemma 5.3 at $y = \pi u$ gives

$$p(u) - q(u) \geq u^2\beta(u), \qquad \beta(u) = (1-u)\Big(\frac{s}{3} + \frac{s^2u^2}{45}\Big) - \frac{5}{2} + u + \frac{u^2}{2}.$$

β increases with s. At $s = 9.869$ it is decreasing in u on $[0,\frac{1}{2}]$, with $\beta(\frac{1}{2}) \approx 0.0404$; so $\beta > 0$. If $u \geq \frac{1}{2}$, write $t = 1-u \leq \frac{1}{2}$. Then $p(1-t) = t^2(\frac{7}{2} - 3t + \frac{t^2}{2})$, and the upper bound of Lemma 5.3 at $y = \pi t$ gives $q(u) \leq (1-t)(\frac{st^2}{3} + \frac{s^2t^4}{32})$. So

$$p(u) - q(u) \geq t^2\gamma(t), \qquad \gamma(t) = \frac{7}{2} - 3t + \frac{t^2}{2} - (1-t)\Big(\frac{s}{3} + \frac{s^2t^2}{32}\Big) \geq 0.21 + 0.29t - 2.54428\, t^2 + 3.04428\, t^3,$$

where we used $s < 9.87$. The cubic has minimum ≈ 0.0994 on $[0,\frac{1}{2}]$, at $t \approx 0.4927$. Both polynomial inequalities are proved in Lean by `nlinarith`. □

Lemma 5.5 (`upperKernel_le_affine`)**.** *For $0 \leq \xi \leq \frac{1}{2}$ and $0 < u < 1$, $\kappa_\xi(u) \leq (1-u) + \xi p(u)$.*

Proof. Use $\cos \leq 1$, $\rho \geq 0$, $0 \leq \sin(\pi\xi u) \leq \pi\xi u$, and then $q \leq p$. □

5.3. **Interpolation in ξ.** Let $k_0(u) = 1-u$ and $k_h(u) = 1 - u + \frac{1}{2}p(u)$, so that $(1-u) + \xi p(u) = (1-2\xi)k_0(u) + 2\xi k_h(u)$. For a polynomial k and $0 < b \leq \alpha$ put

$$G_k(\alpha; b) = \int_0^1 k(u)\, W_b\big(\tfrac{\pi u}{\alpha}\big)\, \mathrm{d}u,$$
$$F_0(\alpha; b) = c(\alpha) - G_{k_0}(\alpha; b), \qquad F_h(\alpha; b) = c(\alpha) + c(\alpha/2) - G_{k_h}(\alpha; b).$$

Lemma 5.6 (`mul_c_le_c_mul`)**.** *For $0 \leq \lambda \leq 1$ and $t \geq 0$, $\lambda c(t) \leq c(\lambda t)$.*

Proof. $c(\lambda t) = \lambda \int_0^t \varphi(\lambda s)\, \mathrm{d}s \geq \lambda \int_0^t \varphi(s)\, \mathrm{d}s$, since φ decreases on $[0,\infty)$. □

Proposition 5.7. *For $0 < b \leq a < 0.675$, $F_0(a; b) \geq 0$ and $F_h(a; b) \geq 0$.*

Proof of Theorem 5.2 from Proposition 5.7. Since $W_b \geq 0$, Lemma 5.5 gives $Z_b(a,\xi) \leq (1-2\xi)G_{k_0}(a;b) + 2\xi G_{k_h}(a;b) \leq c(a) + 2\xi c(a/2)$. By Lemma 5.6 with $\lambda = 2\xi$ and $t = a/2$, $2\xi c(a/2) \leq c(a\xi)$. □

Proposition 5.7 is proved in two steps: on the diagonal $\alpha = b$ (Lemma 5.10), and along the ray $\alpha \geq b$ at fixed b (Lemma 5.12).

5.4. **A Wallis inequality.** For real $t \geq 0$ let $V(t) = \int_0^{1/2} \cos(\pi u)^t \, du$ (`wallisV`).

Lemma 5.8 (`wallisV_sq_le`). *For every real $N \geq \frac{1}{2}$, $V(N)^2 \leq \dfrac{1}{2\pi(N+\frac{1}{2})}$.*

Proof. (i) $\log V$ is convex on $[0,\infty)$, by Hölder's inequality (`wallisV_holder`). (ii) Integration by parts gives $(t+2)V(t+2) = (t+1)V(t)$. (iii) Hence $G(t) = (t+1)V(t)V(t+1)$ satisfies $G(t+1) = G(t)$. Since $V(0) = \frac{1}{2}$ and $V(1) = \frac{1}{\pi}$, $G(m) = \frac{1}{2\pi}$ for every integer $m \geq 0$. (iv) Let $m \in \mathbb{N}$ and $\vartheta \in [0,1]$. By (i), $V(m+\vartheta) \leq V(m)^{1-\vartheta} V(m+1)^{\vartheta}$ and $V(m+1+\vartheta) \leq V(m+1)^{1-\vartheta} V(m+2)^{\vartheta}$. Multiplying, and using (iii),

$$\begin{aligned} V(m+\vartheta)V(m+1+\vartheta) &\leq \big(V(m)V(m+1)\big)^{1-\vartheta}\big(V(m+1)V(m+2)\big)^{\vartheta} \\ &= \frac{(m+1)^{\vartheta-1}(m+2)^{-\vartheta}}{2\pi} \leq \frac{1}{2\pi(m+1)}. \end{aligned}$$

So $G(\vartheta) = G(m+\vartheta) \leq \frac{m+1+\vartheta}{2\pi(m+1)}$ for every m. Letting $m \to \infty$ gives $G(\vartheta) \leq \frac{1}{2\pi}$, and by periodicity $G \leq \frac{1}{2\pi}$ on $[0,\infty)$ (`wallisG_le`). (v) By (i), $V(N)^2 \leq V(N-\frac{1}{2})V(N+\frac{1}{2}) = G(N-\frac{1}{2})/(N+\frac{1}{2}) \leq 1/(2\pi(N+\frac{1}{2}))$. □

Remark 5.9. Since $V(N) = \Gamma(\frac{N+1}{2})/(2\sqrt{\pi}\,\Gamma(\frac{N}{2}+1))$, Lemma 5.8 is equivalent to $\Gamma(y+1)/\Gamma(y+\frac{1}{2}) \geq (y+\frac{1}{4})^{1/2}$ for $y = N/2 \geq \frac{1}{4}$. This is Watson's inequality, the case $s = \frac{1}{2}$ of Kershaw's bound [9], which [6, Proposition 9.1] uses in the same way to prove $\mathbb{P}(X=0) \leq H(a_1)$. The proof above avoids the Gamma function and is the one formalized.

5.5. **The diagonal.**

Lemma 5.10 (`diag_zero_le`, `diag_half_le`). *If $b > 0$ and $b^2 \leq \frac{1}{2}$, then $F_0(b;b) \geq 0$ and $F_h(b;b) \geq 0$.*

Proof. Let $N = 1/b^2 \geq 2$ and $\beta = b^2$. On the diagonal $W_b(\pi u/b) = |\cos \pi u|^N$, which is invariant under $u \mapsto 1-u$. Since $k_0(u) + k_0(1-u) = 1$ and $p(u) + p(1-u) = 1 + u^2(1-u)^2$ (`kernelPoly_add_reflect`),

$$G_{k_0}(b;b) = V(N), \qquad G_{k_h}(b;b) = \tfrac{3}{2}V(N) + \tfrac{1}{2}K, \qquad K = \int_0^{1/2} u^2(1-u)^2 \cos(\pi u)^N \, du$$

(`diag_zero_eq`, `diag_half_eq`). We use three bounds.

(a) $V(N)^2 \leq \varphi_0^2 \beta/(1+\beta/2)$, by Lemma 5.8.

(b) $K \leq \varphi_0 b^3/\pi^2$ (`diagJ_le`). Indeed $(1-u)^2 \leq 1$ and $\cos(\pi u)^N \leq e^{-\pi^2 u^2/(2b^2)}$ on $[0,\frac{1}{2}]$, and $\int_0^\infty u^2 e^{-\pi^2 u^2/(2b^2)} \, du = (b/\pi)^3 \int_0^\infty s^2 g(s) \, ds = (b/\pi)^3 \pi \varphi_0$.

(c) $c(t) \geq \varphi_0(t - t^3/6)$ for $t \geq 0$.

For F_0: by (a) and (c) it suffices that $(1-\frac{\beta}{6})^2(1+\frac{\beta}{2}) \geq 1$, and indeed $(1-\frac{\beta}{6})^2(1+\frac{\beta}{2}) - 1 = \beta(12 - 10\beta + \beta^2)/72 \geq 0$. For F_h: by (c), $c(b) + c(b/2) \geq \varphi_0 b(\frac{3}{2} - \frac{3\beta}{16})$, and by (b), $\frac{1}{2}K \leq \varphi_0 b\beta/(2\pi^2)$. So it suffices that $\frac{3}{2}V(N) \leq \varphi_0 b R$ with $R = \frac{3}{2} - \frac{3\beta}{16} - \frac{\beta}{2\pi^2} \geq \frac{3}{2} - \frac{35\beta}{144} > 0$. By (a) this follows from $R^2(1+\frac{\beta}{2}) \geq \frac{9}{4}$. Finally

$$\Big(\frac{3}{2} - \frac{35\beta}{144}\Big)^2\Big(1+\frac{\beta}{2}\Big) - \frac{9}{4} = \frac{57}{144}\beta - \frac{6335}{20736}\beta^2 + \frac{1225}{41472}\beta^3 > 0 \qquad (0 < \beta \leq \tfrac{1}{2}).$$ □

At $\xi = 0$ the diagonal bound for F_0 is the statement $\mathbb{P}(X=0) \leq H(a)$ for equal weights, and the computation for F_0 is that of [6, Proposition 9.1].

5.6. **The ray.**

Lemma 5.11 (`hasDerivAt_profileIntegral`, `profile_ray_mono`)**.** *Let $b > 0$ with $b^2 \le \frac{1}{2}$, let k be a polynomial with $k(1) = 0$, and let $j(u) = uk(u)$. Then $\alpha \mapsto G_k(\alpha; b)$ is differentiable on $(0, \infty)$, and*

$$\frac{\partial}{\partial \alpha} G_k(\alpha; b) = \frac{1}{\alpha} \int_0^1 j'(u)\, W_b\big(\tfrac{\pi u}{\alpha}\big)\, \mathrm{d}u.$$

Proof. $W_b = (\cos^2(b\,\cdot))^{1/(2b^2)}$ is C^1 because $1/(2b^2) \ge 1$, and $|W_b'| \le 1/b$. Differentiation under the integral sign (dominated convergence) gives $\partial_\alpha G_k = -\int_0^1 k(u) \frac{\pi u}{\alpha^2} W_b'(\frac{\pi u}{\alpha})\, \mathrm{d}u$. Since $\partial_u[W_b(\pi u/\alpha)] = \frac{\pi}{\alpha} W_b'(\pi u/\alpha)$, this equals $-\frac{1}{\alpha}\int_0^1 j(u)\, \partial_u[W_b(\pi u/\alpha)]\, \mathrm{d}u$. Integrate by parts, using $j(0) = j(1) = 0$. □

Lemma 5.12 (`ray_zero`, `ray_half`)**.** *For $0 < b \le \alpha < 0.675$, $\partial_\alpha F_0(\alpha; b) \ge 0$ and $\partial_\alpha F_h(\alpha; b) \ge 0$. Hence $F_0(a; b) \ge F_0(b; b)$ and $F_h(a; b) \ge F_h(b; b)$ for $b \le a < 0.675$.*

Proof. Suppose $j'(u) \le c_1 - 2u$ on $[0, \frac{1}{2}]$ for some $c_1 \ge 1$, and $j' \le 0$ on $[\frac{1}{2}, 1]$. For $u \le \frac{1}{2}$ we have $b\pi u/\alpha \le \pi/2$, hence $0 \le W_b(\pi u/\alpha) \le e^{-\pi^2 u^2/(2\alpha^2)}$, while $c_1 - 2u \ge 0$. By Lemma 5.11,

$$\alpha\, \partial_\alpha G_k \le \int_0^{1/2} (c_1 - 2u) e^{-\pi^2 u^2/(2\alpha^2)}\, \mathrm{d}u \le c_1 \alpha \varphi_0 - \frac{2\alpha^2}{\pi^2}\Big(1 - e^{-\pi^2/(8\alpha^2)}\Big),$$

using $\int_0^\infty e^{-\pi^2 u^2/(2\alpha^2)}\, \mathrm{d}u = \alpha\varphi_0$ and $\int_0^{1/2} u e^{-\pi^2 u^2/(2\alpha^2)}\, \mathrm{d}u = \frac{\alpha^2}{\pi^2}(1 - e^{-\pi^2/(8\alpha^2)})$.

For k_0, $j_0' = 1 - 2u$, so $c_1 = 1$. For k_h,

$$j_h'(u) = \tfrac{3}{2} - 2u - \tfrac{15}{4}u^2 + 2u^3 + \tfrac{5}{4}u^4,$$

which is at most $\frac{3}{2} - 2u$ on $[0, \frac{1}{2}]$ and nonpositive on $[\frac{1}{2}, 1]$, so $c_1 = \frac{3}{2}$. For $\alpha < 0.675$ we have $\pi^2/(8\alpha^2) > 2.7$, so $e^{-\pi^2/(8\alpha^2)} < \frac{1}{10}$. We also have $\varphi(\alpha) \ge \varphi_0(1 - \alpha^2/2)$ and $\varphi_0 \alpha \pi^2 < 3$. Therefore

$$\partial_\alpha F_0 \ge \varphi_0(1 - \tfrac{\alpha^2}{2}) - \varphi_0 + \frac{9\alpha}{5\pi^2} = \alpha\Big(\frac{9}{5\pi^2} - \frac{\varphi_0 \alpha}{2}\Big) \ge 0,$$

since $\varphi_0 \alpha \pi^2 \le \frac{18}{5}$. Similarly, with $\varphi(\alpha) + \frac{1}{2}\varphi(\alpha/2) \ge \frac{3}{2}\varphi_0 - \frac{9}{16}\varphi_0 \alpha^2$,

$$\partial_\alpha F_h \ge \alpha\Big(\frac{9}{5\pi^2} - \frac{9}{16}\varphi_0 \alpha\Big) \ge 0,$$

since $\varphi_0 \alpha \pi^2 \le \frac{16}{5}$. □

Proof of Proposition 5.7. Since $b \le a < 0.675$, we have $b^2 < \frac{1}{2}$. Lemma 5.12 reduces the claim to the diagonal, which is Lemma 5.10. □

Remark 5.13. In [6] the window near the origin is covered by 728 137 certified cells for $a_1 \ge 0.07$. In [3] it is treated analytically for $a_1 \le 0.07$. Since (U) is elementary for $a_1 \ge 0.675$, the window needs no computation at all. Two attempts that did not work are described in Section 10: monotonicity of $b \mapsto Z_b(a, \xi)$ is false, and the exact kernel along the ray requires a three-parameter certificate.

6. Small coefficients

This section proves (P4) for $a \le \frac{1}{10}$ (Theorem 6.6) and (P6) for $a \le \frac{3}{10}$ (Proposition 6.8). For (P4) we use Goyal's decomposition of J [3], which was designed for the upper half, at the negative location $y = -x$. In Section 6.4 we use it at $y = x$ to prove Theorem 4.6, following [3].

6.1. **Goyal's decomposition.** For $a>0$ and $y,\xi\in\mathbb{R}$ let

$$S(a,y)=\int_0^1 \kappa_{y/a}(u)\,g\big(\tfrac{\pi u}{a}\big)\,\mathrm{d}u=\frac{a}{\pi}\int_0^{\pi/a}\kappa_{y/a}\big(\tfrac{as}{\pi}\big)\,g(s)\,\mathrm{d}s,$$

$$\Lambda(a,\xi)=\frac{a}{\pi}\int_0^{1/a}\Big(-\kappa_\xi\big(\tfrac{as}{\pi}\big)\Big)_+\Big(g(s)-\cos(as)^{1/a^2}\Big)\,\mathrm{d}s,$$

$$A(a,\xi)=\frac{a}{\pi}\int_{\theta/a}^{\pi/a}\Big|\kappa_\xi\big(\tfrac{as}{\pi}\big)\Big|\,W_a(s)\,\mathrm{d}s$$

(`scaledGaussianKernelIntegral`, `primaryLoss`, `zoneBAliasIntegral`). Thus S is the Gaussian part of J; Λ is the loss of the lower envelope against the Gaussian on $as<1$; and A comes from the range $\theta<as<\pi$, where $|\varphi_X|$ may return close to 1.

Lemma 6.1 (Goyal [3]; `JUpper_le_gaussian_primary_alias_real`)**.** *For $0<a\le\frac12$ and $y\in\mathbb{R}$, with $\xi=y/a$,*

$$J(a,y)\le S(a,y)+\Lambda(a,\xi)+A(a,\xi)+6e^{-1/(2a^2)}.$$

Proof. We follow [3]. Write $\varepsilon=e^{-1/(2a^2)}=g(1/a)$, and for $0<u<1$ let $t=\pi u/a$, $K=\kappa_\xi(u)$, and let $d(u)$ be the integrand of J minus $K\,g(t)$. Then $J(a,y)-S(a,y)=\int_0^1 d(u)\,\mathrm{d}u$. We bound d on three intervals.

(i) For $0<u<1/\pi$ we have $at=\pi u<\frac{\pi}{2}$, so $U_a(t)=g(t)$ and $L_a(t)=\cos(at)^{1/a^2}$, and $L_a(t)\le g(t)$ by (i) in the proof of Lemma 3.3. Since $\max(KL,KU)=KU+(-K)_+(U-L)$ when $L\le U$, we get $d(u)=(-K)_+(g(t)-\cos(at)^{1/a^2})$. The substitution $u=as/\pi$ turns $\int_0^{1/\pi}d$ into $\Lambda(a,\xi)$.

(ii) For $1/\pi\le u<1$ we have $|K|\le 1+\frac{1}{\pi u}\le 2$ by Lemma A.1(b), and $t\ge 1/a$, so $|K|\,g(t)\le 2\varepsilon$. If moreover $u\le\theta/\pi$, then $U_a(t)=g(t)$ and $|L_a(t)|\le\varepsilon$. Indeed, if $at\le\frac{\pi}{2}$, then $0\le L_a(t)\le g(t)$ as in (i); and if $\frac{\pi}{2}<at\le\theta$, then $|L_a(t)|\le e^{-(1-a^2)t^2/2}$ with $(1-a^2)t^2\ge\frac34\big(\frac{\pi}{2a}\big)^2>\frac{1}{a^2}$, because $a\le\frac12$. Hence $d(u)\le\max(|K||L_a(t)|,|K|g(t))+|K|g(t)\le 4\varepsilon$, and, since the interval has length less than 1, $\int_{1/\pi}^{\theta/\pi}d\le 4\varepsilon$.

(iii) For $\theta/\pi<u<1$ we have $L_a(t)=-W_a(t)$ and $U_a(t)=W_a(t)$, so the integrand of J is $|K|\,W_a(t)$ and $d(u)\le|K|\,W_a(t)+2\varepsilon$ by (ii). The substitution $u=as/\pi$ turns $\int_{\theta/\pi}^1|K|\,W_a(t)\,\mathrm{d}u$ into $A(a,\xi)$, so $\int_{\theta/\pi}^1 d\le A(a,\xi)+2\varepsilon$.

Adding (i)–(iii) gives the claim. Goyal states the lemma for $y\ge0$ (`JUpper_le_gaussian_primary_alias`), but the argument does not use the sign of y. □

Let

$$D(x)=\int_0^\infty\sin(sx)g(s)\,\mathrm{d}s,\qquad M(x)=\int_0^\infty s\cos(sx)g(s)\,\mathrm{d}s,\qquad N(x)=\int_0^\infty s^2\sin(sx)g(s)\,\mathrm{d}s$$

(`gaussianD`, `gaussianM`, `gaussianN`). Integration by parts gives $M=1-xD$, and D and N are odd.

Lemma 6.2 (Goyal [3]; `scaledGaussianKernelIntegral_le`)**.** *For $0<a\le\frac12$ and $y\in\mathbb{R}$,*

$$S(a,y)-c(y)-c(a)\le\Gamma(a,y)-\frac{a^2}{\pi^2}+\frac{a^4}{15\pi}+\Big(1+\frac{a^2}{2\pi}\Big)e^{-1/(2a^2)},$$

where $\Gamma(a,y)=a\varphi(y)-c(a)-\frac13a^2y\varphi(y)+\frac{1}{\pi^2}a^2yD(y)+\frac{1}{3\pi^2}a^3N(y)$ (`zoneBCommon`).

Proof. We follow [3]. With $u=as/\pi$ we have $\pi\xi u=sy$, and with $Q(t)=\cot t-\frac1t+\frac t3$ (`cotangentQ`),

$$\rho\big(\tfrac{as}{\pi}\big)=\frac{1}{as}-\frac{as}{3}+\frac{a^2s^2}{3\pi}+\Big(1-\frac{as}{\pi}\Big)Q(as).\tag{6.1}$$

Extend the integrals of the explicit terms from $(0,\pi/a)$ to $(0,\infty)$ and use

$$\int_0^\infty \cos(sy)g(s)\,\mathrm{d}s = \pi\varphi(y), \qquad \int_0^\infty \frac{\sin(sy)}{s}\,g(s)\,\mathrm{d}s = \pi c(y), \qquad \int_0^\infty s\sin(sy)g(s)\,\mathrm{d}s = \pi y\varphi(y).$$

This gives the value $B = c(y)+a\varphi(y)-\frac{a^2}{\pi^2}M(y)-\frac{1}{3}a^2y\varphi(y)+\frac{1}{3\pi^2}a^3N(y)$ (`gaussianBaselineValue`), and $B-c(y)-c(a) = \Gamma(a,y)-a^2/\pi^2$ because $M = 1-yD$ (`gaussianBaselineValue_eq_common`). It remains to bound the term with Q and the tails of the explicit terms beyond $s=\pi/a$. Let $\varepsilon = e^{-1/(2a^2)} = g(1/a)$.

The term with Q is $\frac{a}{\pi}\int_0^{\pi/a}(1-\frac{as}{\pi})Q(as)\sin(sy)\,g(s)\,\mathrm{d}s$. By Lemma A.1(d) with $u = as/\pi$, the weight satisfies $|(1-\frac{as}{\pi})Q(as)| \le (as)^3/30$ for $s \le 1/a$ and $|(1-\frac{as}{\pi})Q(as)| \le as/2$ for $s > 1/a$. Since $\int_0^\infty s^3 g(s)\,\mathrm{d}s = 2$ and $\int_{1/a}^\infty s\,g(s)\,\mathrm{d}s = \varepsilon$, the term is at most $\frac{a^4}{15\pi}+\frac{a^2}{2\pi}\varepsilon$ in absolute value (`abs_cotangentRemainderIntegral_le`).

For $s \ge \pi/a$ we have $|\sin(sy)|/(as) \le 1/\pi$, so the explicit terms of the integrand are bounded by $\big[(1+\frac{1}{\pi})+(\frac{a}{\pi}+\frac{a}{3})s+\frac{a^2}{3\pi}s^2\big]g(s)$ (`abs_gaussianBaselineIntegrand_tail`). With $R=\pi/a$ we have $\int_R^\infty g \le g(R)/R$, $\int_R^\infty s\,g = g(R)$ and $\int_R^\infty s^2 g \le (R+\frac{1}{R})g(R)$, so the tails contribute at most

$$a^2\Big(\frac{2}{\pi^2}+\frac{1}{\pi^3}+\frac{2}{3\pi}+\frac{a^2}{3\pi^3}\Big)g\Big(\frac{\pi}{a}\Big)$$

in absolute value (`integral_gaussianBaselineTailMajorant_le`). Since $\pi > 3$ and $a \le \frac{1}{2}$, the factor in front of $g(\pi/a)$ is less than 1, so the tails contribute at most $g(\pi/a) \le \varepsilon$ (`abs_gaussianBaseline_tail_le`). □

Lemma 6.3 (`zoneBAliasIntegral_compensated` from [3]; `zoneBAliasIntegral_le_neg`)**.** *For $a > 0$ and $\eta \ge 0$, with $\eta_0 = \frac{1}{2}+\frac{3}{\pi^2} \in [\frac{4}{5},\frac{5}{6}]$,*

$$A(a,\pm\eta) - \frac{a^2}{\pi^2} \le \frac{\varphi_0^2}{2}(\eta+\eta_0)^2a^4 + \frac{6(\eta+1)a^4}{\pi^4}.$$

Proof. We follow [3], who proves the bound for $A(a,\eta)$; the same argument applies to $A(a,-\eta)$. Let $\beta = \pi\eta$. Substituting $u = as/\pi$ and then $u = 1-t$, and using $|\cos\pi(1-t)| = \cos\pi t$ for $t \le \frac{1}{2}$,

$$A(a,\pm\eta) = \int_{\theta/\pi}^1 |\kappa_{\pm\eta}(u)|\,|\cos\pi u|^{1/a^2}\,\mathrm{d}u = \int_0^{1-\theta/\pi} |\kappa_{\pm\eta}(1-t)|\cos(\pi t)^{1/a^2}\,\mathrm{d}t.$$

Here $1-\theta/\pi < \frac{1}{2}$, and $\cos(\pi t)^{1/a^2} \le e^{-\pi^2t^2/(2a^2)}$ by (i) in the proof of Lemma 3.3. Next, $\kappa_{\pm\eta}(1-t) = t\cos(\beta(1-t)) \pm \rho(1-t)\sin(\beta(1-t))$. The addition formulas and $|\sin\beta t| \le \beta t$ give $|\cos(\beta(1-t))| \le |\cos\beta| + \beta t|\sin\beta|$ and $|\sin(\beta(1-t))| \le |\sin\beta| + \beta t$, and $0 \le \rho(1-t) \le \frac{4}{3}t^2$ by Lemma A.1(e). Hence, for $0 < t \le \frac{1}{2}$,

$$|\kappa_{\pm\eta}(1-t)| \le t|\cos\beta| + (\beta+\tfrac{4}{3})t^2|\sin\beta| + \tfrac{4}{3}\beta t^3$$

(`abs_upperKernel_alias_bound`, `abs_upperKernel_neg_alias_bound`). We extend the integral to $(0,\infty)$ and use $\int_0^\infty t^k e^{-\pi^2t^2/(2a^2)}\,\mathrm{d}t = \frac{a^2}{\pi^2}$, $\frac{\varphi_0 a^3}{\pi^2}$, $\frac{2a^4}{\pi^4}$ for $k = 1,2,3$. This gives $A(a,\pm\eta) \le V(a,\eta)$, where

$$V(a,\eta) = Q\,|\cos\pi\eta| + K\,|\sin\pi\eta| + \frac{8\eta a^4}{3\pi^3}, \qquad Q = \frac{a^2}{\pi^2}, \quad K = \frac{\varphi_0}{\pi}(\eta+\delta)a^3, \quad \delta = \frac{4}{3\pi}$$

(`zoneBAliasValue`, `zoneBAliasIntegral_le`, `zoneBAliasIntegral_le_neg`).

Goyal's compensation of $Q|\cos\pi\eta|$ against $Q = \frac{a^2}{\pi^2}$ is the following. By the Cauchy–Schwarz inequality, $Q|\cos z| + K|\sin z| \le \sqrt{Q^2+K^2} \le Q + \frac{K^2}{2Q}$, and $\frac{K^2}{2Q} = \frac{\varphi_0^2}{2}(\eta+\delta)^2a^4$. It therefore remains to show

$$\frac{\varphi_0^2}{2}(\eta+\delta)^2 + \frac{8\eta}{3\pi^3} \le \frac{\varphi_0^2}{2}(\eta+\eta_0)^2 + \frac{6(\eta+1)}{\pi^4}.$$

Since $\delta \le \frac{4}{9} < \frac{4}{5} \le \eta_0$, we have $(\eta+\eta_0)^2 - (\eta+\delta)^2 \ge 2\eta(\eta_0-\delta)$, so it suffices that $\varphi_0^2(\eta_0-\delta) + \frac{6}{\pi^4} \ge \frac{8}{3\pi^3}$. This holds because $\varphi_0 > 0.39$, $\eta_0 - \delta \ge \frac{16}{45}$, $\pi^4 \le 100$ and $\pi^3 \ge 27$, so that $\varphi_0^2(\eta_0-\delta) + \frac{6}{\pi^4} \ge 0.1521 \cdot \frac{16}{45} + \frac{3}{50} > \frac{8}{81} \ge \frac{8}{3\pi^3}$ (`zoneBAliasValue_compensated`). □

6.2. The lower half for $a \le \frac{1}{10}$.

Lemma 6.4 (`negKernel_negPart_le`). *For $\eta \ge 1$ and $0 < u < 1$, $\big(-\kappa_{-\eta}(u)\big)_+ \le (\eta-1) + u + \frac{1}{2}(\pi\eta u)^2$.*

Proof. Let $t = \pi\eta u \ge 0$. Then $-\kappa_{-\eta}(u) = -(1-u)\cos t + \rho(u)\sin t$. Use $\cos t \ge 1 - \frac{t^2}{2}$, $\sin t \le t$, and $\rho(u)\, t = \eta\, q(u) \le \eta$. This gives $-\kappa_{-\eta}(u) \le -(1-u) + (1-u)\frac{t^2}{2} + \eta \le (\eta-1) + u + \frac{t^2}{2}$, and the right-hand side is nonnegative. □

Lemma 6.5 (`primaryLoss_le_neg`, `primaryLoss_le_large_real`). *Let $a > 0$. For $\eta \ge 1$,*

$$\Lambda(a,-\eta) \le (\eta-1)\varphi_0\Big(\frac{a^3}{4} + \frac{3a^5}{4}\Big) + \frac{a^4}{\pi^2}\Big(\frac{2}{3} + \frac{12a^2}{5}\Big) + \eta^2\varphi_0\Big(\frac{5a^5}{8} + \frac{21a^7}{8}\Big),$$

and for every $\xi \in \mathbb{R}$, $\Lambda(a,\xi) \le \frac{1}{4}\varphi_0 a^3 + \frac{a^2}{6\pi} + \frac{3}{4}\varphi_0 a^5 + \frac{2a^4}{5\pi}$.

Proof. For $0 \le s \le 1/a$, Goyal's profile bound (Lemma A.2; `cosine_profile_loss_bound` in [3]) gives

$$0 \le g(s) - \cos(as)^{1/a^2} \le g(s)\Big(\frac{a^2 s^4}{12} + \frac{a^4 s^6}{20}\Big).$$

For the first bound, multiply by Lemma 6.4 at $u = as/\pi$ and integrate, using the half moments $\int_0^\infty s^k g(s)\,\mathrm{d}s = 3\pi\varphi_0,\ 8,\ 15\pi\varphi_0,\ 48,\ 105\pi\varphi_0$ for $k = 4,\dots,8$. For the second, use $|\kappa_\xi(u)| \le 1 + \frac{1}{\pi u}$ (Lemma A.1(b)) and the moments for $k = 3,\dots,6$. □

Theorem 6.6 (`lowerFourier_small`). *Let $0 < a \le \frac{1}{10}$, $x \ge \frac{3}{2}a$ and $c(a) < e^{-x^2/2}$. Then $J(a,-x) \le c(a) - c(x)$.*

Proof. Let $\eta = x/a \ge \frac{3}{2}$. By Lemmas 6.1 and 6.2 at $y = -x$, and the parity of φ, D and N,

$$J(a,-x) - c(a) + c(x) \le \Gamma_- + \Lambda(a,-\eta) + \Big(A(a,-\eta) - \frac{a^2}{\pi^2}\Big) + \frac{a^4}{15\pi} + 9e^{-1/(2a^2)}, \tag{6.2}$$

where $\Gamma_- = \Gamma(a,-x) = a\varphi(x) - c(a) + \frac{1}{3}a^2 x\varphi(x) + \frac{1}{\pi^2}a^2 x D(x) - \frac{1}{3\pi^2}a^3 N(x)$. Since $e^{-1/(2a^2)} \le 8!\,(2a^2)^8 = 10\,321\,920\,a^{16}$, the last two terms are at most $a^4/40$ for $a \le \frac{1}{10}$ (`gaussian_remainder_budget_tenth`). The alias term is bounded by Lemma 6.3. There are two cases.

Case $x \le 1$. We bound every term by $\varphi_0 a^3$ times a polynomial in a and η:

- $a\varphi(x) - c(a) \le \varphi_0 a^3(\frac{1}{6} - \frac{\eta^2}{2} + \frac{a^2\eta^4}{8})$, from $g(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{8}$ and $c(a) \ge \varphi_0(a - \frac{a^3}{6})$;
- $\frac{1}{3}a^2 x\varphi(x) \le \varphi_0 a^3 \frac{\eta}{3}$; $\frac{1}{\pi^2}a^2 x D(x) \le \frac{a^4\eta^2}{\pi^2}$, since $0 \le D(x) \le x$; and $N(x) \ge 0$ for $0 \le x \le 1$;
- $\frac{1}{\pi^2} \le \frac{100}{351}\varphi_0$, $\frac{\varphi_0^2}{2} \le \frac{\varphi_0}{5}$, $\frac{6}{\pi^4} \le \frac{200}{1053}\varphi_0$ and $\frac{1}{40} \le \frac{5}{78}\varphi_0$.

With Lemmas 6.3 and 6.5 and $\eta_0 \le \frac{5}{6}$, the right-hand side of (6.2) is at most $\varphi_0 a^3 P(a,\eta)$, where

$$P(a,\eta) = \Big(\frac{1}{6} - \frac{\eta^2}{2} + \frac{a^2\eta^4}{8}\Big) + \frac{\eta}{3} + (\eta-1)\Big(\frac{1}{4} + \frac{3a^2}{4}\Big) + \eta^2\Big(\frac{5a^2}{8} + \frac{21a^4}{8}\Big) + a\Big[\frac{100}{351}\Big(\eta^2 + \frac{2}{3} + \frac{12a^2}{5}\Big) + \frac{1}{5}\Big(\eta + \frac{5}{6}\Big)^2 + \frac{200}{1053}(\eta+1) + \frac{5}{78}\Big].$$

At $a = 0$ the first line equals $-\frac{1}{12}(6\eta-1)(\eta-1)$. An elementary argument (`lowerSmallPoly_le`, by `nlinarith`) shows $P \le 0$ whenever $0 \le a \le \frac{1}{10}$, $\eta \ge \frac{3}{2}$ and $a\eta \le 1$.

Case $x > 1$. First, $ax^2 \le \frac{4}{5}$ (`mul_sq_le_of_lt_exp`). Otherwise $t = \frac{2}{5a} \ge 4$ satisfies $\frac{x^2}{2} > t$, and $e^t \ge \frac{25}{3}t$ gives $e^{-x^2/2} < \frac{3}{25t} = \frac{3a}{10} \le c(a)$, a contradiction. In particular $ax \le \frac{4}{5}$. Now use the second bound of Lemma 6.5 together with

$$a\varphi(x) - c(a) \le -\tfrac{117}{800}a + \tfrac{1}{15}a^3, \quad \tfrac{1}{3}a^2x\varphi(x) \le \tfrac{1}{12}a^2, \quad \tfrac{1}{\pi^2}a^2xD(x) \le \tfrac{2}{9}a^2, \quad -\tfrac{1}{3\pi^2}a^3N(x) \le \tfrac{2}{45}a^3.$$

These follow from $\varphi(x) \le \varphi(1) \le \varphi_0 - \frac{117}{800}$, $xg(x) \le \frac{5}{8}$, $\varphi_0 \le \frac{2}{5}$, $xD(x) = 1 - M(x) \le 2$ and $|N| \le \pi\varphi_0$. By $ax^2 \le \frac{4}{5}$ and $ax \le \frac{4}{5}$, the two alias terms are at most $\frac{2}{25}(\frac{4}{5}a + \frac{4}{3}a^2 + \frac{25}{36}a^4)$ and $\frac{2}{27}(\frac{4}{5}a^2 + a^4)$. Altogether the right-hand side of (6.2) is at most a times

$$-\frac{117}{800} + \frac{8}{125} + a\Big(\frac{1}{12} + \frac{2}{9} + \frac{1}{18} + \frac{8}{75} + \frac{8}{135}\Big) + a^2\Big(\frac{1}{15} + \frac{2}{45} + \frac{1}{10}\Big) + a^3\Big(\frac{2}{15} + \frac{1}{18} + \frac{2}{27} + \frac{1}{40}\Big) + \frac{3a^4}{10},$$

which is negative for $a \le \frac{1}{10}$. □

Remark 6.7. Section 8 of [6] also proves the lower half for $a_1 \le 0.1$ analytically, with the same frequency π/a_1, through a different explicit bound. There the final polynomial inequalities are checked in ball arithmetic on 8200 and 2000 subintervals, and the range $a_1 < x \le 1.8a_1$ uses a local concentration bound [6, Lemma 4.2]. Here the polynomial inequalities are proved symbolically, and the range $a < x \le \frac{3}{2}a$ is handled by peeling (Proposition 6.8).

6.3. **The lower half near the diagonal.**

Proposition 6.8 (`peelLNear_small`)**.** *If* $0 < a \le \frac{3}{10}$ *and* $a < x \le \frac{3}{2}a$*, then* PeelL(a, x) *holds.*

Proof. Let $\sigma = \sqrt{1 - a^2}$ and $d = x - a \in (0, \frac{a}{2}]$. Since $\varphi \le \varphi_0$, $H(x) - H(a) \le 2d\varphi_0$. On the other side $\mathrm{lo}(a, x - a) \ge 0$, and since $\ell \le a/\sigma$ and φ decreases on $[0, \infty)$,

$$\mathrm{lo}(a, x + a) \ge H\Big(\frac{x+a}{\sigma}\Big) - H\Big(\frac{a}{\sigma}\Big) \ge \frac{2x}{\sigma}\,\varphi\Big(\frac{x+a}{\sigma}\Big) \ge 2x\varphi_0\Big(1 - \frac{(x+a)^2}{2\sigma^2}\Big),$$

where we used $\sigma \le 1$ and $g(v) \ge 1 - \frac{v^2}{2}$. It therefore suffices that $4d(1-a^2) \le x\big(2(1-a^2) - (x+a)^2\big)$ (`peelLNear_poly`). With $e = \frac{a}{2} - d \ge 0$, the difference of the two sides is

$$a\Big(1 - \frac{83}{8}a^2\Big) + 2e + 6a^2e + 5ae\Big(\frac{a}{2} + d\Big) + e\Big(\frac{a^2}{4} + \frac{ad}{2} + d^2\Big),$$

and every term is nonnegative for $a \le \frac{3}{10}$. □

6.4. **The upper half for** $a \le 0.07$**.**

We now prove Theorem 4.6, following [3]. Besides the lemmas above, the proof uses a bound for the negative part of κ_ξ when $\xi \ge 0$.

Lemma 6.9 (`upperKernel_negativePart_le`, `primaryLoss_le_small` from [3])**.** *For* $\xi \ge 0$ *let* $k(\xi) = \frac{\xi^2}{2} + \frac{\xi^3}{6} + \frac{\xi}{2}$ *(`kernelLossCoeff`). Then* $\big(-\kappa_\xi(u)\big)_+ \le k(\xi)\,(\pi u)^2$ *for* $0 < u < 1$*, and*

$$\Lambda(a, \xi) \le \varphi_0 a^5\Big(\frac{5}{4} + \frac{21a^2}{4}\Big)k(\xi) \qquad (a > 0).$$

Proof. Let $y = \pi u$ and $h = \rho(u) - \frac{1}{y}$. Then $-\frac{y}{2} \le h \le 0$ by Lemma A.1(b),(c), and

$$\kappa_\xi(u) = (1 - u)\cos(\xi y) + \frac{\sin(\xi y)}{y} + h\sin(\xi y).$$

Since $\xi y \ge 0$, we have $\cos(\xi y) \ge 1 - \frac{\xi^2 y^2}{2}$ and $\sin(\xi y) \ge \xi y - \frac{\xi^3 y^3}{6}$; and $h\sin(\xi y) \ge h\,\xi y \ge -\frac{\xi y^2}{2}$, because $h \le 0$ and $\sin(\xi y) \le \xi y$. Hence

$$\kappa_\xi(u) \ge (1 - u)\Big(1 - \frac{\xi^2 y^2}{2}\Big) + \xi - \frac{\xi^3 y^2}{6} - \frac{\xi y^2}{2} \ge 1 - u + \xi - k(\xi)\,y^2,$$

and $1-u+\xi > 0$. This proves the first claim. For the second, let $0 < s < 1/a$ and $u = as/\pi$, so that $\pi u = as$. By the first claim and Lemma A.2, the integrand of $\Lambda(a,\xi)$ is at most $k(\xi)\, a^2 s^2\, g(s)\,(\frac{a^2 s^4}{12} + \frac{a^4 s^6}{20})$. Integrating over $(0,\infty)$, with $\int_0^\infty s^6 g(s)\,\mathrm{d}s = 15\pi\varphi_0$ and $\int_0^\infty s^8 g(s)\,\mathrm{d}s = 105\pi\varphi_0$, gives the bound. □

Proof of Theorem 4.6. Let $0 < a \le 0.07$ and $x \ge 0$. As in [3], we consider three ranges of x.

The window $x \le a/2$. This is the case $a \le 0.07$ of Theorem 5.1. Goyal proves the case $a \le 0.07$ of Theorem 5.2 by a different argument (`holderProfile_le_c_pair` in `ZoneATheorem.lean` [3]), and the Lean proof of Theorem 4.6 uses his argument, through `rademacher_concentration_zoneA`.

The far range $ax^2 \ge 1$. Then $x^2 \ge \frac{1}{a} > 1$, so $x > 1 > \varphi_0$. Hence

$$\frac{1}{2} - c(x) = \int_x^\infty \varphi(s)\,\mathrm{d}s \le \int_x^\infty \frac{s}{x}\,\varphi(s)\,\mathrm{d}s = \frac{\varphi(x)}{x} \le g(x) \le e^{-1/(2a)} \le 48a^3 \le \frac{a}{3} \le c(a).$$

Here we used $x^2 \ge \frac{1}{a}$; $e^v \ge \frac{v^3}{6}$ with $v = \frac{1}{2a}$; $a^2 \le \frac{1}{144}$; and $c(a) \ge \varphi_0(a - \frac{a^3}{6}) \ge \frac{a}{3}$, which holds because $\varphi_0 > 0.39$. Therefore $H(x) + H(a) \ge 1 \ge \mathbb{P}(|X| \le x)$ (`one_le_H_add_H_of_large`).

The main range $x \ge a/2$, $ax^2 \le 1$. Let $\xi = x/a \ge \frac{1}{2}$ and $\varepsilon = e^{-1/(2a^2)}$. By Proposition 3.4 it suffices to prove $J(a,x) \le c(a) + c(x)$. By Lemmas 6.1 and 6.2 at $y = x$,

$$J(a,x) - c(a) - c(x) \le \Gamma(a,x) + \Lambda(a,\xi) + \Big(A(a,\xi) - \frac{a^2}{\pi^2}\Big) + \frac{a^4}{15\pi} + \Big(7 + \frac{a^2}{2\pi}\Big)\varepsilon.$$

Here $\frac{a^4}{15\pi} + (7 + \frac{a^2}{2\pi})\varepsilon \le \frac{a^4}{45} + 9\varepsilon$, and $\varepsilon \le 6!\,(2a^2)^6 = 46\,080\,a^{12}$ gives $9\varepsilon \le \frac{a^4}{1000}$ for $a \le 0.07$ (`gaussian_remainder_budget`). Since $\frac{1}{45} + \frac{1}{1000} < \frac{1}{40}$, Lemma 6.3 at $\eta = \xi$ gives

$$J(a,x) - c(a) - c(x) \le \Gamma(a,x) + \Lambda(a,\xi) + \frac{\varphi_0^2}{2}(\xi + \eta_0)^2 a^4 + \frac{6(\xi+1)a^4}{\pi^4} + \frac{a^4}{40} \tag{6.3}$$

(`JUpper_le_master_base`). Recall that $\Gamma(a,x) = a\varphi(x) - c(a) - \frac{1}{3}a^2 x\varphi(x) + \frac{1}{\pi^2}a^2 x D(x) + \frac{1}{3\pi^2}a^3 N(x)$.

Case $x \le 1$. Then $a\xi \le 1$. We bound every term on the right of (6.3) by $\varphi_0 a^3$ times a polynomial in a and ξ, using $x = a\xi$:

- $a\varphi(x) - c(a) - \frac{1}{3}a^2 x\varphi(x) \le \varphi_0 a^3(\frac{1}{6} - \frac{\xi^2}{2} - \frac{\xi}{3} + a^2(\frac{\xi^4}{8} + \frac{\xi^3}{6}))$, from $1 - \frac{x^2}{2} \le g(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{8}$ and $c(a) \ge \varphi_0(a - \frac{a^3}{6})$;
- $\frac{1}{\pi^2}a^2 x D(x) \le \frac{a^4\xi^2}{\pi^2}$ and $\frac{1}{3\pi^2}a^3 N(x) \le \frac{2a^4\xi}{3\pi^2}$, because $D(x) \le x$ and $N(x) \le 2x$, by $|\sin(sx)| \le sx$, $\int_0^\infty s\,g(s)\,\mathrm{d}s = 1$ and $\int_0^\infty s^3 g(s)\,\mathrm{d}s = 2$;
- $\Lambda(a,\xi) \le \frac{32}{25}\varphi_0 a^5 k(\xi)$, by Lemma 6.9 and $\frac{5}{4} + \frac{21a^2}{4} \le \frac{32}{25}$;
- $\frac{1}{\pi^2} \le \frac{100}{351}\varphi_0$, $\frac{2}{3\pi^2} \le \frac{200}{1053}\varphi_0$, $\frac{\varphi_0^2}{2} \le \frac{\varphi_0}{5}$, $\frac{6}{\pi^4} \le \frac{200}{1053}\varphi_0$, $\frac{1}{40} \le \frac{5}{78}\varphi_0$ and $\eta_0 \le \frac{5}{6}$, which follow from $\pi^2 \ge 9$ and $0.39 \le \varphi_0 \le \frac{2}{5}$.

Hence the right-hand side of (6.3) is at most $\varphi_0 a^3 P_+(a,\xi)$ (`masterSmall_scaled_le`), where

$$P_+(a,\xi) = \frac{1}{6} - \frac{\xi^2}{2} - \frac{\xi}{3} + a^2\Big(\frac{\xi^4}{8} + \frac{\xi^3}{6} + \frac{32}{25}k(\xi)\Big) + a(C_2\xi^2 + C_1\xi + C_0)$$

with $C_2 = \frac{100}{351} + \frac{1}{5}$, $C_1 = \frac{400}{1053} + \frac{1}{3}$ and $C_0 = \frac{5}{36} + \frac{200}{1053} + \frac{5}{78}$. We show $P_+ \le -\frac{1}{20}$ (`smallPolynomial_le`). Below, the coefficients increase with a, so we bound them at $a = 0.07$; for instance $\frac{1}{6} + aC_0 \le \frac{1}{5}$. If $\xi \le 1$, then $\xi^4 \le \xi^2$ and $\xi^3 \le \xi^2 \le \xi$, so

$$P_+ \le \frac{1}{6} + aC_0 + \Big(-\frac{1}{2} + aC_2 + \frac{587}{600}a^2\Big)\xi^2 + \Big(-\frac{1}{3} + aC_1 + \frac{121}{150}a^2\Big)\xi \le \frac{1}{5} - \frac{9}{20}\xi^2 - \frac{11}{40}\xi \le -\frac{1}{20},$$

where the last step uses $\xi \ge \frac12$. If $\xi \ge 1$, then $a^2\xi^4 \le \xi^2$, $a^2\xi^3 \le \xi$ and $a^2\xi^3 \le a\xi^2$, because $a\xi \le 1$; so

$$P_+ \le \frac16 + aC_0 + \Big(-\frac38 + aC_2 + \frac{16}{75}a + \frac{16}{25}a^2\Big)\xi^2 + \Big(-\frac16 + aC_1 + \frac{16}{25}a^2\Big)\xi \le \frac15 - \frac{3}{10}\xi^2 - \frac{1}{10}\xi \le -\frac15.$$

In both cases the right-hand side of (6.3) is at most $-\varphi_0 a^3/20 < 0$ (`masterSmall_le`).

Case $x > 1$. Then $ax \le ax^2 \le 1$. We drop the term $-\frac13 a^2 x\varphi(x) \le 0$ and use

$$a\varphi(x) - c(a) \le -\tfrac{117}{800}a + \tfrac{1}{15}a^3, \qquad \tfrac{1}{\pi^2}a^2 xD(x) \le \tfrac29 a^2, \qquad \tfrac{1}{3\pi^2}a^3 N(x) \le \tfrac{2}{45}a^3.$$

These follow from $\varphi(x) \le \frac58\varphi_0 \le \varphi_0 - \frac{117}{800}$ (since $g(x) \le e^{-1/2} \le \frac58$ and $\varphi_0 \ge 0.39$), $c(a) \ge \varphi_0(a - \frac{a^3}{6})$, $\varphi_0 \le \frac25$, $xD(x) = 1 - M(x) \le 2$, $|N| \le \int_0^\infty s^2 g(s)\,\mathrm{d}s = \pi\varphi_0$ and $\pi > 3$. Since $(\xi+\eta_0)^2a^4 = a^2x^2 + 2\eta_0 a^3 x + \eta_0^2 a^4$ and $(\xi+1)a^4 = a^3x + a^4$, the bounds $ax^2 \le 1$, $ax \le 1$, $\eta_0 \le \frac56$, $\frac{\varphi_0^2}{2} \le \frac{2}{25}$ and $\pi^4 \ge 81$ give

$$\frac{\varphi_0^2}{2}(\xi+\eta_0)^2a^4 \le \frac{2}{25}\Big(a + \frac53 a^2 + \frac{25}{36}a^4\Big), \qquad \frac{6(\xi+1)a^4}{\pi^4} \le \frac{2}{27}(a^2 + a^4).$$

With these bounds, and the second bound of Lemma 6.5 together with $\varphi_0 \le \frac25$ and $\pi > 3$, the right-hand side of (6.3) is at most $a(-\frac{117}{800} + R_+(a))$ (`masterLarge_le`), where

$$R_+(a) = \frac{2a}{9} + \frac{a}{18} + \frac{a^2}{15} + \frac{2a^2}{45} + \frac{a^2}{10} + \frac{2a^3}{15} + \frac{3a^4}{10} + \frac{2}{25}\Big(1 + \frac{5a}{3} + \frac{25a^3}{36}\Big) + \frac{2}{27}(a + a^3) + \frac{a^3}{40}$$

(`largeRemainder`). The polynomial R_+ is increasing on $[0,\infty)$, and $R_+(0.07) < 0.1152 < \frac{117}{800} - \frac{1}{40}$. So the right-hand side of (6.3) is at most $-a/40 < 0$ (`largeRemainder_le`).

In both cases $J(a,x) \le c(a) + c(x)$ (`JUpper_le_c_pair_of_zoneB`), and Proposition 3.4 gives $\mathbb{P}(|X| \le x) \le H(a) + H(x)$ (`rademacher_concentration_zoneB`). □

7. The master decomposition

The computer-assisted inputs (P3) and (P4) (the latter for $a \ge 0.1$) are inequalities $J(a,y) \le c(a) + c(y)$, at $y = x \ge a/2$ and at $y = -x \le -\frac32 a$. Both sides are at least of order a, while the margin is of order a^3: in floating point it is at least about $0.30a^3$ on the range of (P3) and $0.26a^3$ on that of (P4) (these numbers are not used in the proof). Enclosing J directly would therefore require relative accuracy of order a^2. Instead, we remove the Gaussian part of J in closed form, as in Section 6, and move the Gaussian moment term $a^2M(y)/\pi^2$ to the range $\theta/\pi < u < 1$, where it compensates the contribution of $|\varphi_X|$ near $u = 1$. What remains is the integral of an explicit function of the size of the margin.

Let $0 < a \le 1$, $y \in \mathbb{R}$ and $\xi = y/a$. For $0 \le t < \frac{\pi}{2}$ let $E(t) = -\log\cos t - \frac{t^2}{2}$ (`Num.Efun`); then $E \ge 0$ and $\cos(t)^{1/a^2} = g(t/a)\,e^{-E(t)/a^2}$. Let $Q(t) = \cot t - \frac1t + \frac t3$ as in (6.1). For $0 < u < 1$ write

$$K = \kappa_\xi(u), \qquad w_1 = g\Big(\frac{\pi u}{a}\Big), \qquad w_2 = g\Big(\frac{\pi(1-u)}{a}\Big),$$

and define (`envelopeDifference`, `cotU`, `mU`, `gU`)

$$\Delta(u) = \max\Big\{K\,L_a\big(\tfrac{\pi u}{a}\big),\ K\,U_a\big(\tfrac{\pi u}{a}\big)\Big\} - K\,w_1, \qquad \gamma(u) = (1-u)\,Q(\pi u)\,\sin(\pi\xi u)\,w_1,$$
$$\mu(u) = (1-u)\cos\big(\pi\xi(1-u)\big)\,w_2, \qquad \mathcal{G}(u) = \Delta(u) + \gamma(u) - \mathbf{1}_{\{u > \theta/\pi\}}\,\mu(u).$$

Finally let (`masterTarget`, `tailBound`, `aliasTailBound`)

$$\Theta(a,y) = c(a) - a\varphi(y) + \tfrac{1}{3}a^2 y\varphi(y) - \frac{a^3 N(y)}{3\pi^2},$$
$$\tau(a) = a^2\Big(\frac{2}{\pi^2} + \frac{1}{\pi^3} + \frac{2}{3\pi} + \frac{a^2}{3\pi^3}\Big)\, g\Big(\frac{\pi}{a}\Big), \qquad \tau'(a) = \frac{a^2}{\pi^2}\, g\Big(\frac{\pi-\theta}{a}\Big).$$

Theorem 7.1 (`master_le`). *For $0 < a \le 1$ and $y \in \mathbb{R}$,*

$$J(a,y) - c(a) - c(y) \le \int_0^1 \mathcal{G}(u)\,\mathrm{d}u - \Theta(a,y) + \tau(a) + \tau'(a).$$

Proof. By definition, $J(a,y) = \int_0^1 \Delta + S(a,y)$, with S as in Section 6. The proof of Lemma 6.2 rests on the identity (`scaledGaussianKernelIntegral_eq`)

$$S(a,y) = B - T_1 + \int_0^1 \gamma(u)\,\mathrm{d}u,$$

where B is the closed-form value obtained there, T_1 is the integral of the explicit terms of (6.1) over $s > \pi/a$, and the last integral is the contribution of Q, written in the variable $u = as/\pi$. An explicit majorant of the explicit terms gives $|T_1| \le \tau(a)$ (`abs_baselineTail_le`). The substitution $u = 1 - as/\pi$ gives (`gaussianM_split`)

$$\frac{a^2}{\pi^2}M(y) = \int_{\theta/\pi}^1 \mu(u)\,\mathrm{d}u + T_2, \qquad T_2 = \frac{a^2}{\pi^2}\int_{(\pi-\theta)/a}^\infty s\cos(sy)\,g(s)\,\mathrm{d}s,$$

and $|T_2| \le \frac{a^2}{\pi^2}\int_{(\pi-\theta)/a}^\infty s\,g(s)\,\mathrm{d}s = \tau'(a)$ (`abs_aliasTail_le`). Since $B - c(a) - c(y) = -\Theta(a,y) - \frac{a^2}{\pi^2}M(y)$, we obtain the identity

$$J(a,y) - c(a) - c(y) = \int_0^1 \mathcal{G}(u)\,\mathrm{d}u - \Theta(a,y) - T_1 - T_2,$$

which implies the claim. □

On each envelope region, $\mathcal{G}$ has an explicit form (`gU_R1`, `gU_R2`, `gU_R3`):

$$\mathcal{G}(u) = (-K)_+\, w_1\big(1 - e^{-E(\pi u)/a^2}\big) + \gamma(u) \qquad \text{for } 0 < u < \tfrac{1}{2}, \tag{7.1}$$

$$\mathcal{G}(u) = (-K)_+\, w_1\big(1 + |\cos \pi u|\, e^{(\pi u)^2/2}\big) + \gamma(u) \qquad \text{for } \tfrac{1}{2} < u \le \tfrac{\theta}{\pi}, \tag{7.2}$$

$$\mathcal{G}(u) = |K|\, w_2\big(e^{-E(\pi(1-u))/a^2} - 1\big) + \max(b_+, b_-)\, w_2 - K\, w_1 + \gamma(u) \qquad \text{for } \tfrac{\theta}{\pi} < u < 1, \tag{7.3}$$

where

$$b_+(u) = -2(1-u)\sin\big(\tfrac{\pi\xi}{2}\big)\sin\big(\pi\xi(u - \tfrac{1}{2})\big) + \rho(u)\sin(\pi\xi u),$$
$$b_-(u) = -2(1-u)\cos\big(\tfrac{\pi\xi}{2}\big)\cos\big(\pi\xi(u - \tfrac{1}{2})\big) - \rho(u)\sin(\pi\xi u).$$

Indeed, for $u < \frac{1}{2}$ we have $U_a = g$ and $L_a = g\,e^{-E/a^2} \le U_a$ at $\pi u/a$, so $\Delta = (-K)_+(U_a - L_a)$. For $\frac{1}{2} < u \le \theta/\pi$ the same formula holds with $U_a - L_a = w_1(1 + |\cos \pi u| e^{(\pi u)^2/2})$. For $u > \theta/\pi$ we have $\max(KL_a, KU_a) = |K|\,|\cos \pi u|^{1/a^2} = |K|\, w_2\, e^{-E(\pi(1-u))/a^2}$, because $|\cos \pi u| = \cos(\pi(1-u))$. By the sum-to-product formulas, $b_+ = K - (1-u)\cos(\pi\xi(1-u))$ and $b_- = -K - (1-u)\cos(\pi\xi(1-u))$, so $\max(b_+, b_-) = |K| - (1-u)\cos(\pi\xi(1-u))$ (`abs_upperKernel_sub`). This gives (7.3).

Remark 7.2. Every term of $\mathcal{G}$ is small. On $u < \frac{1}{2}$ the envelope loss $1 - e^{-E(\pi u)/a^2} \le E(\pi u)/a^2$, with $E(t) = O(t^4)$, is weighted by $w_1 = e^{-\pi^2 u^2/(2a^2)}$, so it contributes $O(a^3)$ to the integral; for $u > \frac{1}{2}$, $w_1 \le e^{-\pi^2/(8a^2)}$. Since $Q(t) = O(t^3)$, the term γ contributes $O(a^4)$. Near $u = 1$, where $|\varphi_X|$ may return close to 1, the first term of (7.3) is nonpositive, and $\rho(u) = O((1-u)^2)$ gives

$$\max(b_+, b_-) = -(1-u)\big(1 - |\cos \pi\xi|\big) + O\big((1-u)^2\big).$$

Since $\int_{\theta/\pi}^{1}(1-u)\,w_2\,\mathrm{d}u \approx a^2/\pi^2$, the transported term μ turns the alias contribution $\frac{a^2}{\pi^2}|\cos\pi\xi|$ of J into $-\frac{a^2}{\pi^2}(1-|\cos\pi\xi|) \le 0$, to leading order. This is Goyal's compensation (Lemma 6.3), carried out pointwise in u. On the other side, $c(a) \ge \varphi_0(a - a^3/6)$ gives

$$c(a) - a\varphi(y) \ge -\frac{\varphi_0 a^3}{6} + a\varphi_0(1-g(y)),$$

a lower bound without cancellation. So both sides of the inequality $\int_0^1 \mathcal{G} \le \Theta - \tau - \tau'$ are of order a^3, and enclosures with relative accuracy of order one suffice.

8. Certified numerics in the Lean kernel

This section describes the checks behind (P1), (P3), (P4) for $a \ge 0.1$, (P5), (P6) for $a \ge 0.3$, and (P7). Each check is a function with values in `Bool`, defined by structural recursion on natural numbers and lists, together with a soundness theorem: if the function returns `true` on given data, then a stated real inequality holds. That it returns `true` is proved by `decide +kernel`, which asks the kernel to reduce the closed Boolean term to `true`. The kernel does this with its built-in arbitrary-precision arithmetic on natural-number literals; no compiled code is run.

8.1. **Ball arithmetic.** A ball is a triple (p, n, r) of natural numbers. It stands for the interval of real numbers x with $|2^{64}x-(p-n)| \le r$ (`Num/Ball.lean`). Sums and negatives are exact. Products are rescaled by 2^{-64} with truncation, and the radius absorbs the propagated radii and two units of truncation error; products of wide balls are formed from the four products of endpoints (`imul`). Every operation has a soundness lemma (`mem_add`, `mem_mul`, `mem_imul`, ...): if the inputs contain x and y, the output contains the corresponding real result. A comparison returns `true` only if it holds for all elements of the balls (`le_of_leB`).

8.2. **Elementary functions.** The enclosures are in `Num/Series.lean`, `Num/Elem.lean` and `Num/GaussCDF.lean`.

- π is a ball of radius 2^{-64}, from Mathlib's 20-digit bounds (`pi_mem`).
- $\cos(\pi c)$ and $\sin(\pi c)$ at a dyadic point c: reduction by quarter periods to an argument $|z| \le 1$, then Taylor polynomials with 12 terms, whose error is at most 2^{-64} by two-sided Taylor bounds. On a ball with centre c and radius r, the addition formulas are applied with $\cos(\pi\delta) \in [1-\frac{1}{2}(\pi r)^2, 1]$ and $|\sin\pi\delta| \le \pi r$ (`trigBox_mem`).
- e^x: the argument is halved until its modulus is at most $\frac{1}{4}$, the Taylor polynomial with 16 terms is evaluated (error at most 2^{-64}), and the result is squared back. Arguments ≤ -45 give a ball of radius 2^{-64} about 0 (`mem_expB`).
- $\log x$ for $0 < x \le 1$: with $w = 1 - 2^k x$, where k is chosen so that $|w| \le 0.51$, $\log x = -k\log 2 - \sum_{i\ge1} w^i/i$, summed to 70 terms (`mem_logB`). The ball for $\log 2$ is certified by one more `decide +kernel` proof (`LN2_check`).
- Square roots (used only in the peeling checks): Newton's iteration with a fixed number of steps, whose result r is accepted only after checking $r^2 \le m$, respectively $m < (r+1)^2$ (`mem_sqrtKB`). Reciprocals of balls not containing 0 (`mem_invB`).
- $c(t)$ for $t \ge 0$, from $c(t) = \varphi_0 \sum_{k\ge0}(-1)^k t^{2k+1}/(2^k k!\,(2k+1))$. The partial sums with 40 and 41 terms bracket $c(t)/\varphi_0$, because the Taylor polynomials of e^{-v}, $v \ge 0$, alternate around it (`mem_cB`). The constant φ_0 is a fixed ball, certified by squaring its endpoints (`phi0_mem`).

8.3. **The cell bound.** A box consists of an interval $a \in [a_0, a_1]$, an interval $|\xi| \in [m_0, m_1]$ with $m_0 > 0$, and a sign: $y = a\xi$ for (U) and $y = -a\xi$ for (L). It comes with a grid $0 = u_0 < u_1 < \cdots < u_K = 1$ of dyadic points. For each cell (u_{k-1}, u_k), the function `cellHi` (`Num/CellDefs.lean`) computes an upper bound for $\mathcal{G}(u)$ that is uniform over the cell and the box. It uses (7.1) if $u_k \le \frac{1}{2}$; (7.2) if $u_{k-1} \ge \frac{1}{2}$ and $\pi u_k \le 1.7$; (7.3) if $\pi u_{k-1} \ge 1.8$; and the larger of the bounds

from (7.2) and (7.3) otherwise. Recall that $1.7 < \theta < 1.8$. The ingredients are enclosures of the following quantities.

- The kernel K: the intersection of the enclosures of $(1-u)\cos(\pi\xi u) + \xi\, q(u)\,\frac{\sin(\pi\xi u)}{\pi\xi u}$, which remains bounded as $u \to 0$, and of $(1-u)\cos(\pi\xi u) + \rho(u)\sin(\pi\xi u)$. Here q and ρ are enclosed by values of δ at the cell endpoints, using the monotonicity of δ on $(0,\pi)$ and the identities $q(u) = 1-(1-u)\delta(\pi u) = u\,\delta(\pi(1-u))$ and $\rho(u) = \delta(\pi(1-u))/\pi$.
- The Gaussian factors w_1 and w_2, by monotonicity in u and in a; the factors e^{-E/a^2}, by monotonicity of E; and $|\cos \pi u|\, e^{(\pi u)^2/2}$, which increases on $(\frac{1}{2}, 1)$.
- $\max(b_+, b_-)$, through its product formulas, which avoid the cancellation in $|K| - (1-u)\cos(\pi\xi(1-u))$.
- The cotangent term. Since $(1-u)Q(\pi u) \le 0$ (Lemma A.1(d)),

$$\gamma(u) \le \big|(1-u)Q(\pi u)\big|\, \max(0, -\sin \pi\xi u)\, w_1.$$

  The first factor is bounded by $(\pi u_k)^3/30$ if $\pi u_k \le 1$ (Lemma A.1(d)), and otherwise by the identity $|(1-u)Q(\pi u)| = \frac{1}{\pi u} - \frac{(1-u)\pi u}{3} - \rho(u)$, evaluated at the cell endpoints.

The soundness theorem is `gU_le_cellHi`: $\mathcal{G}(u)$ is at most the value of `cellHi` for every u in the open cell and every (a,ξ) in the box.

8.4. **The box check.** The function `boxOk` adds up the cell bounds weighted by the cell lengths, adds an upper bound for $\tau(a) + \tau'(a)$ on $[a_0, a_1]$ (with θ replaced by 1.8, which is allowed because g decreases), and compares the sum with a lower bound for $\Theta(a,y)$ on the box. For the latter, Remark 7.2 gives $c(a) - a\varphi(y) \ge -\varphi_0 a_1^3/6 + a_0\varphi_0(1 - g(a_0 m_0))$; the term $\frac{1}{3}a^2 y\varphi(y)$ is bounded using the monotonicity of each factor; and N is bounded by $|N| \le \pi\varphi_0$ and, for $0 \le x \le 1$, by $x - (1-x^2)\pi\varphi_0 \le N(x) \le 2x$. The lower bound follows from $N = x + (1-x^2)D$ and $|D| \le \pi\varphi_0$, the upper bound from $\sin t \le t$ and $\int_0^\infty s^3 g(s)\,\mathrm{d}s = 2$. By Theorem 7.1, a successful box check gives $J(a,y) \le c(a) + c(y)$ for all (a,ξ) in the box (`boxOk_master`, `boxOk_upper`, `boxOk_lower`).

8.5. **Covers.** A group $(k_0, k_1, n, [m_0, \dots, m_K])$ consists of an interval $a \in [k_0 2^{-32}, k_1 2^{-32}]$, a number n of grid cells per lobe, and breakpoints $m_0 2^{-16} < \cdots < m_K 2^{-16}$ of $|\xi|$, where $m_0 2^{-16} = \frac{1}{2}$ for (U) and $\frac{3}{2}$ for (L) (`groupOk`). All boxes of a group use the same grid (`gridGen`): $n = 40$ equal cells on each of $[0, L]$ and $[1-L, 1]$, where L is $\min(3.2a_0, \frac{1}{2})$ rounded down to a multiple of 2^{-20}; the middle cell; and the additional points $\frac{1}{2}$, 0.541015625 and 0.572998046875, which bracket $\theta/\pi \in (1.7/\pi, 1.8/\pi)$. Each box has 82 or 84 cells. Beyond the last breakpoint, an end condition at $x_e = a_0 m_K 2^{-16}$ shows that the remaining x lie outside the range of (P3) or (P4) (`endOk`):

- for (U), $g(x_e)/x_e \le a_0 - a_0^3/6$. With the Mills ratio bound $1 - \Phi(x) \le \varphi(x)/x$ and $c(a) \ge \varphi_0(a - a^3/6)$, this gives $H(x) + H(a) \ge 1$ for $x \ge a\, m_K 2^{-16}$;
- for (L), $g(x_e) \le \varphi_0(a_0 - a_0^3/6)$, which gives $e^{-x^2/2} \le c(a)$ for $x \ge a\, m_K 2^{-16}$.

The boxes of a group are checked in slices of at most six consecutive boxes, with one `decide +kernel` proof per slice, because the kernel keeps all intermediate terms of a declaration in memory. The end condition is one more `decide +kernel` proof. Slices are glued in ξ (`sliceU_glue`, `sliceL_glue`), and groups with adjacent a-intervals are glued in a (`cover_glue`). The resulting theorems `upper_certified` and `lower_certified` state:

- $J(a,x) \le c(a) + c(x)$ whenever $k2^{-32} \le a \le \frac{1}{2}$, $x \ge a/2$ and $H(x) + H(a) < 1$, where $k = 300\,647\,710$, so that $k2^{-32} < 0.07$;
- $J(a,-x) \le c(a) - c(x)$ whenever $k'2^{-32} \le a \le k''2^{-32}$, $x \ge \frac{3}{2}a$ and $c(a) < e^{-x^2/2}$, where $k' = 429\,496\,729$ and $k'' = 2\,362\,232\,013$, so that $k'2^{-32} < 0.1$ and $k''2^{-32} > 0.55$.

Restricted to decimal ranges, they are `upperFourier_certified` and `lowerFourier_certified`, that is, (P3), and (P4) for $0.1 \le a \le 0.55$.

8.6. **The peeling inequalities.** The inequalities (P5)–(P7) involve only H, g and $\sigma(a) = \sqrt{1-a^2}$. They are checked on boxes $[a_0, a_1] \times [d_0, d_1]$, with $x = a + d$, for PeelL, and on boxes $[a_0, a_1] \times [x_0, x_1]$ for PeelU (`peelLBox`, `peelUBox`). Each box check evaluates the functions at corners, using monotonicity: σ decreases and ℓ increases with a; lower bounds for lo use $\sigma(a_0)$, the level at a_1 and the smallest argument; upper bounds for up use $\sigma(a_1)$, the level at a_1 and the largest argument; and $H(a+d) - H(a)$ decreases in $a \geq 0$ and increases in $d \geq 0$. In lower bounds, the arguments of H and g are capped at 4, where the series for c is still accurate. The soundness theorems are `peelLBox_sound` and `peelUBox_sound`. Groups are glued as before, with one `decide +kernel` proof for all boxes of a group and one or two for its end conditions: $g(a_0 + d_K) \leq c(a_0)$ for (P5); $2d_K \geq a_1$ for (P6); and $2x_0 \leq a_0$ and $H(x_K) + H(a_0) \geq 1$ for (P7). Finally, (P1) is the single check $c(0.675) \geq \frac{1}{4}$ (`c675_certified`).

input	range of a	groups	boxes	cell bounds	kernel proofs
(P3)	[0.07, 0.5]	67	790	65 736	225
(P4)	[0.1, 0.55]	58	913	75 584	229
(P5)	[0.55, 1]	5	20	–	10
(P6)	[0.3, 0.55]	8	31	–	16
(P7)	[0.5, 0.675]	6	54	–	18
(P1)	–	–	–	–	1
$\log 2$	–	–	–	–	1
total		144	1808	141 320	500

TABLE 3. The kernel-checked certificates. For (P3) and (P4) the kernel proofs are 158 and 171 slices of at most six boxes, plus one end condition per group.

8.7. **Finding the certificates.** The groups were found by search programs written in Lean (`lean/certs/search/FourierSearch.lean` and `lean/certs/search/PeelSearch.lean`). They run the same checkers with `#eval`, outside the kernel, and bisect the ξ-intervals of failing boxes. For (P3) and (P4) the a-intervals form a geometric sequence: most of these groups have relative width 0.03 in a, and none is narrower than 0.015. The search programs are not part of the proof. Their output is a list of numbers (`lean/certs/*.txt`), from which two scripts (`gen_lean.py` and `gen_peel.py`) generate the certificate modules; only the kernel's evaluation of these modules is trusted. Table 3 lists the sizes.

9. THE FORMALIZATION

9.1. **Structure.** The formalization is a Lake project for Lean 4.19.0 and Mathlib (commit `c44e0c8`), in the directory `lean`, which is distributed with this paper as `keller-klein-lean.zip` and listed file by file in Appendix B; build instructions are in `lean/README.md`. It has three parts.

- The 61 modules of Goyal's formalization [3] (about 6 900 lines), byte for byte as published. Three of them are adapted from the CausalSmith library [16].
- The modules in `KellerKlein/Full` (17 files, about 4 000 lines): the reduction, the Prawitz lower bound, Hoeffding's bound, peeling and the induction, the window near the origin, the small-coefficient lower half, the master decomposition, and the final theorem.
- The modules in `KellerKlein/Num` (12 files, about 4 700 lines): ball arithmetic, elementary functions, the cell and box checks with their soundness proofs, covers and peeling checks; and 33 generated certificate modules in `KellerKlein/Num/Cert` (about 2 500 lines).

The root module `KellerKlein.lean` imports the three modules that Goyal's root module imports; his two modules `FarTail` and `ZoneAAudit`, which no other module imports; and `KellerKlein/Full.lean`, through which all new modules are imported. So `lake build` compiles every module of the project, 125 in all. Table 4 maps the results of this paper to Lean declarations.

9.2. **Trusted base.** The module `Full/FinalAudit.lean` prints the axioms of the final theorems:

```
#print axioms KellerKlein.keller_klein
-- 'KellerKlein.keller_klein' depends on axioms: [propext, Classical.choice, Quot.sound]
```

The same holds for `keller_klein_max` and `pieces`. The sources contain no `sorry`, `admit`, `native_decide`, `implemented_by`, `extern` or `axiom` declaration; these words occur only in three docstrings of [3], which state that no axiom is added. The constant θ is defined by choice from a proved existence theorem.

All 500 computational steps are proofs by `decide +kernel`: 454 in `Num/Cert`, 45 in `Num/PeelCert.lean` and one in `Num/Elem.lean`. Such a proof is checked by the kernel's reduction of a closed Boolean term to `true`. The kernel uses its built-in arbitrary-precision arithmetic on natural-number literals, but neither the compiler nor `Lean.ofReduceBool` is involved. The trusted base therefore consists of the Lean kernel, including its arithmetic on natural numbers, and the three axioms above.

9.3. **Building.** On a machine with 8 cores and 47 GB of memory, downloading the Mathlib cache took 38 seconds and `lake build` took 18 minutes of wall-clock time (111 minutes of CPU time). The 32 certificate modules dominate. The largest, `Num/Cert/U00.lean`, takes about 3.3 minutes and peaks at 6.9 GB of resident memory. The axiom audit is part of the build.

9.4. **Independent replay.** The tool lean4checker [10] with the option `--fresh` re-checks every declaration in the environment of a module, including all of Mathlib, in a new kernel environment, and reports any declaration that the kernel rejects. With lean4checker at the tag `v4.19.0`, the run `lean4checker --fresh KellerKlein.Full.FinalAudit` finished without reporting an error (exit code 0). The module `KellerKlein.Full.FinalAudit` imports the final theorem and all of Mathlib, so this replays every declaration on which Theorem 1.1 depends, including the 500 kernel computations. On the machine above, for the build described in this section, the run took 2 hours and 15 minutes of wall-clock time and 132 minutes of CPU time on a single core, with about 6.9 GB of resident memory.

Table 4: Results of this paper and their Lean declarations. Paths are relative to `lean/KellerKlein`, and the declarations are in the namespace `KellerKlein` or in its subnamespaces `Num`, `ZoneA` and `ZoneB`. Files marked [3] are from Goyal's formalization and are used without change.

result	Lean declarations	file
Theorem 1.1	`keller_klein`, `keller_klein_max`	`Full/Final.lean`
axiom audit	`#print axioms` of the above and of `pieces`	`Full/FinalAudit.lean`
Definition 2.1, monotonicity in the level	`UpperHalf`, `LowerHalf`; `upperHalf_mono`, `lowerHalf_mono`	`Full/Reduction.lean`; `Full/Induction.lean`
Proposition 2.2	`cdf_bounds_of_halves`	`Full/Reduction.lean`
Lemma 3.1	`prawitz_indicator_majorant`	`PrawitzRecurrence.lean`, `PrawitzOscillation.lean`, `PrawitzLimit.lean`, `PrawitzMajorant.lean` [3]

result	Lean declarations	file
Lemma 3.2	`closed_interval_indicator_le_prawitz`; `open_interval_indicator_ge_prawitz`	`PrawitzFinite.lean` [3]; `Full/LowerPrawitz.lean`
θ, $1.7 < \theta < 1.8$	`crossoverRoot`, `crossoverRoot_spec`; `crossoverRoot_bounds`	`Envelopes.lean`; `CrossoverBounds.lean` [3]
Lemma 3.3	`logCosRatio_antitone`, `logCosRatio_monotone_after_half_pi`, `cosProduct_enclosed`	`Envelopes.lean` [3]
Proposition 3.4	`rademacher_concentration_le_JUpper`; `rademacher_open_concentration_ge_JUpper`, `upperHalf_at_of_JUpper`, `lowerHalf_at_of_JUpper`	`PrawitzEnvelope.lean` [3]; `Full/LowerPrawitz.lean`
Lemma 4.1	`hoeffding_open`, `lowerHalf_at_of_hoeffding`	`Full/Hoeffding.lean`
Lemma 4.2	`peel_closed_ge`, `peel_closed_lt`, `peel_closed_lt_le_half`, `peel_open_gt`	`Full/Peeling.lean`
Lemma 4.3	`peel_remainder_bounds`	`Full/Induction.lean`
Definition 4.4, Theorem 4.5	`PeelL`, `PeelU`, `Pieces`, `halves_of_pieces`; `keller_klein_of_pieces`	`Full/Induction.lean`; `Full/MainConditional.lean`
Theorem 4.6	`small_coefficient_upper_concentration`	`Main.lean` [3]
Theorems 5.1 and 5.2	`zoneA_piece`, `holderProfile_le_general`	`Full/ZoneAGeneral.lean`
Hölder reduction (Section 5)	`integral_cosProduct_le_uniform_profile`; `upperKernel_nonneg_closed`	`ProfileMixture.lean`, `PrawitzEnvelope.lean` [3]
Lemma 5.3	`cotangent_upper_quartic`; `cotangent_lower_product`; `cotangent_remainder_bound`	`Full/ZoneAPoly.lean`; `Elementary.lean`, `CotangentRemainder.lean` [3]
Lemmas 5.4 to 5.6	`pi_mul_rho_le_poly`, `upperKernel_le_affine`, `mul_c_le_c_mul`, `kernelPoly_add_reflect`	`Full/ZoneAPoly.lean`
Lemma 5.8	`wallisV_holder`, `wallisG_le`, `wallisV_sq_le`	`Full/Wallis.lean`
Lemma 5.10	`diag_zero_eq`, `diag_half_eq`, `diagJ_le`, `diag_zero_le`, `diag_half_le`	`Full/ZoneADiagonal.lean`
Lemmas 5.11 and 5.12	`hasDerivAt_profileIntegral`, `ray_zero`, `ray_half`, `profile_ray_mono`	`Full/ZoneARay.lean`
Lemma 6.1	`JUpper_le_gaussian_primary_alias_real`; `envelopeDifference_first`, `envelopeDifference_middle_le`, `envelopeDifference_alias_le`; `JUpper_le_gaussian_primary_alias`	`Full/LowerSmall.lean`; `ZoneBRegions.lean`, `ZoneBDecomposition.lean` [3]
Lemma 6.2	`scaledGaussianKernelIntegral_eq`, `abs_gaussianBaselineIntegrand_tail`, `integral_gaussianBaselineTailMajorant_le`, `abs_gaussianBaseline_tail_le`, `scaledGaussianKernelIntegral_le`; `abs_cotangentRemainderIntegral_le`; `gaussianBaselineValue_eq_common`	`GaussianBaseline.lean`; `ZoneBRemainder.lean`; `ZoneBTheorem.lean` [3]
Lemma 6.3	`zoneBAliasIntegral_le_neg`, `abs_upperKernel_neg_alias_bound`; `abs_upperKernel_alias_bound`, `zoneBAliasIntegral_le`; `trig_linear_quadratic_bound`, `zoneBAlias_budget`, `zoneBAliasValue_compensated`, `zoneBAliasIntegral_compensated`	`Full/LowerSmall.lean`; `ZoneBAlias.lean`; `ZoneBAliasCompensation.lean` [3]
Lemmas 6.4 and 6.5	`negKernel_negPart_le`, `primaryLoss_le_neg`, `primaryLoss_le_large_real`	`Full/LowerSmall.lean`
Theorem 6.6	`lowerFourier_small`, `lowerSmallPoly_le`, `mul_sq_le_of_lt_exp`	`Full/LowerSmall.lean`

result	Lean declarations	file
Proposition 6.8	`peelLNear_small`, `peelLNear_poly`	`Full/PeelNearSmall.lean`
Lemma 6.9	`kernelLossCoeff`, `upperKernel_negativePart_le`; `primaryLoss_le_small`	`ZoneBKernel.lean`; `ZoneBIntegral.lean` [3]
proof of Theorem 4.6	`rademacher_concentration_zoneA`; `gaussian_remainder_budget`, `one_le_H_add_H_of_large`; `JUpper_le_master_base`, `JUpper_le_c_pair_of_zoneB`, `rademacher_concentration_zoneB`; `masterSmall_scaled_le`, `smallPolynomial_le`, `masterSmall_le`, `largeRemainder_le`, `masterLarge_le`	`PrawitzEnvelope.lean`; `GaussianError.lean`; `ZoneBTheorem.lean`; `ZoneBClosure.lean` [3]
Theorem 7.1, (7.1)–(7.3)	`master_le`, `gU_R1`, `gU_R2`, `gU_R3`, `abs_upperKernel_sub`	`Full/Master.lean`
ball arithmetic	`Ball`, `mem_add`, `mem_mul`, `mem_imul`, `le_of_leB`	`Num/Ball.lean`
elementary functions	`pi_mem`, `trigBox_mem`, `mem_expB`, `mem_logB`, `LN2_check`, `log2_mem`, `mem_invB`; `mem_cB`; `phi0_mem`; `mem_sqrtKB`	`Num/Elem.lean`; `Num/GaussCDF.lean`; `Num/CellSound.lean`; `Num/Peel.lean`
cell bound	`cellHi`; `gU_le_cellHi`	`Num/CellDefs.lean`; `Num/CellSound.lean`
box check	`boxOk`; `boxOk_master`, `boxOk_upper`, `boxOk_lower`	`Num/CellDefs.lean`; `Num/CellSound.lean`
covers	`gridGen`, `endOk`, `groupOk`; `groupOk_upper`, `groupOk_lower`, `sliceU_glue`, `sliceL_glue`, `cover_glue`	`Num/Cover.lean`; `Num/CoverSound.lean`
(P3)	`upper_certified`; `upperFourier_certified`	`Num/Cert/All.lean`; `Num/Fourier.lean`
(P4)	`lower_certified`; `lowerFourier_certified`; `lowerFourier_piece`	`Num/Cert/All.lean`; `Num/Fourier.lean`; `Full/Final.lean`
peeling checks	`peelLBox`, `peelUBox`, `peelLBox_sound`, `peelUBox_sound`	`Num/Peel.lean`
(P1), (P5)–(P7)	`c675_certified`, `peelLLarge_cover`, `peelLNear_cover`, `peelU_cover`; `c675_piece`, `peelLLarge_piece`, `peelLNear_piece`, `peelU_piece`	`Num/PeelCert.lean`; `Full/PeelPieces.lean`
all seven inputs	`pieces`	`Full/Final.lean`
Lemma A.1	`sin_sub_mul_cos_nonneg`, `cotangent_upper_product`; `rho_nonneg`, `cot_le_inv_sub_third`, `rho_upper_quadratic`, `rho_le_inv`, `rho_lower_linear`, `rho_reflect`, `rho_alias_bound`; `cotangent_remainder_lower`; `abs_upperKernel_le`, `rho_remainder_eq`, `weighted_cotangentQ_nonpos`, `abs_weighted_cotangentQ_le`, `abs_weighted_cotangentQ_small`; `rho_alias_four_thirds`	`Elementary.lean`; `KernelBounds.lean`; `CotangentRemainder.lean`; `ZoneBKernel.lean`; `ZoneBAlias.lean` [3]
Lemma A.2	`tan_upper_quintic_unit`, `neg_log_cos_upper_unit`, `cosine_profile_loss_bound`	`ZoneBProfile.lean` [3]

10. Approaches that did not work

The following observations come from floating-point experiments made while choosing the architecture. They are not used in any proof.

(1) *Monotonicity in b.* The window near the origin would reduce to the diagonal if $b \mapsto Z_b(a,\xi)$ were monotone. It is not: $W_b(\pi u/a)$ decreases in b on the main peak, but the alias peak near $u = 1$ enters only for b larger than about $a(1 - a/\pi)$.
(2) *The exact kernel along the ray.* Running the argument of Lemma 5.12 with κ_ξ in place of its affine majorant works numerically, with 3–17% slack, but it requires a certificate in the three parameters (a, b, ξ) and derivatives of parametric integrals. The affine majorant reduces the window to two explicit endpoint inequalities.
(3) *Moment expansions in the window.* At $b \approx a$ and $\xi = 0$ the margin of Theorem 5.2 is only about $0.033a^3$, so the moments of the profile would be needed to relative accuracy about 10^{-4}.
(4) *Analytic bounds for larger a.* The expansions in a behind Section 6 have rapidly growing coefficients, and the crude constants of the middle and alias regions dominate the $O(a^3)$ margin already at $a \approx 0.15$. The analytic lower-half bound of [6, Section 8] extends at most to about $a \approx 0.14$.
(5) *Peeling in the upper half near the origin.* Peeling fails for every $a < 0.675$ near $x = 0$ (at $a = \frac{1}{2}$, for instance, for $\xi < 0.25$). In general it loses about $a^3\varphi(a)$, the order of the true margin, so it cannot replace the Fourier analysis for small or moderate a; two steps of peeling, or sharper bounds for the window applied to Y, gain only about $0.07a^3$.
(6) *Quadrature as in* [6]. Blocks of a of width $O(a^3)$ and point evaluations of x on a mesh of size $O(a^3)$, each a full-precision integral, amount to 10^5–10^6 certified integrals. This is far beyond kernel evaluation: the 141 320 cell bounds of Section 8 already take most of the 111 minutes of CPU time of the build.
(7) *Optimizing the frequency.* Once the alias term is compensated as in Section 7, the single frequency $T = \pi/a$ works for both halves on all certified ranges.

Appendix A. Elementary bounds for the kernel

The bounds in this appendix and their proofs are from [3]; in his formalization they are in the files `Elementary.lean`, `KernelBounds.lean`, `CotangentRemainder.lean`, `ZoneBKernel.lean`, `ZoneBAlias.lean` and `ZoneBProfile.lean`. Recall that $\rho(u) = (1-u)\cot(\pi u) + \frac{1}{\pi}$ and $\delta(y) = 1 - y\cot y$, and that $Q(y) = \cot y - \frac{1}{y} + \frac{y}{3}$ for $0 < y < \pi$.

Lemma A.1 ([3])**.**

(a) For $0 \le y \le \pi$, $y\cos y \le (1 - \frac{y^2}{3})\sin y$. Hence $Q(y) \le 0$ and $\delta(y) \ge \frac{y^2}{3}$ for $0 < y < \pi$, and $\rho \ge 0$ on $(0,1)$.

For the remaining parts let $0 < u < 1$.

(b) $\rho(u) \le \frac{1}{\pi u} - \frac{(1-u)\pi u}{3} \le \frac{1}{\pi u}$. Hence $|\kappa_\xi(u)| \le (1-u) + \rho(u) \le 1 + \frac{1}{\pi u}$ for every $\xi \in \mathbb{R}$.
(c) $\rho(u) \ge \frac{1}{\pi u} - \frac{\pi u}{2}$.
(d) $0 \le -(1-u)Q(\pi u) \le \frac{\pi u}{2}$, and $-(1-u)Q(\pi u) \le \frac{(\pi u)^3}{30}$ if $\pi u \le 1$.
(e) If $0 < t \le \frac{1}{2}$, then $0 \le \rho(1-t) = \frac{1}{\pi}\delta(\pi t) \le \frac{\pi t^2}{3} + t^4 \le \frac{4}{3}t^2$.

Proof. (a) Let $n(y) = \sin y - y\cos y$ and $m(y) = (1 - \frac{y^2}{3})\sin y - y\cos y$. Then $n(0) = m(0) = 0$, $n'(y) = y\sin y \ge 0$ and $m'(y) = \frac{y}{3}\,n(y)$ on $[0,\pi]$, so $n \ge 0$ and $m \ge 0$ there. For $0 < y < \pi$, dividing $m(y) \ge 0$ by $y\sin y > 0$ gives $\cot y \le \frac{1}{y} - \frac{y}{3}$, that is, $Q(y) \le 0$ and $\delta(y) \ge \frac{y^2}{3}$. Finally $\rho(u) = \delta(\pi(1-u))/\pi \ge 0$ (`cotangent_upper_product`, `cot_le_inv_sub_third`, `rho_nonneg`).

(b) Let $y = \pi u$. By (a), $\rho(u) \le (1-u)(\frac{1}{y} - \frac{y}{3}) + \frac{1}{\pi}$, and $\frac{1-u}{y} + \frac{1}{\pi} = \frac{1}{y}$; this is the first claim (`rho_upper_quadratic`, `rho_le_inv`). The second follows from $|\kappa_\xi(u)| \le (1-u)|\cos(\pi\xi u)| + \rho(u)|\sin(\pi\xi u)|$ and $\rho(u) \ge 0$ (`abs_upperKernel_le`).

(c) Let $y = \pi u$. If $y^2 > 2$, the right-hand side is negative, and the claim follows from (a). If $y^2 \le 2$, then $y\cos y \ge y(1-\frac{y^2}{2}) \ge (1-\frac{y^2}{2})\sin y$, so $\cot y \ge \frac{1}{y} - \frac{y}{2}$ and $\rho(u) \ge (1-u)(\frac{1}{y}-\frac{y}{2}) + \frac{u}{y} = \frac{1}{y} - \frac{y}{2} + \frac{uy}{2}$ (`rho_lower_linear`).

(d) By the definitions, $(1-u)Q(\pi u) = \rho(u) - \frac{1}{\pi u} + \frac{(1-u)\pi u}{3}$ (`rho_remainder_eq`). The first inequality follows from (a), since $1-u > 0$, and the second from (c). If $y = \pi u \le 1$, then $y \le \frac{\pi}{2}$, and the upper bound of Lemma 5.3 gives $Q(y) = (\frac{y^2}{3} - \delta(y))/y \ge -\frac{y^3}{32}$; hence $-(1-u)Q(y) \le \frac{y^3}{32} \le \frac{y^3}{30}$ (`weighted_cotangentQ_nonpos`, `abs_weighted_cotangentQ_le`, `abs_weighted_cotangentQ_small`).

(e) Since $\cot(\pi - z) = -\cot z$, we have $\rho(1-t) = \frac{1}{\pi} - t\cot(\pi t) = \frac{1}{\pi}\delta(\pi t)$, which is nonnegative by (a). Since $\pi t \le \frac{\pi}{2}$, the upper bound of Lemma 5.3 gives $\rho(1-t) \le \frac{\pi t^2}{3} + \frac{\pi^3 t^4}{32} \le \frac{\pi t^2}{3} + t^4$, because $\pi^3 \le 32$. Finally $t^4 \le \frac{t^2}{4}$ and $\frac{\pi}{3} + \frac{1}{4} \le \frac{4}{3}$ (`rho_reflect`, `rho_alias_bound`, `rho_alias_four_thirds`). □

Lemma A.2 ([3]). *For $0 \le t \le 1$, the function $E(t) = -\log\cos t - \frac{t^2}{2}$ satisfies $0 \le E(t) \le \frac{t^4}{12} + \frac{t^6}{20}$. Consequently, for $a > 0$ and $0 \le s \le 1/a$,*

$$0 \le g(s) - \cos(as)^{1/a^2} \le g(s)\Big(\frac{a^2 s^4}{12} + \frac{a^4 s^6}{20}\Big).$$

Proof. Let $0 \le t \le 1$. We use the Taylor bounds $\sin t \le t - \frac{t^3}{6} + \frac{t^5}{120}$ and $\cos t \ge 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$, and the identity

$$\Big(t + \frac{t^3}{3} + \frac{3t^5}{10}\Big)\Big(1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}\Big) - \Big(t - \frac{t^3}{6} + \frac{t^5}{120}\Big) = t^5\Big(\frac{1}{6} - \frac{11t^2}{80} + \frac{13t^4}{1080} - \frac{t^6}{2400}\Big).$$

Its right-hand side is nonnegative, because $\frac{1}{6} - \frac{11}{80} - \frac{1}{2400} > 0$. Hence $\sin t \le (t + \frac{t^3}{3} + \frac{3t^5}{10})\cos t$, that is, $\tan t \le t + \frac{t^3}{3} + \frac{3t^5}{10}$ (`tan_upper_quintic_unit`). Since $(-\log\cos)' = \tan$, integration gives $E(t) \le \frac{t^4}{12} + \frac{t^6}{20}$ (`neg_log_cos_upper_unit`), and $E(t) \ge 0$ by (i) in the proof of Lemma 3.3. For the second claim let $t = as \le 1$. Then $\cos(as)^{1/a^2} = g(s)\,e^{-E(as)/a^2}$, and $0 \le 1 - e^{-v} \le v$ for $v \ge 0$ (`cosine_profile_loss_bound`). □

Appendix B. The Lean project

The complete Lean project is distributed with this paper as the archive `keller-klein-lean.zip`; with this paper it forms a self-contained proof of Theorem 1.1. The archive contains every file of the project and no build products. Lean is pinned to `leanprover/lean4:v4.19.0` in `lean-toolchain`, and Mathlib to commit `c44e0c8ee63ca166450922a373c7409c5d26b00b` in `lakefile.lean` and `lake-manifest.json`. To check the proof, install elan, unpack the archive and run

```
cd lean
lake exe cache get
lake build
```

The build ends with `Build completed successfully.` and prints the axioms of `KellerKlein.keller_klein`, as described in Section 9. The main theorem is `KellerKlein.keller_klein` in `lean/KellerKlein/Full/Final.lean`; its statement is shown in Section 1.

Table 5 lists the 127 Lean source files and the Lake configuration file `lakefile.lean`, with their numbers of lines (18 333 lines in all). Table 6 lists the remaining files of the archive, with their sizes in bytes: the documentation, the licence of [3], the toolchain and manifest files, the certificate data with the scripts that generate the certificate modules, and the three CausalSmith

sources [16] from which three of Goyal's modules are adapted, kept for reference with the suffix `.lean.txt`.

Table 5: The Lean files of the project.

File	Lines
`lean/KellerKlein.lean`	6
`lean/KellerKlein/Audit.lean`	34
`lean/KellerKlein/CotangentRemainder.lean`	32
`lean/KellerKlein/CrossoverBounds.lean`	85
`lean/KellerKlein/Elementary.lean`	198
`lean/KellerKlein/EnvelopeIntegral.lean`	59
`lean/KellerKlein/EnvelopeRegularity.lean`	73
`lean/KellerKlein/Envelopes.lean`	457
`lean/KellerKlein/FarTail.lean`	33
`lean/KellerKlein/Full.lean`	3
`lean/KellerKlein/Full/Final.lean`	60
`lean/KellerKlein/Full/FinalAudit.lean`	16
`lean/KellerKlein/Full/Hoeffding.lean`	154
`lean/KellerKlein/Full/Induction.lean`	311
`lean/KellerKlein/Full/LowerPrawitz.lean`	152
`lean/KellerKlein/Full/LowerSmall.lean`	737
`lean/KellerKlein/Full/MainConditional.lean`	30
`lean/KellerKlein/Full/Master.lean`	392
`lean/KellerKlein/Full/PeelNearSmall.lean`	137
`lean/KellerKlein/Full/PeelPieces.lean`	41
`lean/KellerKlein/Full/Peeling.lean`	301
`lean/KellerKlein/Full/Reduction.lean`	227
`lean/KellerKlein/Full/Wallis.lean`	349
`lean/KellerKlein/Full/ZoneADiagonal.lean`	269
`lean/KellerKlein/Full/ZoneAGeneral.lean`	120
`lean/KellerKlein/Full/ZoneAPoly.lean`	210
`lean/KellerKlein/Full/ZoneARay.lean`	471
`lean/KellerKlein/GaussianBaseline.lean`	293
`lean/KellerKlein/GaussianError.lean`	130
`lean/KellerKlein/GaussianFourier.lean`	321
`lean/KellerKlein/GaussianMoments.lean`	183
`lean/KellerKlein/GaussianTail.lean`	93
`lean/KellerKlein/GaussianTarget.lean`	36
`lean/KellerKlein/Imports.lean`	29
`lean/KellerKlein/IntervalCompat.lean`	22
`lean/KellerKlein/KernelBounds.lean`	169
`lean/KellerKlein/KernelDefs.lean`	21
`lean/KellerKlein/KernelMeasurability.lean`	56
`lean/KellerKlein/KernelRegularity.lean`	34
`lean/KellerKlein/Main.lean`	61
`lean/KellerKlein/Nonvacuity.lean`	31
`lean/KellerKlein/NormalCDF.lean`	43
`lean/KellerKlein/Num/Ball.lean`	607
`lean/KellerKlein/Num/CellDefs.lean`	264
`lean/KellerKlein/Num/CellSound.lean`	1239
`lean/KellerKlein/Num/Cert/All.lean`	46
`lean/KellerKlein/Num/Cert/L00.lean`	90
`lean/KellerKlein/Num/Cert/L01.lean`	65
`lean/KellerKlein/Num/Cert/L02.lean`	57
`lean/KellerKlein/Num/Cert/L03.lean`	78
`lean/KellerKlein/Num/Cert/L04.lean`	78

File	Lines
lean/KellerKlein/Num/Cert/L05.lean	78
lean/KellerKlein/Num/Cert/L06.lean	78
lean/KellerKlein/Num/Cert/L07.lean	66
lean/KellerKlein/Num/Cert/L08.lean	66
lean/KellerKlein/Num/Cert/L09.lean	66
lean/KellerKlein/Num/Cert/L10.lean	92
lean/KellerKlein/Num/Cert/L11.lean	80
lean/KellerKlein/Num/Cert/L12.lean	67
lean/KellerKlein/Num/Cert/L13.lean	80
lean/KellerKlein/Num/Cert/L14.lean	93
lean/KellerKlein/Num/Cert/L15.lean	80
lean/KellerKlein/Num/Cert/U00.lean	78
lean/KellerKlein/Num/Cert/U01.lean	57
lean/KellerKlein/Num/Cert/U02.lean	57
lean/KellerKlein/Num/Cert/U03.lean	78
lean/KellerKlein/Num/Cert/U04.lean	53
lean/KellerKlein/Num/Cert/U05.lean	66
lean/KellerKlein/Num/Cert/U06.lean	83
lean/KellerKlein/Num/Cert/U07.lean	66
lean/KellerKlein/Num/Cert/U08.lean	83
lean/KellerKlein/Num/Cert/U09.lean	67
lean/KellerKlein/Num/Cert/U10.lean	67
lean/KellerKlein/Num/Cert/U11.lean	80
lean/KellerKlein/Num/Cert/U12.lean	93
lean/KellerKlein/Num/Cert/U13.lean	93
lean/KellerKlein/Num/Cert/U14.lean	99
lean/KellerKlein/Num/Cert/U15.lean	87
lean/KellerKlein/Num/Cover.lean	108
lean/KellerKlein/Num/CoverSound.lean	314
lean/KellerKlein/Num/Elem.lean	724
lean/KellerKlein/Num/Fourier.lean	31
lean/KellerKlein/Num/GaussCDF.lean	172
lean/KellerKlein/Num/Peel.lean	578
lean/KellerKlein/Num/PeelCert.lean	209
lean/KellerKlein/Num/Series.lean	286
lean/KellerKlein/Num/Special.lean	200
lean/KellerKlein/PrawitzEnvelope.lean	117
lean/KellerKlein/PrawitzFinite.lean	122
lean/KellerKlein/PrawitzLimit.lean	129
lean/KellerKlein/PrawitzMajorant.lean	58
lean/KellerKlein/PrawitzOscillation.lean	55
lean/KellerKlein/PrawitzRecurrence.lean	124
lean/KellerKlein/ProfileMixture.lean	81
lean/KellerKlein/Rademacher.lean	318
lean/KellerKlein/ShiftedGaussianMoments.lean	73
lean/KellerKlein/SignedRademacher.lean	92
lean/KellerKlein/Sinc.lean	34
lean/KellerKlein/Vendor/PrawitzKernel.lean	51
lean/KellerKlein/Vendor/PrawitzSignIntegrability.lean	104
lean/KellerKlein/Vendor/TriangularCosineIntegral.lean	63
lean/KellerKlein/ZoneAAlias.lean	66
lean/KellerKlein/ZoneAAliasIntegral.lean	109
lean/KellerKlein/ZoneAAliasScaled.lean	88
lean/KellerKlein/ZoneAAudit.lean	6
lean/KellerKlein/ZoneAConstants.lean	143
lean/KellerKlein/ZoneAFinalAlgebra.lean	170

File	Lines
lean/KellerKlein/ZoneAIntegral.lean	60
lean/KellerKlein/ZoneAIntegralBound.lean	86
lean/KellerKlein/ZoneAKernel.lean	121
lean/KellerKlein/ZoneAProfile.lean	91
lean/KellerKlein/ZoneAScaled.lean	123
lean/KellerKlein/ZoneATail.lean	137
lean/KellerKlein/ZoneATailBound.lean	58
lean/KellerKlein/ZoneATheorem.lean	60
lean/KellerKlein/ZoneBAlias.lean	196
lean/KellerKlein/ZoneBAliasCompensation.lean	107
lean/KellerKlein/ZoneBClosure.lean	356
lean/KellerKlein/ZoneBDecomposition.lean	155
lean/KellerKlein/ZoneBIntegral.lean	209
lean/KellerKlein/ZoneBKernel.lean	115
lean/KellerKlein/ZoneBProfile.lean	103
lean/KellerKlein/ZoneBRegions.lean	215
lean/KellerKlein/ZoneBRemainder.lean	134
lean/KellerKlein/ZoneBTheorem.lean	123
lean/certs/search/FourierSearch.lean	99
lean/certs/search/PeelSearch.lean	94
lean/lakefile.lean	10

Table 6: The other files of the project.

File	Bytes
lean/.gitignore	22
lean/LICENSE-dvgyl-keller-klein	11357
lean/README.md	12678
lean/certs/c_0_0.txt	3325
lean/certs/c_0_17.txt	2357
lean/certs/c_0_34.txt	1785
lean/certs/c_0_51.txt	1384
lean/certs/c_1_0.txt	3347
lean/certs/c_1_15.txt	2668
lean/certs/c_1_30.txt	2047
lean/certs/c_1_45.txt	1613
lean/certs/gen_lean.py	5156
lean/certs/gen_peel.py	4172
lean/certs/log_0_0.txt	1487
lean/certs/log_0_17.txt	1497
lean/certs/log_0_34.txt	1481
lean/certs/log_0_51.txt	1392
lean/certs/log_1_0.txt	1317
lean/certs/log_1_15.txt	1327
lean/certs/log_1_30.txt	1327
lean/certs/log_1_45.txt	1157
lean/certs/peel_L_large.txt	304
lean/certs/peel_L_near.txt	465
lean/certs/peel_U.txt	600
lean/lake-manifest.json	3548
lean/lean-toolchain	25
lean/third_party/causalsmith/LICENSE	11341
lean/third_party/causalsmith/NOTICE	719
lean/third_party/causalsmith/PrawitzKernel.lean.txt	2118

File	Bytes
lean/third_party/causalsmith/PrawitzSignIntegrability.lean.txt	4165
lean/third_party/causalsmith/TriangularCosineIntegral.lean.txt	2716
lean/third_party/causalsmith/UPSTREAM.txt	1686

Acknowledgments

This paper builds on the publicly available work of Dev D. Goyal [3]:

https://github.com/dvgyl/math-and-theory/tree/main/papers/mathematics/keller-klein

From it we use the proof of the small-coefficient upper half (Theorem 4.6, proved in Section 6.4), the decomposition of the Prawitz integral in Section 6, Prawitz's inequality and the envelope bounds in the form of Section 3, the Hölder reduction of Section 5, and the kernel bounds of Appendix A. His complete Lean formalization of these results is included without change, in 61 modules.

The recurrence-and-limit argument for Prawitz's inequality (Lemma 3.1) comes from the public CoolRmal/BerryEsseen project [2], as [3] records in its Lean sources; the Lean proof in [3] is an independent formalization of that argument:

https://github.com/CoolRmal/BerryEsseen

Three of Goyal's modules are adapted from the CausalSmith library of Jiyuan Tan [16], under the Apache-2.0 license. The formalization relies throughout on Mathlib [12], and I thank its contributors.

Disclosure

This proof was generated on October 30, 2026, by Grok Bot using Cursor's agents, guided by several prompts from Paata Ivanisvili.

Parts of the proofs in this paper, the Lean formalization of the new parts, the certified numerics and the programs that search for the certificates, and much of the text were generated with AI assistance, under the direction of the author. The earlier manuscripts [6, 3] on which the paper builds were also prepared with AI assistance; [3] states that its proof and formalization were produced by AI research agents. The Lean statements listed in Table 4 have been checked by the Lean kernel as described in Section 9, and the statement of Theorem 1.1 is shown in Lean in Section 1. The correspondence between the other statements of the paper and their Lean versions was checked by reading, not mechanically. The remarks on earlier work, the floating-point observations and the comparisons in Table 1 are not machine-checked. This version incorporates the revisions requested by an independent referee of an earlier version; it has not been peer-reviewed by a journal.

References

[1] K. Ball, Cube slicing in $\mathbb{R}^n$, Proc. Amer. Math. Soc. **97** (1986), no. 3, 465–473.
[2] CoolRmal, BerryEsseen, public project, https://github.com/CoolRmal/BerryEsseen, commit bfb2260.
[3] D. D. Goyal, The small-coefficient upper half of the Keller–Klein conjecture: analytic completion and Lean 4 formalization, preprint, 30 October 2026, https://github.com/dvgyl/math-and-theory/tree/main/papers/mathematics/keller-klein.
[4] U. Haagerup, The best constants in the Khintchine inequality, Studia Math. **70** (1981), no. 3, 231–283.
[5] W. Hoeffding, Probability inequalities for sums of bounded random variables, J. Amer. Statist. Assoc. **58** (1963), 13–30.
[6] P. Ivanisvili, The Keller–Klein Berry–Esseen conjecture: the reachable half for all Rademacher sums, and the full conjecture for $a_1 \geq 0.07$, manuscript, 21 October 2026, https://paatai.github.io/ai-proofs/keller-klein/keller-klein.pdf.

[7] F. Johansson, Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, IEEE Trans. Comput. **66** (2017), 1281–1292.
[8] N. Keller and O. Klein, Proof of Tomaszewski's conjecture on randomly signed sums, Adv. Math. **407** (2022), 108558; arXiv:2006.16834.
[9] D. Kershaw, Some extensions of W. Gautschi's inequalities for the gamma function, Math. Comp. **41** (1983), 607–611.
[10] lean4checker, a tool for replaying Lean environments in the kernel, https://github.com/leanprover/lean4checker.
[11] L. Ma, L. Ye and X. Han, Improved Berry–Esseen bound for Rademacher sum, Chinese J. Appl. Probab. Statist. **40** (2024), no. 6, 910–941, doi:10.12460/j.issn.1001-4268.aps.2024.2022121.
[12] The mathlib Community, The Lean mathematical library, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), 367–381.
[13] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in: Automated Deduction – CADE 28, Lecture Notes in Comput. Sci. **12699**, 2021, 625–635.
[14] F. L. Nazarov and A. N. Podkorytov, Ball, Haagerup, and distribution functions, in: Complex analysis, operators, and related topics, Oper. Theory Adv. Appl. **113**, 2000, 247–267.
[15] H. Prawitz, Limits for a distribution, if the characteristic function is given in a finite domain, Skand. Aktuarietidskr. 1972, 138–154.
[16] J. Tan, CausalSmith, a Lean library, https://github.com/Jiyuan-Tan/CausalSmith.