A COMPARISON INEQUALITY FOR GAUSSIAN VECTORS AND PRODUCT FUNCTIONS

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Abstract. Let Σ_0 and Σ_1 be covariance matrices of Gaussian vectors $\xi = (\xi_1, \dots, \xi_n)$ whose blocks $\xi_j \in \mathbb{R}^{k_j}$ are standard Gaussian, with Σ_1 positive definite, and let $p > 1$. We prove that

$$\mathbb{E}_{\Sigma_0}\Big[\prod_j f_j(\xi_j)\Big] \le \Big(\mathbb{E}_{\Sigma_1}\Big[\prod_j f_j(\xi_j)^p\Big]\Big)^{1/p} \qquad \text{for all measurable } f_j \ge 0$$

whenever some block-diagonal matrix D satisfies $\frac{p}{p-1}(\Sigma_0 - \Sigma_1) \le D \le \Sigma_0$. The necessary condition is $\Sigma_0 \le p\Sigma_1$, and the block-diagonal condition implies it. The two conditions are equivalent when Σ_0 lies on the segment between Id and Σ_1, and for scalar blocks when, after changing the signs of some coordinates, all correlations are non-positive and those of Σ_0 are no larger in absolute value than those of Σ_1. For $\Sigma_0 = \mathrm{Id}$ the inequality is the reverse Hölder inequality of Chen, Dafnis and Paouris with equal exponents. In general $\Sigma_0 \le p\Sigma_1$ is not sufficient, already for two functions whose correlation decreases from Σ_0 to Σ_1, and Gaussian functions are not extremal: a computer-assisted example has all Gaussian test functions satisfying the inequality while it fails. Applied to $\Sigma_0 = X\Sigma_1 X + \mathrm{Id} - X^2$, the results give the forward inequality of Ivanisvili and Kalantzopoulos with outer exponent $\alpha = 1/p$ and unequal parameters x_j, a case excluded from their theorem. Their covariance condition is then necessary and sufficient for two functions with any parameters and any cross-covariance, and for any number of scalar functions whose correlations become non-positive after sign changes, provided $x_i x_j \ge 0$ whenever the corresponding correlation is non-zero. The proof is a flow along the segment from Σ_0 to Σ_1, with the exponent rising from 1 to p, controlled by a weighted reverse log-Sobolev inequality. The flow also works along curved paths. In an explicit example, where the block-diagonal condition holds only for $p \ge 3$ and $\Sigma_0 \le p\Sigma_1$ holds for $p \ge 2.175\ldots$, such a path, with two of the blocks merged on its first stretch and an exact rational certificate, gives the inequality for $p \ge 2.1925$. We show that these flows stay bounded away from $2.175\ldots$ there; the range in between is open.

1. Introduction and results

Let $k_1, \dots, k_n \ge 1$ and $K = k_1 + \cdots + k_n$. We write matrices on $\mathbb{R}^K = \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$ in blocks $A = (A_{ij})_{i,j=1}^n$, $A_{ij} \in \mathbb{R}^{k_i \times k_j}$, and call A *block-diagonal* if $A_{ij} = 0$ for $i \ne j$. Let $\mathcal{C}$ be the set of positive semi-definite $K \times K$ matrices whose diagonal blocks are identities. For $\Sigma \in \mathcal{C}$, $\mathbb{E}_\Sigma$ denotes expectation with respect to a centred Gaussian vector $\xi = (\xi_1, \dots, \xi_n)$ with covariance Σ; so $\xi_j \sim \gamma_{k_j}$, the standard Gaussian measure on $\mathbb{R}^{k_j}$. For $p > 0$ put $\|W\|_{L^p(\Sigma)} = (\mathbb{E}_\Sigma |W|^p)^{1/p}$. Products of values in $[0, \infty]$ are taken with $0 \cdot \infty = 0$. The order $A \le B$ is the Loewner order.

Theorem 1.1. *Let $\Sigma_0, \Sigma_1 \in \mathcal{C}$ with Σ_1 positive definite, and let $p > 1$. Suppose there is a block-diagonal symmetric matrix D with*

$$\frac{p}{p-1}(\Sigma_0 - \Sigma_1) \;\le\; D \;\le\; \Sigma_0. \tag{1}$$

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Then for all measurable $f_j : \mathbb{R}^{k_j} \to [0,\infty]$,

$$\mathbb{E}_{\Sigma_0}\Big[\prod_{j=1}^n f_j(\xi_j)\Big] \;\le\; \Big\|\prod_{j=1}^n f_j(\xi_j)\Big\|_{L^p(\Sigma_1)}. \tag{2}$$

Conversely, if (2) *holds for all* f_j *of the form* $e^{\langle a_j,\cdot\rangle}$, *then* $\Sigma_0 \le p\Sigma_1$; *and* (1) *implies* $\Sigma_0 \le p\Sigma_1$.

Remark 1.2. (a) Condition (1) is a linear matrix inequality in the diagonal blocks of D, so its feasibility is a semidefinite programme. Any such D is positive semi-definite: the diagonal blocks of $\Sigma_0 - \Sigma_1$ vanish, so those of D are ≥ 0.

(b) Let $p^* = \lambda_{\max}(\Sigma_1^{-1/2}\Sigma_0\Sigma_1^{-1/2})$, the smallest p with $\Sigma_0 \le p\Sigma_1$. Since the diagonal entries of Σ_0 and Σ_1 agree, $p^* \ge 1$, and $p^* = 1$ only if $\Sigma_0 = \Sigma_1$ (then $\Sigma_1 - \Sigma_0 \ge 0$ has trace 0). By Lyapunov's inequality, if (2) holds for some p, it holds for every larger p. If it holds for every $p > p_0$, it also holds at p_0: for f_j with values in $[\varepsilon, M]$, $0 < \varepsilon < M < \infty$, the right side of (2) is continuous in p, and general f_j follow by the truncation in Section 4.4. By Proposition 1.3 below, (2) holds when $\lambda_{\max}(\Sigma_0) \le p\,\lambda_{\min}(\Sigma_1)$. So the set of $p \ge 1$ for which (2) holds is an interval $[p_c,\infty)$ with $p^* \le p_c \le \lambda_{\max}(\Sigma_0)/\lambda_{\min}(\Sigma_1)$.

(c) Condition (1) is monotone in p: if D satisfies it at p and $p' > p$, then $\frac{c'}{c}D$ satisfies it at p', where $c = \frac{p}{p-1} > c' = \frac{p'}{p'-1}$, because $D \ge 0$. The feasible D lie in the compact set $\{0 \le D \le \Sigma_0\}$, so the set of $p > 1$ at which (1) is feasible is closed in $(1,\infty)$. Theorem 1.1 gives $p_c \le p$ for every such p, and $p_c = p^*$ when (1) is feasible at $p = p^*$. The set can be empty (Remark 5.3).

(d) If the diagonal blocks A_j of Σ_0 and Σ_1 agree but are not identities (A_j positive definite), the substitution $\xi_j \mapsto A_j^{-1/2}\xi_j$ reduces to $\mathcal{C}$, since it maps product functions to product functions. Grouping several blocks into one is allowed when the corresponding parts of Σ_0 and Σ_1 agree, and makes (1) weaker, since D may then have larger blocks.

Theorem 4.1 in Section 4.6 is a version of Theorem 1.1 along curved paths, with a matrix D that may vary along the path; it contains Theorem 1.1. The following elementary bound complements both. It is a direct consequence of the theorem of Chen, Dafnis and Paouris [3], which we use in the form [6, Theorem B]: for $\Sigma \in \mathcal{C}$ and $p_j \in \mathbb{R}\setminus\{0\}$, the inequality $\prod_j(\mathbb{E}F_j(\xi_j))^{p_j} \le \mathbb{E}_\Sigma \prod_j F_j(\xi_j)^{p_j}$ holds for all measurable $F_j : \mathbb{R}^{k_j} \to (0,\infty)$ if and only if $\mathrm{diag}(p_j^{-1}\mathrm{Id}_{k_j}) \le \Sigma$, and the reverse inequality holds for all such F_j if and only if $\mathrm{diag}(p_j^{-1}\mathrm{Id}_{k_j}) \ge \Sigma$.

Proposition 1.3. *Let* $\Sigma_0, \Sigma_1 \in \mathcal{C}$ *and* $p \ge 1$. *If there are numbers* $s_1,\dots,s_n > 0$ *with*

$$\Sigma_0 \;\le\; S \;\le\; p\,\Sigma_1, \qquad S = \mathrm{diag}(s_1\mathrm{Id}_{k_1},\dots,s_n\mathrm{Id}_{k_n}), \tag{3}$$

then (2) *holds for all measurable* $f_j \ge 0$. *In particular, it holds if* $\lambda_{\max}(\Sigma_0) \le p\,\lambda_{\min}(\Sigma_1)$.

Proof. Let f_j take values in $[\varepsilon, M]$, $0 < \varepsilon < M < \infty$, and put $F_j = f_j^{s_j}$. The reverse form of the theorem quoted above, with $\Sigma = \Sigma_0$ and $p_j = 1/s_j$, gives $\mathbb{E}_{\Sigma_0}\prod_j f_j(\xi_j) \le \prod_j \|f_j\|_{L^{s_j}(\gamma_{k_j})}$, since $S \ge \Sigma_0$. The direct form, with $\Sigma = \Sigma_1$ and $p_j = p/s_j$, gives $\prod_j \|f_j\|^p_{L^{s_j}(\gamma_{k_j})} \le \mathbb{E}_{\Sigma_1}\prod_j f_j(\xi_j)^p$, since $\frac{1}{p}S \le \Sigma_1$. Together they give (2); general $f_j \ge 0$ follow by the truncation in Section 4.4. For the last statement take all $s_j = \lambda_{\max}(\Sigma_0)$: then $\Sigma_0 \le S = \lambda_{\max}(\Sigma_0)\mathrm{Id} \le p\lambda_{\min}(\Sigma_1)\mathrm{Id} \le p\Sigma_1$. □

Proposition 1.3 passes through the independent coupling, so it suits "forward" situations, where correlations decrease from Σ_0 to Σ_1; for $\Sigma_1 = \mathrm{Id}$ it is sharp (Remark 5.3). Theorem 1.1 suits "reverse" situations, where correlations increase. For instance, for $n = 2$, $k_j = 1$, $\mathrm{cov}_{\Sigma_1}(\xi_1,\xi_2) = \rho \in (0,1)$ and $\mathrm{cov}_{\Sigma_0}(\xi_1,\xi_2) = t\rho$ with $t \in (0,1)$, (3) needs $1 + t\rho \le p(1-\rho)$ (by

symmetry and convexity one may take $S = s\mathrm{Id}$), while Corollary 1.4(a) below gives (2) under the sharp condition $1 - t\rho \le p(1-\rho)$.

1.1. **Cases in which the condition is sharp.**

Corollary 1.4. *In each of the following cases, (2) holds if and only if $\Sigma_0 \le p\Sigma_1$, for every $p > 1$.*

(a) $\Sigma_1 \in \mathcal{C}$ is positive definite and $\Sigma_0 = (1-t)\mathrm{Id} + t\Sigma_1$ with $t \in [0,1]$. Here $\Sigma_0 \le p\Sigma_1$ reads $1 - t + t\lambda \le p\lambda$, $\lambda = \lambda_{\min}(\Sigma_1)$.

(b) All $k_j = 1$, Σ_1 is positive definite, and there are signs $\varepsilon_j \in \{\pm 1\}$ with

$$\varepsilon_i\varepsilon_j(\Sigma_1)_{ij} \;\le\; \varepsilon_i\varepsilon_j(\Sigma_0)_{ij} \;\le\; 0 \qquad (i \ne j). \tag{4}$$

(c) $n = 2$, $\Sigma_1 = \left(\begin{smallmatrix} \mathrm{Id} & C \\ C^T & \mathrm{Id} \end{smallmatrix}\right)$ is positive definite and $\Sigma_0 = \left(\begin{smallmatrix} \mathrm{Id} & tC \\ tC^T & \mathrm{Id} \end{smallmatrix}\right)$ with $t \le 0$ and $|t|\,\|C\| \le 1$. Here $\Sigma_0 \le p\Sigma_1$ reads $(p-t)\|C\| \le p - 1$.

At $p = 1$ both (2) and $\Sigma_0 \le \Sigma_1$ mean $\Sigma_0 = \Sigma_1$: exponential f_j in (2) give $\Sigma_0 \le \Sigma_1$, and then the trace argument of Remark 1.2(b) applies.

For $t = 0$, (a) is the reverse Hölder inequality of Chen, Dafnis and Paouris [3] with equal exponents ([6, Theorem B] with $p_j = p$): $\prod_j \mathbb{E}f_j \le \|\prod_j f_j(\xi_j)\|_{L^p(\Sigma_1)}$ iff $p\lambda_{\min}(\Sigma_1) \ge 1$. Inequalities of this direction are also called inverse Brascamp–Lieb inequalities [2]; Barthe's reverse Brascamp–Lieb inequality [1] can be obtained from them by a limiting argument [6]. For $t = 1$, (a) is Lyapunov's inequality. Case (a) contains every pair with $n = 2$ of the form $\Sigma_1 = \left(\begin{smallmatrix} \mathrm{Id} & C \\ C^T & \mathrm{Id} \end{smallmatrix}\right)$, $\Sigma_0 = \left(\begin{smallmatrix} \mathrm{Id} & tC \\ tC^T & \mathrm{Id} \end{smallmatrix}\right)$, $t \in [0,1]$, since then $\Sigma_0 = (1-t)\mathrm{Id} + t\Sigma_1$; together with (c), every $t \le 1$ is covered. For some $t > 1$ the condition $\Sigma_0 \le p\Sigma_1$ is not sufficient (Proposition 1.9). Condition (4) says that after changing the signs of some coordinates all correlations are non-positive, and those of Σ_0 are no larger in absolute value than those of Σ_1. Signs making the off-diagonal entries of Σ_1 non-positive exist if and only if every cycle in the graph of its non-zero off-diagonal entries contains an even number of positive entries. (Along a cycle the product of the signs $-\varepsilon_i\varepsilon_j$ is $(-1)^{\text{length}}$; conversely, choose the signs along a spanning tree of each component.) This holds for instance when $n = 2$, when the graph is a forest, and when all correlations are non-positive. Three variables with pairwise positive correlations are a non-example.

By Remark 1.2(d), (a) and (c) also apply to grouped blocks. Let Σ_1 be positive definite, and suppose the index set is partitioned into groups such that Σ_0 and Σ_1 agree on each group, and $(\Sigma_0)_{ij} = t(\Sigma_1)_{ij}$ whenever i and j lie in different groups. If $t \in [0,1]$, or if $t < 0$, there are two groups and $\Sigma_0 \ge 0$, then (2) holds if and only if $\Sigma_0 \le p\Sigma_1$. Indeed, after the normalisation of Remark 1.2(d) on each group (a congruence, which preserves the Loewner order), Σ_0 becomes $(1-t)\mathrm{Id} + t\Sigma_1$, respectively the matrix of (c).

1.2. **The forward inequality with outer exponent $1/p$.**

For $x \in [-1,1]$ let $T_x f(y) = \mathbb{E}f(xy + \sqrt{1-x^2}\,G)$, $G \sim \gamma_k$, be the Ornstein–Uhlenbeck (Mehler) operator. Ivanisvili and Kalantzopoulos [6, Thm. 9] consider, for $\Sigma \in \mathcal{C}$, $r_j \in [-1,1]$, $p_j \ne 0$ and $\alpha \in \mathbb{R}\setminus\{0\}$, the forward inequality

$$\Big\| \prod_j \big(T_{r_j} f_j(\xi_j)\big)^{p_j} \Big\|_{L^\alpha(\Sigma)} \le \Big\| \prod_j f_j(\xi_j)^{p_j} \Big\|_{L^1(\Sigma)} \tag{5}$$

for positive f_j. They show that for $\alpha \in (-\infty, 0) \cup [1, \infty)$ it holds iff

$$\mathrm{diag}\Big(\frac{1 - r_j^2}{p_j}\mathrm{Id}_{k_j}\Big) \le \big((1 - \alpha r_u r_v)\Sigma_{uv}\big)_{u,v=1}^{n} \tag{6}$$

([6, (2.16), (2.17)]). The range $\alpha \in (0,1)$ is excluded. The companion note [5] settles it for equal parameters $r_j = x$ and equal exponents $p_j = p$ ([5, Thm. 1.1] for $p \ge 1$, and [5, Cor. 1.4] for all

$p \neq 0$ when Σ is non-singular), and for two ρ-correlated functions when $\alpha = 1/p$, for all r_1, r_2 [5, Prop. 6.1]. It also shows that for two functions with unequal parameters and any $\alpha p > 2$, (6) is not sufficient in general [5, Props. 6.3, 6.5], while for $\alpha p > 1$ and positive ρ, r_1, r_2 a flow with separate noise parameters gives a sufficient condition under which it is sharp [5, Thm. 6.9].

Corollary 1.5. *Let $\Sigma \in \mathcal{C}$ be positive definite, $p > 1$, $x_j \in [-1, 1]$, and $X = \mathrm{diag}(x_j \mathrm{Id}_{k_j})$. Then (5) with $r_j = x_j$, $p_j = p$ and $\alpha = 1/p$ is the inequality (2) for $\Sigma_0 = X\Sigma X + \mathrm{Id} - X^2$ and $\Sigma_1 = \Sigma$, and (6) is $\Sigma_0 \leq p\Sigma$. So (6) is necessary, and it is sufficient whenever (1) or (3) is feasible. In particular, (5)$\Leftrightarrow$(6) in each of the following cases:*

- *(i) $x_i x_j = t \geq 0$ for all $i \neq j$ with $\Sigma_{ij} \neq 0$; for instance, all x_j are equal;*
- *(ii) all $k_j = 1$, $x_i x_j \geq 0$ whenever $\Sigma_{ij} \neq 0$, and some signs ε_j satisfy $\varepsilon_i \varepsilon_j \Sigma_{ij} \leq 0$ for all $i \neq j$;*
- *(iii) $n = 2$, with any x_1, x_2, any k_1, k_2 and any cross-covariance Σ_{12};*
- *(iv) $x_j = 1$ for all j in a set G of indices, and either $x_j = 0$ for all $j \notin G$, or G contains all indices but one.*

In case (i), $\Sigma_0 = (1-t)\mathrm{Id} + t\Sigma$ is the noised covariance for the equal parameter $x = \sqrt{t}$, so (i) is the case $r = 1$ of [5, Thm. 1.1], which allows arbitrary block sizes. Case (iii) with $x_1 x_2 \geq 0$ is contained in (i), and case (iv) with a non-negative parameter (x_{j_0}, or 0) follows from the same theorem after grouping and normalising as in Remark 1.2(d); with a negative parameter both follow from [3] through Proposition 1.3. For two ρ-correlated functions case (iii) is [5, Prop. 6.1]. We include (iii) and (iv) for completeness. Case (ii) with $n \geq 3$ and genuinely unequal parameters is new as far as we know.

1.3. **Limits of the method.**

Proposition 1.6. *Let $n = 3$, $k_j = 1$, Σ_1 the equicorrelated matrix with correlation $\frac{1}{2}$, and $x = (1, \frac{1}{2}, 0)$, so $\Sigma_0 = X\Sigma_1 X + \mathrm{Id} - X^2 = \begin{pmatrix} 1 & 1/4 & 0 \\ 1/4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $p^* = \frac{11+\sqrt{41}}{8} = 2.175\ldots$. Condition (1) is feasible if and only if $p \geq 3$, and condition (3) if and only if $p \geq \frac{5}{2}$. So (2) holds for $p \geq \frac{5}{2}$, while the necessary condition only requires $p \geq p^*$.*

Proposition 1.8 below improves $\frac{5}{2}$ to 2.1925, and we do not know whether (2) holds for $p \in [p^*, 2.1925)$ here. The example is not covered by Corollaries 1.4 and 1.5: all correlations of Σ_1 are positive, so (4) fails, and Σ_0 and Σ_1 agree on no group of two or more coordinates. Numerically, no Gaussian test function violates (2) at $p = p^*$ (Section 6), and by the following remark (2) holds near constant functions, in the uniform norm, for every $p > p^*$.

Remark 1.7. Let all $k_j = 1$, let $(h_d)_{d \geq 0}$ be the normalised Hermite polynomials, and let g_j be bounded with $\mathbb{E} g_j(\xi_j) = 0$, so that $g_j = \sum_{d \geq 1} a_{j,d} h_d$ in $L^2(\gamma_1)$. Put $a_d = (a_{1,d}, \ldots, a_{n,d})$ and let $\Sigma^{\circ d}$ be the entrywise d-th power. For $f_j = 1 + \varepsilon g_j$, the right side of (2) minus the left side equals

$$\frac{\varepsilon^2}{2} \sum_{d \geq 1} a_d^T \left(p\Sigma_1^{\circ d} - \Sigma_0^{\circ d} \right) a_d + O(\varepsilon^3) \qquad (\varepsilon \to 0).$$

Indeed, $\mathbb{E}_{\Sigma_0} \prod_j (1 + \varepsilon g_j) = 1 + \varepsilon^2 \sum_{i<j} \mathbb{E}_{\Sigma_0}[g_i g_j] + O(\varepsilon^3)$, and $N(\varepsilon) = \mathbb{E}_{\Sigma_1} \prod_j (1 + \varepsilon g_j)^p$ is smooth near 0 with $N(0) = 1$, $N'(0) = 0$ and $N''(0) = p(p-1) \sum_j \mathbb{E} g_j^2 + 2p^2 \sum_{i<j} \mathbb{E}_{\Sigma_1}[g_i g_j]$, so that $N^{1/p} = 1 + \frac{\varepsilon^2}{2p} N''(0) + O(\varepsilon^3)$. Mehler's formula gives $\mathbb{E}_\Sigma[g_i(\xi_i) g_j(\xi_j)] = \sum_d a_{i,d} a_{j,d} \Sigma_{ij}^d$. In particular $p\Sigma_1^{\circ d} \geq \Sigma_0^{\circ d}$ for all $d \geq 1$ is necessary for (2) (truncate h_d as in [5, Prop. 6.2]); $d = 1$ is $\Sigma_0 \leq p\Sigma_1$. In Proposition 1.6 the off-diagonal entries of $p\Sigma_1^{\circ d} - \Sigma_0^{\circ d}$ are $p2^{-d} - 4^{-d}$, $p2^{-d}$ and $p2^{-d}$, so for $d \geq 2$ and $p > 2$ Gershgorin's theorem gives $\lambda_{\min}(p\Sigma_1^{\circ d} - \Sigma_0^{\circ d}) \geq p - 1 - 2p\, 2^{-d} \geq \frac{p}{2} - 1 > 0$.

For $p > p^*$ also $p\Sigma_1 - \Sigma_0 > 0$. Hence for every $p > p^*$ the quadratic form is at least $c\sum_j \mathbb{E}g_j^2$ with $c > 0$. The remainder is uniform: if $\|g_j\|_\infty \le M$ and $|\varepsilon|M \le \frac{1}{2}$, it is at most $C|\varepsilon|^3 M \sum_j \mathbb{E}g_j^2$ with $C = C(n,p)$, because every term of order $k \ge 3$ contains two factors g_i, g_j whose product is bounded by $\frac{1}{2}(g_i^2+g_j^2)$ and $k-2$ factors bounded by M. So for every $p > p^*$ there is $\delta(p) > 0$ such that (2) holds whenever $\|f_j/\mathbb{E}f_j - 1\|_\infty \le \delta(p)$ for all j: a violation for some $p \in (p^*, 2.1925)$, if there is one, must be far from constants in the uniform norm.

Proposition 1.8. *In the example of Proposition 1.6, (2) holds for every* $p \ge \frac{877}{400} = 2.1925$.

Proof (computer-assisted, exact arithmetic). Let $b = \frac{9}{40}$, $C_1 = \begin{pmatrix} 1 & 1/4 & b \\ 1/4 & 1 & b \\ b & b & 1 \end{pmatrix}$ and $p_1 = \frac{139783}{100000}$. The matrices Σ_0 and C_1 agree on the groups $\{1,2\}$ and $\{3\}$, and Σ_0 vanishes between them, so by the grouping paragraph after Corollary 1.4 (case (a) with $t = 0$), (2) holds from Σ_0 to C_1 with exponent p_1 as soon as $\Sigma_0 \le p_1 C_1$; this is checked exactly (the threshold is $1/(1-\sqrt{8/5}\,b) = 1.3978291\ldots$). Next, Theorem 4.1 applies to a piecewise-linear path through rational correlation matrices $K_0 = C_1, K_1, \ldots, K_{256} = \Sigma_1$, with scalar blocks, pieces of length $\ell = \frac{4501}{2560000}$ and a rational diagonal matrix D_k on the k-th piece. On a piece $K(s) - D_k$ is affine in s and $K'(s) = (K_{k+1} - K_k)/\ell$, so (12) holds there if $D_k \ge 0$, $K_k - D_k \ge 0$, $K_{k+1} - D_k \ge 0$ and $(K_{k+1} - K_k) + \ell D_k \ge 0$; positive definiteness of all K_k gives that of $K(s)$. These 1281 conditions are verified exactly (principal minors) by `numerics/comparison_chain_certificate.py`, which stores the rational data and re-checks them without any solver. This gives (2) from C_1 to Σ_1 with exponent $e^{256\ell} = e^{0.4501}$. Applying it to the functions $f_j^{p_1}$ after the first step, the exponents multiply:

$$\mathbb{E}_{\Sigma_0}\Big[\prod_j f_j\Big] \le \Big(\mathbb{E}_{C_1}\Big[\prod_j f_j^{p_1}\Big]\Big)^{1/p_1} \le \Big(\mathbb{E}_{\Sigma_1}\Big[\prod_j f_j^{p_1 e^{0.4501}}\Big]\Big)^{1/(p_1 e^{0.4501})}.$$

Finally $p_1 e^{0.4501} = 2.19245\ldots < \frac{877}{400}$ in ball arithmetic, and Lyapunov's inequality gives every $p \ge \frac{877}{400}$. □

The first step is the segment flow of Theorem 1.1 for the groups, with $D = \Sigma_0$ (for this D, condition (1) is exactly $\Sigma_0 \le p_1 C_1$), so by Remark 4.2(a) the whole construction is one admissible path for Theorem 4.1 with grouped blocks on its first piece. The data were found by semidefinite programming (Section 6); this construction was suggested by an independent referee of this note. Optimising b gives 2.1949 and 2.1931 with 64 and 128 scalar pieces, which suggests about 2.191 in the limit. For comparison (all numerical): along the straight segment, letting $\tilde{D}$ vary as in Remark 4.2(d) already gives 2.3148 instead of 3 (`numerics/comparison_segment_varying_D.py`); curved paths with scalar blocks give at best about 2.287 (2.2957, 2.2913 and 2.2891 for 64, 128 and 256 pieces; $\frac{229}{100}$ is certified the same way by `numerics/comparison_path_certificate.py 256 1657/2000 229/100`), since the optimal scalar paths are not straight: the correlation of ξ_1 and ξ_2 first decreases slightly. A discretised first stretch with frozen c_{12} gives $\frac{221}{100}$ (`numerics/comparison_multistage_certificate.py GS 32 0.4,0.6 7929/10000 221/100`). So most of the gain over Theorem 1.1 comes from letting $\tilde{D}$ vary, and the rest from merging blocks. By Proposition 4.4, no such flow reaches p^* here, nor any exponent close to it; numerically they bottom out near 2.191.

The second variation also shows that the necessary condition is not sufficient in general.

Proposition 1.9. *Let* $n = 2$, $k_j = 1$, $\operatorname{cov}_{\Sigma_1}(\xi_1,\xi_2) = \rho \in [0,1)$ *and* $\operatorname{cov}_{\Sigma_0}(\xi_1,\xi_2) = \theta \in (\rho, 1)$. *Then* $p^* = \frac{1+\theta}{1+\rho}$, *and (2) fails for every* $p < \sup_{d\ge 1} \frac{1+\theta^d}{1+\rho^d}$. *If* $(1+\theta)(1+\rho) > 2$, *this supremum exceeds* p^* *(already the term* $d = 2$ *does), so* $\Sigma_0 \le p\Sigma_1$ *does not imply (2).*

Proof. $p\Sigma_1 - \Sigma_0 \geq 0$ iff $|p\rho - \theta| \leq p - 1$. If $p\rho \leq \theta$ this reads $p \geq \frac{1+\theta}{1+\rho}$; if $p\rho > \theta$ it reads $p(1-\rho) \geq 1 - \theta$, which holds for all $p \geq 1$. Since $\frac{1+\theta}{1+\rho} \leq \frac{\theta}{\rho}$, $p^* = \frac{1+\theta}{1+\rho}$. By Remark 1.7, (2) requires $p\Sigma_1^{\circ d} \geq \Sigma_0^{\circ d}$ for every d, and the vector $(1,1)$ gives $p - 1 + p\rho^d - \theta^d \geq 0$, i.e. $p \geq \frac{1+\theta^d}{1+\rho^d}$. Finally $(1+\theta^2)(1+\rho) - (1+\theta)(1+\rho^2) = (\theta - \rho)\big((1+\theta)(1+\rho) - 2\big)$, which is positive under the hypothesis. □

For $\rho = 0$ the hypothesis cannot hold, in accordance with the forward Hölder inequality. For instance, for $\theta = 0.9$ and $\rho = 0.5$ the supremum is attained at $d = 4$, and (2) fails on $[\frac{19}{15}, 1.5587\ldots)$. By Remark 1.7, $f_1 = f_2 = 1 + \varepsilon(y^2 - 1)/\sqrt{2}$ violates it for every $p < 1.448$ once ε is small enough (depending on p); the expansion there also holds for this unbounded perturbation, since $f_j \geq 1 - \varepsilon/\sqrt{2} > 0$ and all moments are finite; quadrature confirms this at $p = \frac{19}{15}$ and $p = 1.35$ (`numerics/comparison_hermite_scan.py`). Gaussian test functions do not detect the higher levels: for these θ, ρ, a numerical optimisation over all $f_j = e^{a_j y - b_j y^2/2}$ finds no violation for $p \geq 1.497$. The following proposition makes this rigorous at another point. The Hermite levels do not determine the threshold either: at $p = 1.56$, where all levels are positive semi-definite (smallest margin 0.0014, at $d = 4$), an optimised non-Gaussian pair f_1, f_2 violates (2) by $4 \cdot 10^{-6}$ in the logarithm, stably under refinement of the quadrature, while nothing is found for $p \geq 1.6$ (`numerics/comparison_n2_forward_attack.py`; floating point). So for $\theta = 0.9$, $\rho = 0.5$ there is a violation at $p = 1.56$ and none was found for $p \geq 1.6$ (floating point), while Proposition 1.3 only gives $p \geq 3.8$ and Theorem 4.1 $p \geq 5$.

In the setting of Corollary 1.5 the correlations of $\Sigma_0 = X\Sigma X + \mathrm{Id} - X^2$ are no larger in absolute value than those of Σ, so Proposition 1.9 does not apply there. In 196515 random instances of that setting with $n \in \{3, 4\}$ all levels $2 \leq d \leq 8$ of Remark 1.7 were positive semi-definite at p^* (`numerics/comparison_hermite_scan.py`), and a targeted minimisation of the smallest level margin for $n \leq 5$, with $\lambda_{\min}(\Sigma) \geq 0.05$, found only positive values (`numerics/comparison_hermite_target.py`).

Proposition 1.10. *Let $n = 2$, $k_j = 1$, $\mathrm{cov}_{\Sigma_1}(\xi_1, \xi_2) = \frac{3}{5}$, $\mathrm{cov}_{\Sigma_0}(\xi_1, \xi_2) = \frac{93}{100}$ and $p = \frac{3}{2}$. Then every Gaussian test function $f_j(y) = c_j e^{a_j y - b_j y^2/2}$ ($c_j > 0$, $a_j, b_j \in \mathbb{R}$) satisfies (2), but (2) fails for some measurable f_j with values in $[\frac{1}{2}, \frac{3}{2}]$. So Gaussian functions are not extremal for (2).*

Inequalities of Brascamp–Lieb type bound a multilinear Gaussian form by a product of norms of the individual functions, and under the hypotheses of [9, 2, 4, 11] Gaussian functions are extremal for them. Inequality (2) compares two multilinear forms; unless $\Sigma_1 = \mathrm{Id}$ its right side is not a product of norms, and these results do not apply. The functions in Proposition 1.10 are not log-concave; whether Gaussian functions are extremal for (2) among log-concave f_j, the setting of [10], is open.

Proof (computer-assisted: exact arithmetic and ball arithmetic). Write $\theta = \frac{93}{100}$, $\rho = \frac{3}{5}$. Here $p^* = \frac{193}{160} < \frac{3}{2}$, and $(p-1) + p\rho^5 - \theta^5 = -\frac{790483693}{10^{10}} < 0$, so (2) fails by Proposition 1.9 (level $d = 5$; the bounded perturbations $1 + \varepsilon g$ of Remark 1.7 take values in $[\frac{1}{2}, \frac{3}{2}]$ for small ε).

Let $B = \mathrm{diag}(b_1, b_2)$, $K_0 = \Sigma_0^{-1} + B$ and $K_1 = \Sigma_1^{-1} + pB$. If K_1 is not positive definite, the right side of (2) is infinite. Otherwise $K_0 \geq \frac{1}{p} K_1 > 0$, because $\Sigma_0 \leq p\Sigma_1$, and Gaussian integrals give

$$\log \frac{\mathbb{E}_{\Sigma_0}[f_1 f_2]}{\|f_1 f_2\|_{L^p(\Sigma_1)}} = \frac{1}{2}\Big(\frac{1}{p}\log\det(\mathrm{Id} + p\Sigma_1 B) - \log\det(\mathrm{Id} + \Sigma_0 B)\Big) + \frac{1}{2} a^T\big(K_0^{-1} - pK_1^{-1}\big)a,$$

where the last term is ≤ 0. With $A(b) = \det(\mathrm{Id} + \Sigma_0 B) = 1 + b_1 + b_2 + \frac{1351}{10^4} b_1 b_2$ and $\tilde{B}(b) = \det(\mathrm{Id} + p\Sigma_1 B) = 1 + \frac{3}{2}(b_1 + b_2) + \frac{36}{25} b_1 b_2$, it remains to show $P = A^3 - \tilde{B}^2 \geq 0$ on $\mathcal{D} = \{b :$

$K_1 > 0\}$. In the variables $u_i = b_i + \frac{25}{24}$, $\mathcal{D} = \{u_1, u_2 > 0,\ u_1u_2 > \frac{25}{64}\}$, $\tilde{B} = \frac{36}{25}u_1u_2 - \frac{9}{16}$, and $A = \kappa u_1u_2 + \frac{8249}{9600}(u_1+u_2) - \frac{8633}{9216}$ with $\kappa = \frac{1351}{10^4}$. Since $u_1+u_2 \geq \frac{5}{4}$ on $\mathcal{D}$, $A \geq \kappa u_1u_2 + 0.137$ there. On the boundary $\tilde{B} = 0$, so $P = A^3 > 0$. Along any sequence in $\mathcal{D}$ with $|u| \to \infty$, $P \to \infty$: where u_1u_2 stays bounded, $\tilde{B}$ stays bounded and $A \to \infty$; where $u_1u_2 \to \infty$, $P \geq \kappa^3(u_1u_2)^3 - (\frac{36}{25}u_1u_2)^2$. So if $\inf_{\mathcal{D}} P \leq 0$, it is attained at a critical point of P in $\mathcal{D}$. The partial derivatives of P are coprime, and a lexicographic Gröbner basis of the ideal they generate is $\{b_2 + d(b_1),\, h(b_1)\}$ with $\deg h = 9$. The polynomial h has exactly three real roots, isolated in exact arithmetic. They give the critical points $(0,0)$, where $P = 0$; $b \approx (-0.3024, -0.3024) \in \mathcal{D}$, where $P \in 0.0173028661872 \pm 2 \cdot 10^{-14}$; and $b \approx (-39.83, -39.83) \notin \mathcal{D}$. Hence $P \geq 0$ on $\mathcal{D}$ (`numerics/certify_gaussian_not_extremal.py`). □

2. Preliminaries

Let $\mathcal{S}$ be the set of C^∞ functions f on some $\mathbb{R}^k$ with $\inf f > 0$ such that f and all its derivatives are bounded. If g is measurable with values in $[\varepsilon, M]$, $0 < \varepsilon < M < \infty$, and $s \in [0,1)$, then $T_s g \in \mathcal{S}$: differentiate the Gaussian density in $T_s g(y) = \int g(w)\phi_{1-s^2}(w - sy)\,dw$ for $s > 0$, where ϕ_v is the density of $N(0, v\mathrm{Id}_k)$ (for $s = 0$, $T_0 g$ is constant).

Gaussian calculus. Let $C(h)$, $h \in (h_0, h_1)$, be a C^1 family of positive definite $K \times K$ matrices, and let H be a C^2 function on $\mathbb{R}^K$ which, with its derivatives up to order two, is bounded. Then

$$\frac{d}{dh}\mathbb{E}_{C(h)}H(\zeta) = \frac{1}{2}\sum_{a,b}\dot{C}_{ab}(h)\,\mathbb{E}_{C(h)}\big[\partial_a\partial_b H(\zeta)\big], \tag{7}$$

by the heat equation $\partial_h \phi_{C(h)} = \frac{1}{2}\sum_{a,b}\dot{C}_{ab}\partial_a\partial_b\phi_{C(h)}$ for the Gaussian density (compare Fourier transforms) and two integrations by parts. If $\dot{C}$ has zero diagonal blocks and $H(\zeta) = \prod_j G_j(\zeta_j)$ with $G_j \in \mathcal{S}$, this reads

$$\frac{d}{dh}\mathbb{E}_{C(h)}\Big[\prod_j G_j(\zeta_j)\Big] = \sum_{i<j}\mathbb{E}_{C(h)}\Big[\big\langle \nabla G_i(\zeta_i), \dot{C}_{ij}\nabla G_j(\zeta_j)\big\rangle \prod_{l\neq i,j} G_l(\zeta_l)\Big]. \tag{8}$$

Entropy. For a random variable $W > 0$ with $\mathbb{E}W\log W < \infty$ put $\mathrm{Ent}(W) = \mathbb{E}W\log W - \mathbb{E}W\log\mathbb{E}W \geq 0$. If η is a random vector, then $\mathrm{Ent}(W) = \mathbb{E}[\mathrm{Ent}(W\,|\,\eta)] + \mathrm{Ent}(\mathbb{E}[W\,|\,\eta]) \geq \mathbb{E}[\mathrm{Ent}(W\,|\,\eta)]$. If $W = \prod_j W_j$ with independent $W_j > 0$, then $\mathrm{Ent}(W) = \sum_j \mathrm{Ent}(W_j)\prod_{l\neq j}\mathbb{E}W_l$ (expand $\log W$).

3. A weighted reverse log-Sobolev inequality

Lemma 3.1. *Let ζ be a centred Gaussian vector on $\mathbb{R}^K$ with covariance C, let $D = \mathrm{diag}(D_1, \dots, D_n)$ be block-diagonal and positive semi-definite with $C \geq D$, and let B be symmetric with zero diagonal blocks and $D + B \geq 0$. Then for $\Psi_j \in \mathcal{S}$ on $\mathbb{R}^{k_j}$,*

$$\mathrm{Ent}\Big(\prod_j \Psi_j(\zeta_j)\Big) \;\geq\; -\sum_{i<j}\mathbb{E}\Big[\big\langle \nabla\Psi_i(\zeta_i), B_{ij}\nabla\Psi_j(\zeta_j)\big\rangle \prod_{l\neq i,j}\Psi_l(\zeta_l)\Big]. \tag{9}$$

For $D = \lambda\mathrm{Id}$ this is [5, Lemma 1.6]; the proof is the same.

Proof. One Gaussian. Let ν be the law of $m + A^{1/2}G$, $G \sim \gamma_k$, with $A \geq 0$, and let $\Psi \in \mathcal{S}$. For $h(w) = \langle\theta, w - m\rangle$ the entropy inequality $\mathbb{E}_\nu[\Psi h] \leq \mathrm{Ent}_\nu(\Psi) + \mathbb{E}_\nu\Psi\log\mathbb{E}_\nu e^h$ holds, $\mathbb{E}_\nu e^h = e^{\langle A\theta,\theta\rangle/2}$, and Gaussian integration by parts in G gives $\mathbb{E}_\nu[\Psi(w)(w-m)] = A\,\mathbb{E}_\nu\nabla\Psi$. With $\theta = \mathbb{E}_\nu\nabla\Psi/\mathbb{E}_\nu\Psi$ we obtain

$$\mathrm{Ent}_\nu(\Psi) \;\geq\; \frac{\langle A\,\mathbb{E}_\nu\nabla\Psi, \mathbb{E}_\nu\nabla\Psi\rangle}{2\,\mathbb{E}_\nu\Psi}. \tag{10}$$

Conditioning. Write $\zeta = \eta + \omega$ with $\eta \sim N(0, C-D)$ and $\omega \sim N(0,D)$ independent. Since D is block-diagonal, given η the blocks $\zeta_j = \eta_j + \omega_j$ are independent with laws $N(\eta_j, D_j)$. Put $u_j(\eta_j) = \mathbb{E}[\Psi_j(\zeta_j)\,|\,\eta]$, so that $u_j \geq \inf \Psi_j > 0$, u_j is C^1 and $\nabla u_j(\eta_j) = \mathbb{E}[\nabla \Psi_j(\zeta_j)\,|\,\eta]$. Put $\beta_j = \nabla u_j / u_j$, $\beta = (\beta_1, \dots, \beta_n)$ and $U = \prod_j u_j(\eta_j)$. By the facts on entropy and (10) with $A = D_j$,

$$\operatorname{Ent}\Big(\prod_j \Psi_j(\zeta_j)\Big) \;\geq\; \mathbb{E}\Big[\sum_j \operatorname{Ent}\left(\Psi_j(\zeta_j)\,|\,\eta\right) \prod_{l\neq j} u_l(\eta_l)\Big] \;\geq\; \frac{1}{2}\,\mathbb{E}\big[U\,\beta^T D\beta\big].$$

By conditional independence, and since B has zero diagonal blocks, the sum on the right of (9) without the minus sign equals $\sum_{i<j} \mathbb{E}[U\langle \beta_i, B_{ij}\beta_j\rangle] = \frac{1}{2}\mathbb{E}[U\beta^T B\beta]$. Finally $\beta^T D\beta \geq -\beta^T B\beta$ pointwise, because $D + B \geq 0$. □

4. Proof of Theorem 1.1

4.1. **The flow.** Let $f_j \in \mathcal{S}$. For $q > 0$ and $C \in \mathcal{C}$ put

$$W = \prod_j f_j(\zeta_j)^q, \qquad \Phi(q, C) = \frac{1}{q}\log \mathbb{E}_C W, \qquad a_j = \frac{\nabla f_j}{f_j}(\zeta_j), \quad a = (a_1, \dots, a_n).$$

Let $C(h)$ be a C^1 family in $\mathcal{C}$ of positive definite matrices and $q(h)$ a C^1 positive function. Then

$$q^2\,\mathbb{E}W\,\frac{d}{dh}\Phi\big(q(h), C(h)\big) = \dot{q}\,\operatorname{Ent}(W) + \frac{q^3}{2}\,\mathbb{E}\big[W\,a^T \dot{C} a\big], \tag{11}$$

all expectations with respect to $C(h)$. Indeed, $\partial_q \Phi = -\frac{1}{q^2}\log \mathbb{E}W + \frac{1}{q}\mathbb{E}[W\sum_j \log f_j]/\mathbb{E}W = \operatorname{Ent}(W)/(q^2\mathbb{E}W)$, since $\log W = q\sum_j \log f_j$. And $\dot{C}$ has zero diagonal blocks, so (8) with $G_j = f_j^q$, $\nabla G_j = qG_j a_j$, gives $\frac{d}{dh}\mathbb{E}_{C(h)}W = \frac{q^2}{2}\mathbb{E}[W a^T \dot{C} a]$ at fixed q. All functions involved are bounded with bounded derivatives, so the differentiations are justified.

4.2. **The path.** Let D satisfy (1). For $h \in [0,1]$ put

$$C(h) = (1-h)\Sigma_0 + h\Sigma_1, \qquad m(h) = 1 - h\Big(1 - \frac{1}{p}\Big), \qquad q(h) = \frac{1}{m(h)}.$$

Then $C(h) \in \mathcal{C}$, $C(h) \geq h\Sigma_1$ is positive definite for $h > 0$, $q(0) = 1$ and $q(1) = p$. Moreover:

- (P1) $C(h) \geq m(h)D$. The difference is affine in h; at $h = 0$ it is $\Sigma_0 - D \geq 0$, and at $h = 1$ it is $\Sigma_1 - \frac{1}{p}D = \big(\Sigma_1 - \Sigma_0 + (1-\frac{1}{p})D\big) + (\Sigma_0 - D) \geq 0$, by both halves of (1).
- (P2) With $\kappa(h) = \frac{p\,m(h)}{p-1}$ and $B = \kappa(h)(\Sigma_1 - \Sigma_0)$, which has zero diagonal blocks, $m(h)D + B = m(h)\big(D - \frac{p}{p-1}(\Sigma_0 - \Sigma_1)\big) \geq 0$.
- (P3) $\dot{q}\,\kappa = q$: indeed $\dot{q} = (1 - \frac{1}{p})/m^2$, so $\dot{q}\kappa = \frac{(p-1)/p}{m^2} \cdot \frac{pm}{p-1} = \frac{1}{m}$.

4.3. **Monotonicity.** Let $h \in (0,1)$ and $\Lambda = q^2\mathbb{E}W\frac{d}{dh}\Phi(q(h), C(h))$. By (11) with $\dot{C} = \Sigma_1 - \Sigma_0$,

$$\Lambda = \dot{q}\,\operatorname{Ent}(W) + \frac{q^3}{2}\,G, \qquad G = \mathbb{E}\big[W\,a^T(\Sigma_1 - \Sigma_0)a\big].$$

Apply Lemma 3.1 with covariance $C(h)$, the block-diagonal matrix $m(h)D$ (admissible by (P1) and Remark 1.2(a)), the matrix B of (P2), and $\Psi_j = f_j^q$. Since $\nabla\Psi_j = q\Psi_j a_j$, the right side of (9) is $-q^2\sum_{i<j}\mathbb{E}[W\langle a_i, B_{ij}a_j\rangle] = -\frac{q^2}{2}\mathbb{E}[W a^T B a] = -\frac{\kappa q^2}{2}G$. Hence $\operatorname{Ent}(W) \geq -\frac{\kappa q^2}{2}G$, whatever the sign of G, and since $\dot{q} > 0$, (P3) gives

$$\Lambda \;\geq\; \frac{q^2 G}{2}\,(q - \dot{q}\kappa) = 0.$$

So $h \mapsto \Phi(q(h), C(h))$ is non-decreasing on $(0,1)$. It is continuous on $[0,1]$: q is continuous, the integrands $\prod_j f_j^q$ are bounded and continuous, and $N(0, C(h))$ depends weakly continuously on h. Therefore

$$\log \mathbb{E}_{\Sigma_0}\Big[\prod_j f_j(\xi_j)\Big] = \Phi(1,\Sigma_0) \le \Phi(p,\Sigma_1) = \log \Big\| \prod_j f_j(\xi_j)\Big\|_{L^p(\Sigma_1)},$$

which is (2) for $f_j \in \mathcal{S}$.

4.4. **Approximation.** Let $f_j : \mathbb{R}^{k_j} \to [\varepsilon, M]$ be measurable, $0 < \varepsilon < M < \infty$, and $f_j^\delta = T_{1-\delta} f_j \in \mathcal{S}$ for $\delta \in (0,1)$. As $\delta \to 0$, $f_j^\delta \to f_j$ in $L^2(\gamma_{k_j})$ ($T_s \to \mathrm{Id}$ strongly on L^2), hence in γ_{k_j}-probability. Under both Σ_0 and Σ_1 each ξ_j has law γ_{k_j}, so $\prod_j f_j^\delta(\xi_j) \to \prod_j f_j(\xi_j)$ in probability, and all values lie in $[\varepsilon^n, M^n]$. Dominated convergence carries (2) from f_j^δ to f_j. For general measurable $f_j \ge 0$ apply this to $\max\{\varepsilon, \min\{f_j, M\}\}$, let $\varepsilon \downarrow 0$ (dominated convergence, values in $[0, M+1]$), and then $M \uparrow \infty$ (monotone convergence on both sides).

4.5. **Necessity.** For $f_j = e^{\langle a_j, \cdot\rangle}$, $\mathbb{E}_{\Sigma_0} e^{\langle a,\xi\rangle} = e^{a^T\Sigma_0 a/2}$ and $\|e^{\langle a,\xi\rangle}\|_{L^p(\Sigma_1)} = e^{pa^T\Sigma_1 a/2}$. So (2) forces $\Sigma_0 \le p\Sigma_1$. If D satisfies (1), then $D \le \Sigma_0$ and $\Sigma_1 - \Sigma_0 + (1-\frac{1}{p})D \ge 0$ give $\Sigma_1 - \frac{1}{p}\Sigma_0 \ge \Sigma_1 - \Sigma_0 + (1-\frac{1}{p})D \ge 0$. □

4.6. **Curved paths.** The argument uses neither the linearity of the path nor the constancy of D. We parametrise by $s = \log q$.

Theorem 4.1. *Let $\Sigma_0, \Sigma_1 \in \mathcal{C}$ and $T > 0$. Let $C : [0,T] \to \mathcal{C}$ be continuous and piecewise C^1, with finitely many pieces, $C(0) = \Sigma_0$, $C(T) = \Sigma_1$, and $C(s)$ positive definite for $0 < s \le T$. Suppose that for every s in the interior of a piece there is a block-diagonal matrix $\tilde{D}(s)$ with*

$$0 \le \tilde{D}(s) \le C(s), \qquad C'(s) + \tilde{D}(s) \ge 0. \tag{12}$$

Then (2) holds with $p = e^T$ for all measurable $f_j \ge 0$. This remains true if, on some pieces, the blocks are grouped: for a partition of $\{1,\dots,n\}$ into groups such that the diagonal blocks of $C(s)$ belonging to the groups are constant on the piece, $\tilde{D}(s)$ may be block-diagonal for the groups.

Proof. Let $f_j \in \mathcal{S}$ and $\tilde{\Phi}(s) = \Phi(e^s, C(s))$. In the interior of a piece, (11) with $q = e^s$ and $\dot{q} = q$ gives $q^2 \mathbb{E} W\, \tilde{\Phi}'(s) = q \operatorname{Ent}(W) + \frac{q^3}{2}\mathbb{E}[W a^T C'(s) a]$. The matrix $C'(s)$ has zero diagonal blocks, so Lemma 3.1 applies with covariance $C(s)$, the matrix $\tilde{D}(s)$, $B = C'(s)$ and $\Psi_j = f_j^q$, and gives $\operatorname{Ent}(W) \ge -\frac{q^2}{2}\mathbb{E}[W a^T C'(s) a]$. Hence $\tilde{\Phi}'(s) \ge 0$. On a piece with grouped blocks, $C'(s)$ has zero diagonal blocks for the groups, and Lemma 3.1, which does not need identity diagonal blocks, applies in the same way to the grouped blocks with $\Psi_G = \prod_{j\in G} f_j^q$. The function $\tilde{\Phi}$ is continuous on $[0,T]$, also at a singular Σ_0: as $s \to s_0$, $\prod_j f_j^{e^s} \to \prod_j f_j^{e^{s_0}}$ uniformly, with values in a fixed compact subset of $(0,\infty)$, and $N(0, C(s)) \to N(0, C(s_0))$ weakly. By the mean value theorem on each closed piece, $\tilde{\Phi}(0) \le \tilde{\Phi}(T)$, which is (2) with $p = e^T$ for $f_j \in \mathcal{S}$; Section 4.4 gives the general case. □

Remark 4.2. (a) Theorem 1.1 is the case of the segment. With $q = e^s = 1/m(h)$, the path of Section 4.2 has $dC/ds = \kappa(\Sigma_1 - \Sigma_0)$, and $\tilde{D} = m(h)D$ satisfies (12) by (P1) and (P2).

(b) Along an admissible path, $C + C' = (C - \tilde{D}) + (C' + \tilde{D}) \ge 0$, so $s \mapsto e^s C(s)$ is non-decreasing and $\Sigma_0 \le e^T \Sigma_1$. Let $T = \log p^*$ and $\Sigma_0 v = p^* \Sigma_1 v$. Then $e^s v^T C(s) v$ is non-decreasing with equal values at $s = 0$ and $s = T$, hence constant, so $v^T(C - \tilde{D})v = 0$ and $(C(s) - \tilde{D}(s))v = 0$ in the interiors of the pieces. For scalar blocks and v without zero entries this forces $\tilde{D}(s) = \operatorname{diag}(C(s)v/v)$, and letting $s \to 0$ gives $\Sigma_0 \ge \operatorname{diag}(\Sigma_0 v/v)$. In Proposition 1.6 this

fails ($\Sigma_0 - \mathrm{diag}(\frac{5}{4}, \frac{5}{4}, 1)$ has the eigenvalue $-\frac{1}{2}$), so no admissible path with scalar blocks has $T = \log p^*$.

(c) For $n = 2$ and $k_j = 1$, write $c(s)$ for the correlation. Some $\tilde{D}(s)$ satisfies (12) iff $|c'(s)| \le 1 - |c(s)|$. Indeed $\tilde{D} = (1-|c|)\mathrm{Id}$ works then, and conversely $|c'|^2 \le \tilde{D}_{11}\tilde{D}_{22} = (1-u_1)(1-u_2) \le (1-\sqrt{u_1u_2})^2 \le (1-|c|)^2$, where $u_i = 1 - \tilde{D}_{ii} \in [0,1]$ and $u_1u_2 \ge c^2$. So paths raise a correlation from $c_0 \ge 0$ to $c_1 \in [c_0, 1)$ iff $p \ge \frac{1-c_0}{1-c_1}$, which is sharp (Corollary 1.4(a)), and lower it from $\theta \in (0,1)$ to 0 iff $p \ge \frac{1}{1-\theta}$. The sharp exponent for lowering is $1+\theta$ (Remark 5.3), while the segment of Theorem 1.1 needs $p \ge \frac{1-\theta}{1-2\theta}$ and $\theta < \frac{1}{2}$.

(d) Along the segment $C(h) = (1-h)\Sigma_0 + h\Sigma_1$ traversed at speed $dh/ds = 1/\sigma(h)$, condition (12) reads $\sigma(h)\tilde{D} \ge \Sigma_0 - \Sigma_1$. So (2) holds with $p = \exp\int_0^1 \sigma(h)\,dh$ whenever $\sigma > 0$ is piecewise continuous and for each h some block-diagonal $\tilde{D}$ with $0 \le \tilde{D} \le C(h)$ satisfies $\sigma(h)\tilde{D} \ge \Sigma_0 - \Sigma_1$. Taking $\tilde{D} = m(h)D$ gives $\sigma = (1 - \frac{1}{p})/m(h)$, with integral $\log p$: this is Theorem 1.1. For $n = 2$ every admissible path is a reparametrised segment, so the gain in (c) over Theorem 1.1 comes entirely from letting $\tilde{D}$ vary.

Remark 4.3. In Proposition 1.6 the grouped form of Theorem 4.1 removes the obstruction of Remark 4.2(b) at $s = 0$: for the groups $G = \{1,2\}$ and $\{3\}$ one may take $\tilde{D} = \Sigma_0$. It does not make p^* reachable there. On a lossless path, $\tilde{D}(s)v = C(s)v$ and $(C'(s) + \tilde{D}(s))v = 0$, so $C'v = -Cv$. On a piece with the groups $G, \{3\}$ we have $C'_{GG} = 0$, and comparing the two components of $C'v = -Cv$ gives $v_G^T C_{GG} v_G = v_3^2$. On the first piece $C_{GG} = (\Sigma_0)_{GG}$, and $v_G^T(\Sigma_0)_{GG}v_G = \frac{5}{2} \ne \frac{21+\sqrt{41}}{8} = v_3^2$ for $v = (1, 1, -\frac{1+\sqrt{41}}{4})$. The other block structures fail on the first piece as well: for the groups $\{1,3\}, \{2\}$ the second component of $\tilde{D}(s)v = C(s)v$ gives $\tilde{D}_{22}(s) = (C(s)v)_2/v_2 \to \frac{5}{4}$ as $s \to 0$, while $\tilde{D}_{22} \le C_{22} = 1$ (similarly for $\{2,3\}, \{1\}$); and for a single group $C' = 0$ on the piece, so the two identities give $C(s)v = 0$, which is impossible. Since the path has finitely many pieces, no admissible path, with any choice of block structures, has $T = \log p^*$ in Proposition 1.6. The following proposition makes this quantitative.

Proposition 4.4. *In Proposition 1.6 there is $\varepsilon > 0$ such that every admissible path in the sense of Theorem 4.1, with any block structures on its pieces, has $e^T \ge p^* + \varepsilon$.*

Proof. For a partition σ of $\{1,2,3\}$ into groups and $C \in \mathcal{C}$, let $\mathcal{A}_\sigma(C)$ be the set of pairs $(V, \tilde{D})$ with V symmetric with zero diagonal blocks for σ, $\tilde{D}$ block-diagonal for σ, $0 \le \tilde{D} \le C$ and $V + \tilde{D} \ge 0$. It is compact, since $\tilde{D}_{ii} \le C_{ii} = 1$ and the 2×2 minors of $V + \tilde{D}$ give $|V_{ij}| \le (\tilde{D}_{ii}\tilde{D}_{jj})^{1/2} \le 1$, and $\{(C, V, \tilde{D}) : (V, \tilde{D}) \in \mathcal{A}_\sigma(C)\}$ is closed. Let $v = (1, 1, -\frac{1+\sqrt{41}}{4})$, so $\Sigma_0 v = p^*\Sigma_1 v$, and put

$$m(C) = \min\left\{v^T(C+V)v : \ \sigma,\ (V,\tilde{D}) \in \mathcal{A}_\sigma(C)\right\}.$$

Since $v^T(C+V)v = v^T(C-\tilde{D})v + v^T(V+\tilde{D})v$, we have $m \ge 0$; the minimum exists, since $(0,0) \in \mathcal{A}_\sigma(C)$, and m is lower semicontinuous, since the bounds above do not depend on C and the graphs are closed. Moreover $m(\Sigma_0) > 0$: otherwise some σ and $(V, \tilde{D}) \in \mathcal{A}_\sigma(\Sigma_0)$ have $(\Sigma_0 - \tilde{D})v = 0$ and $(V + \tilde{D})v = 0$, i.e. $\tilde{D}v = \Sigma_0 v$ and $Vv = -\Sigma_0 v$. For scalar blocks this forces $\tilde{D} = \mathrm{diag}(\frac{5}{4}, \frac{5}{4}, 1) \not\le \Sigma_0$; for the groups $\{1,2\}, \{3\}$, the components of $Vv = -\Sigma_0 v$ give $v_G^T(\Sigma_0)_{GG}v_G = v_3^2$, which is false; for $\{1,3\}, \{2\}$ (resp. $\{2,3\}, \{1\}$) the second (resp. first) component of $\tilde{D}v = \Sigma_0 v$ gives $\tilde{D}_{22} = \frac{5}{4}$ (resp. $\tilde{D}_{11} = \frac{5}{4}$), contradicting $\tilde{D} \le \Sigma_0$; and for a single group $V = 0$ gives $\Sigma_0 v = 0$ (all as in Remarks 4.2(b) and 4.3). Choose $\delta \in (0, \log p^*]$ with $m(C) \ge \eta := \frac{1}{2}m(\Sigma_0)$ whenever $C \in \mathcal{C}$ and $\max_{ij} |C_{ij} - (\Sigma_0)_{ij}| \le \delta$.

Along an admissible path, $(C'(s), \tilde{D}(s)) \in \mathcal{A}_\sigma(C(s))$ in the interior of each piece, so $|C'_{ij}| \le 1$ and $\max_{ij} |C_{ij}(s) - (\Sigma_0)_{ij}| \le s$. The function $s \mapsto e^s v^T C(s) v$ is continuous, piecewise C^1, with

derivative $e^s v^T(C+C')v \geq 0$; comparing its values at 0 and T gives $e^T \geq p^*$, so $T \geq \delta$. Hence

$$e^T v^T \Sigma_1 v - v^T \Sigma_0 v = \int_0^T e^s v^T (C(s) + C'(s)) v \, ds \;\geq\; \int_0^\delta m(C(s))\, ds \;\geq\; \eta\delta,$$

and $e^T \geq p^* + \eta\delta/(v^T\Sigma_1 v)$. The argument does not give an explicit ε; numerically the best value is about 0.016 (see after Proposition 1.8). □

5. Proofs of the corollaries

Lemma 5.1. *Let $p = p^* > 1$ and let $v \neq 0$ satisfy $\Sigma_0 v = p^* \Sigma_1 v$. If D satisfies (1) at $p = p^*$, then $Dv = \Sigma_0 v$.*

Proof. Using $v^T\Sigma_1 v = v^T\Sigma_0 v/p^*$, we get $v^T\big(D - \frac{p^*}{p^*-1}(\Sigma_0 - \Sigma_1)\big)v = v^T D v - \frac{p^*}{p^*-1}\big(1-\frac{1}{p^*}\big)v^T\Sigma_0 v = -v^T(\Sigma_0 - D)v$. Both $D - \frac{p^*}{p^*-1}(\Sigma_0 - \Sigma_1)$ and $\Sigma_0 - D$ are positive semi-definite, so $v^T(\Sigma_0 - D)v = 0$ and hence $(\Sigma_0 - D)v = 0$. □

Lemma 5.2. *Let A be a real symmetric $n \times n$ matrix with $A_{ij} \leq 0$ for $i \neq j$, and let $v \in \mathbb{R}^n$ have positive entries with $Av = 0$. Then $A \geq 0$.*

Proof. For $w \in \mathbb{R}^n$, using $A_{ii}v_i = -\sum_{j\neq i} A_{ij}v_j$ and the symmetry of A, $w^T A w = -\frac{1}{2}\sum_{i\neq j} A_{ij}v_i v_j \big(\frac{w_i}{v_i} - \frac{w_j}{v_j}\big)^2 \geq 0$. □

Proof of Corollary 1.4. In each case necessity is part of Theorem 1.1, and for $p \geq p^*$ it suffices, by Remark 1.2(b), to find D satisfying (1), or S satisfying (3), at $p = p^*$, if $p^* > 1$. If $p^* = 1$ then $\Sigma_0 = \Sigma_1$, and (2) is Lyapunov's inequality.

(a) Let $\lambda = \lambda_{\min}(\Sigma_1) > 0$. The eigenvalues of Σ_0 are $1 - t + t\sigma$, σ an eigenvalue of Σ_1, with the same eigenvectors, and $(1 - t + t\sigma)/\sigma$ decreases in σ. So $\Sigma_0 \leq p\Sigma_1$ iff $1 - t + t\lambda \leq p\lambda$. Assume this with $p > 1$, and let $D = d\,\mathrm{Id}$ with $d = 1 - t + t\lambda$. Then $\Sigma_0 - D = t(\Sigma_1 - \lambda\mathrm{Id}) \geq 0$, and $\frac{p}{p-1}(\Sigma_0 - \Sigma_1) = \frac{p}{p-1}(1-t)(\mathrm{Id} - \Sigma_1) \leq \frac{p}{p-1}(1-t)(1-\lambda)\mathrm{Id}$. The last bound is $\leq d\,\mathrm{Id}$ iff $p(1-t)(1-\lambda) \leq (p-1)(1-t+t\lambda)$, i.e. iff $1 - t + t\lambda \leq p\lambda$.

(b) Conjugating by $\mathrm{diag}(\varepsilon_j)$ (that is, replacing ξ_j by $\varepsilon_j\xi_j$ and f_j by $f_j(\varepsilon_j\,\cdot)$) we may assume $(\Sigma_1)_{ij} \leq (\Sigma_0)_{ij} \leq 0$ for $i \neq j$. If $(\Sigma_1)_{ij} = 0$ then $(\Sigma_0)_{ij} = 0$. So if the graph $\mathcal{G}$ on $\{1,\dots,n\}$ with edges $\{(\Sigma_1)_{ij} \neq 0\}$ is disconnected, the coordinates in different components are independent under both measures, both sides of (2) factorise over components, and $\Sigma_0 \leq p\Sigma_1$ holds iff it holds on each component. So let $\mathcal{G}$ be connected, and let $p^* > 1$. Put $N = p^*\Sigma_1 - \Sigma_0 \geq 0$; it is singular. For $i \neq j$, $N_{ij} = p^*(\Sigma_1)_{ij} - (\Sigma_0)_{ij} \leq (\Sigma_1)_{ij} - (\Sigma_0)_{ij} \leq 0$, with $N_{ij} < 0$ whenever $(\Sigma_1)_{ij} < 0$. So $sI - N$ is entrywise non-negative and irreducible for large s, and by the Perron–Frobenius theorem its largest eigenvalue, $s - \lambda_{\min}(N) = s$, is simple with an eigenvector v having positive entries. Thus $Nv = 0$, i.e. $\Sigma_0 v = p^*\Sigma_1 v$. Let $D = \mathrm{diag}\big((\Sigma_0 v)_i/v_i\big)$. Then $(\Sigma_0 - D)v = 0$, and $\Sigma_0 - D$ has non-positive off-diagonal entries, so $\Sigma_0 - D \geq 0$ by Lemma 5.2. Next, $A' = \Sigma_1 - \Sigma_0 + (1 - \frac{1}{p^*})D$ has the off-diagonal entries $(\Sigma_1 - \Sigma_0)_{ij} \leq 0$, and $A'v = \Sigma_1 v - \frac{1}{p^*}\Sigma_0 v = 0$; so $A' \geq 0$ by Lemma 5.2. Multiplying by $\frac{p^*}{p^*-1}$ gives the left half of (1) at $p = p^*$.

(c) The eigenvalues of a symmetric matrix $\left(\begin{smallmatrix} a\mathrm{Id} & B \\ B^T & a\mathrm{Id}\end{smallmatrix}\right)$ are among a and $a \pm \sigma_i$, σ_i the singular values of B, and $a \pm \sigma_{\max}$ occur. Hence $\lambda_{\max}(\Sigma_0) = 1 + |t|\,\|C\|$, $\lambda_{\min}(\Sigma_1) = 1 - \|C\|$, and $p\Sigma_1 - \Sigma_0 \geq 0$ iff $|p - t|\,\|C\| \leq p - 1$. For $t \leq 0$ this reads $1 + |t|\,\|C\| \leq p(1 - \|C\|)$, i.e. $\lambda_{\max}(\Sigma_0) \leq p\lambda_{\min}(\Sigma_1)$, and Proposition 1.3 applies. □

Proof of Corollary 1.5. Let $A_j \sim \gamma_{k_j}$ be independent of each other and of $\xi \sim N(0,\Sigma)$. Then $\prod_j T_{x_j} f_j(\xi_j) = \mathbb{E}[\prod_j f_j(x_j\xi_j + \sqrt{1-x_j^2}A_j)\,|\,\xi]$, and the vector $(x_j\xi_j + \sqrt{1-x_j^2}A_j)_j$ has covariance $X\Sigma X + \mathrm{Id} - X^2 = \Sigma_0 \in \mathcal{C}$. By Tonelli, $\mathbb{E}_\Sigma[\prod_j T_{x_j} f_j(\xi_j)] = \mathbb{E}_{\Sigma_0}[\prod_j f_j(\xi_j)]$. With

$p_j = p$ and $\alpha = 1/p$, (5) is the p-th power of (2), and (6) reads $\frac{1}{p}(\mathrm{Id} - X^2) \le \Sigma - \frac{1}{p}X\Sigma X$, i.e. $\Sigma_0 \le p\Sigma$. Positive measurable f_j are covered by Theorem 1.1 and Proposition 1.3, and the exponential witnesses of necessity are positive. The off-diagonal blocks of Σ_0 are $(\Sigma_0)_{ij} = x_i x_j \Sigma_{ij}$. In case (i) this gives $\Sigma_0 = (1-t)\mathrm{Id} + t\Sigma$ with $t \in [0,1]$, and Corollary 1.4(a) applies. In case (ii), $\varepsilon_i\varepsilon_j\Sigma_{ij} \le \varepsilon_i\varepsilon_j x_i x_j \Sigma_{ij} \le 0$ because $x_i x_j \in [0,1]$ whenever $\Sigma_{ij} \neq 0$; this is (4), and Corollary 1.4(b) applies. In case (iii), Σ_0 has the off-diagonal block $t\Sigma_{12}$ with $t = x_1 x_2$, and Corollary 1.4(a) (if $t \ge 0$) or (c) (if $t < 0$) applies. In case (iv), $(\Sigma_0)_{ij} = \Sigma_{ij}$ for $i, j \in G$. If $x_j = 0$ for all $j \notin G$, take the groups G and $\{j\}$, $j \notin G$: all blocks of Σ_0 between different groups vanish, which is the grouped form of Corollary 1.4(a) with $t = 0$. If G contains all indices but j_0, take the two groups G and $\{j_0\}$: the block of Σ_0 between them is x_{j_0} times that of Σ, which is the grouped form of (a) or (c) with $t = x_{j_0}$. □

Proof of Proposition 1.6. Exact computation (checked with sympy, `numerics/comparison_example.py`): $\det(\Sigma_0 - p\Sigma_1) = -\frac{1}{16}(2p-3)(4p^2 - 11p + 5)$, so the generalised eigenvalues are $\frac{3}{2}$ and $\frac{11 \pm \sqrt{41}}{8}$, and $p^* = \frac{11+\sqrt{41}}{8}$. Both matrices are invariant under swapping the first two coordinates, and the sets of solutions of (1) and (3) are convex. Averaging a solution with its swap, we may take $D = \mathrm{diag}(d, d, e)$ and $S = \mathrm{diag}(s, s, s_3)$.

Condition (1): $\Sigma_0 - D \ge 0$ iff $d \le \frac{3}{4}$ and $e \le 1$ (eigenvalues $1 - d \pm \frac{1}{4}$ and $1 - e$). With $c = \frac{p}{p-1}$, the matrix $D - c(\Sigma_0 - \Sigma_1)$ has the eigenvector $(1, -1, 0)$ with eigenvalue $d - \frac{c}{4}$, and on the span of $\frac{1}{\sqrt{2}}(1,1,0)$ and $(0,0,1)$ it acts as $\begin{pmatrix} d + c/4 & c/\sqrt{2} \\ c/\sqrt{2} & e \end{pmatrix}$. So it is positive semi-definite iff $d \ge \frac{c}{4}$ and $(d + \frac{c}{4})e \ge \frac{c^2}{2}$. These constraints become weaker as d and e grow, so (1) is feasible iff they hold at $d = \frac{3}{4}$, $e = 1$, i.e. iff $c \le 3$ and $3 + c \ge 2c^2$. Since $(2c-3)(c+1) \le 0$ means $c \le \frac{3}{2}$, this is $p \ge 3$. At $p = 3$, $D = \mathrm{diag}(\frac{3}{4}, \frac{3}{4}, 1)$: $\Sigma_0 - D$ has the eigenvalues $\frac{1}{2}, 0, 0$, and $D - \frac{3}{2}(\Sigma_0 - \Sigma_1)$ has the eigenvalues $\frac{17}{8}, \frac{3}{8}, 0$.

Condition (3): $S - \Sigma_0 \ge 0$ iff $s \ge \frac{5}{4}$ and $s_3 \ge 1$, and $p\Sigma_1 - S$ has the eigenvector $(1, -1, 0)$ with eigenvalue $\frac{p}{2} - s$. So (3) forces $\frac{5}{4} \le s \le \frac{p}{2}$, hence $p \ge \frac{5}{2}$. For $p \ge \frac{5}{2}$ take $S = \mathrm{diag}(\frac{5}{4}, \frac{5}{4}, 1)$: then $S - \Sigma_0$ has the eigenvalues $\frac{1}{2}, 0, 0$, and at $p = \frac{5}{2}$ the matrix $p\Sigma_1 - S$ has the eigenvalues 0 and $2 \pm \frac{3}{4}\sqrt{6}$, all ≥ 0; for larger p add $(p - \frac{5}{2})\Sigma_1 \ge 0$. □

Remark 5.3. For $\Sigma_1 = \mathrm{Id}$, (2) is the forward Brascamp–Lieb (Hölder) inequality $\mathbb{E}_{\Sigma_0} \prod_j f_j \le \prod_j \|f_j\|_{L^p(\gamma_{k_j})}$, which holds iff $\lambda_{\max}(\Sigma_0) \le p$ ([3], see also [8]; this is Proposition 1.3 with $S = p\mathrm{Id}$). Condition (1) is not sharp there. Let $n = 2$, $k_j = 1$ and $\mathrm{cov}_{\Sigma_0}(\xi_1, \xi_2) = \theta \in (0,1)$, so that $p^* = 1 + \theta$. By symmetry and convexity we may take $D = d\mathrm{Id}$, and (1) reads $c\theta \le d \le 1 - \theta$ with $c = \frac{p}{p-1}$. So (1) is feasible iff $\theta < \frac{1}{2}$ and $p \ge \frac{1-\theta}{1-2\theta}$, a threshold larger than $1 + \theta$; for $\theta \ge \frac{1}{2}$ it is never feasible. Thus (1) is not necessary for (2). The reason is that Lemma 3.1 is a reverse inequality.

6. Numerical checks

All of the following is floating point, evidence for the algebra and not part of the proofs. `numerics/check_comparison.py` ($n = 3$, $k_j = 1$, tensor Gauss–Hermite quadrature) checks (A) Lemma 3.1 for 300 random covariances, diagonal D and pairings B, with random smooth positive functions (smallest ratio of the two sides about 1.8); (B) the monotonicity along the path for 60 random pairs (Σ_0, Σ_1) of the form of Corollary 1.5 at which (1) is feasible (no decrease); and (C) the inequality at $p = p^*$ for 25 such pairs with optimised test functions: the optimisers found with 28 nodes per axis exploit the quadrature error, and recomputed with 60 nodes per axis they give log-ratios ≤ 0. `numerics/comparison_example.py` checks Proposition 1.6 exactly (including both thresholds), confirms the threshold $p = 3$ for (1) by

bisection, finds no violation of (2) at $p = p^*$ by Gaussian test functions $e^{a_j y - b_j y^2/2}$ (closed forms), and prints the smallest eigenvalues of $p^*\Sigma_1^{\circ d} - \Sigma_0^{\circ d}$ of Remark 1.7 (0 for $d = 1$, and 0.6102, 0.8982, 1.0381 for $d = 2, 3, 4$). `numerics/comparison_example_attack.py` optimises, in the same example, over test functions $f_j = \exp(a_j t + b_j t^2 + c_j \tanh(w_j(t - d_j)))$ with bounded coefficients at $p \in \{p^*, 2.3, 2.6, 2.9\}$ (only p^* lies in the open range): the best log-ratios are at most $2 \cdot 10^{-8}$ with 30 quadrature nodes per axis, and negative when recomputed with 56. `numerics/comparison_example_attack2.py` uses $f_j = \exp(\sum_{k\le 6} a_{jk} T_k(s))$, T_k Chebyshev polynomials, $s = t/\sqrt{16 + t^2}$, $|a_{jk}| \le 1.5$, optimised with 36 nodes per axis and re-evaluated with 56 and 72: at $p = p^*$ and $p = 2.2$ all 30 optimisers are stable, with best log-ratios $-1.5 \cdot 10^{-7}$ and $-1.4 \cdot 10^{-7}$; a variant with Hermite polynomials and $|a_{jk}| \le 0.3$ gives $-1.8 \cdot 10^{-8}$ and $-2.2 \cdot 10^{-12}$ (`comparison_example_attack2_he.py`). `numerics/comparison_path_flow.py` solves the discretised path problem of Theorem 4.1 (piecewise-linear paths, one diagonal $\tilde{D}$ per piece; a semidefinite programme for fixed T, bisection in T). It reproduces the exponents of Remark 4.2(c) up to the discretisation error, and in Proposition 1.6 it gives 2.4405, 2.3596, 2.3224, 2.3045, 2.2957 for $4, 8, 16, 32, 64$ pieces (and 2.2913, 2.2891 for 128, 256). `numerics/comparison_path_twostage.py` and `comparison_path_multistage.py` do the same with the groups $\{1, 2\}, \{3\}$ on some stretches (Theorem 4.1): two discretised stretches give 2.2152 with $8 + 24$ pieces, and with $32 + 32$ pieces the problem is feasible at $p = 2.21$ but not at 2.20; alternating three or four stretches (16 and 12 pieces each, probed at $p = 2.20$ and 2.21) was not better at the splits we tried. `numerics/comparison_chain_scan.py` scans the construction of Proposition 1.8 over b. Only the exact verifications in Proposition 1.8 are part of a proof. For random $n = 3$ and $n = 4$ instances of Corollary 1.5 where (1) fails at p^*, Gaussian test functions do not violate (2) at p^* either (60 cases each), and the matrix $D = \mathrm{diag}(\Sigma_0 v/v)$ forced by Lemma 5.1 (v had no zero entries in any instance) satisfies (1) in all 2000 random instances with sign pattern (4) for each $n \in \{3, 4, 6\}$, and in none of the random equicorrelated instances with $\rho > 0$, $n \ge 3$ and unequal parameters (`numerics/comparison_gaussian_tests.py`).

7. Novelty and limitations

Novelty. Theorem 1.1 compares a Gaussian expectation under one covariance with a correlated L^p norm under another, for product functions. Its special cases include the reverse Hölder inequality of [3] with equal exponents and Lyapunov's inequality, and Proposition 1.3 is an immediate consequence of [3]. Applied to the forward inequality of [6] with $\alpha = 1/p$, a case excluded from [6, Thm. 9(i)], the results recover the equal-parameter case [5, Thm. 1.1] at $r = 1$ and, for two ρ-correlated functions, [5, Prop. 6.1]. What is new is Theorem 1.1 itself and its path version Theorem 4.1, Corollary 1.4(b), case (ii) of Corollary 1.5 with $n \ge 3$ and unequal parameters, and Propositions 1.8, 1.9, 1.10 and 4.4. Gaussian saturation is known for many inequalities of Brascamp–Lieb type [9, 1, 3, 4, 2, 11]. Comparisons of $\mathbb{E}_{\Sigma_0} F$ with $\mathbb{E}_{\Sigma_1} F$ for a single function (the case $p = 1$) under sign conditions on mixed second derivatives go back to Slepian [12] and Kahane [7], with the interpolation formula (7). Milman, Nakamura and Tsuji [10] prove inequalities for convex sets under Gaussian measures with different covariance matrices, using a Gaussian forward–reverse Brascamp–Lieb inequality for log-concave functions; these results concern convex sets and log-concave functions and do not contain (2) for general non-negative f_j. We did not find a statement covering (2) with two different covariances, but we did not search the literature systematically beyond these works and [8]. Novelty is claimed to the best of this search. The flow generalises the correlation flow of [5, Section 4] in two ways: the covariance moves along an arbitrary segment rather than a ray through Id, and Lemma 3.1 conditions on a block-diagonal rather than a scalar Gaussian component.

Limitations. (1) Condition (1) is sufficient but not necessary for (2): in Remark 5.3, (2) holds at $p^* = 1 + \theta$ while (1) fails there. The necessary condition $\Sigma_0 \le p\Sigma_1$ is not sufficient either (Proposition 1.9), and we do not know the exact threshold in general. The flows of this note (including the construction of Proposition 1.8) do not reach the necessary condition in Proposition 1.6: they give $p \ge 2.1925$ (Proposition 1.8) against $p^* = 2.175\ldots$, the range $[p^*, 2.1925)$ is open, and by Proposition 4.4 the flows are bounded away from p^* there. In the example $\theta = 0.9$, $\rho = 0.5$ after Proposition 1.9 they give only $p \ge 5$ (Proposition 1.3 gives $p \ge 3.8$), while a violation is found at $p = 1.56$ and none for $p \ge 1.6$ (floating point). For the forward inequality (5) with $\alpha = 1/p$ this leaves open, for example, three positively equicorrelated scalar functions with parameters not covered by Corollary 1.5(i) and (iv), such as $x = (1, \frac{1}{2}, 0)$. For scalar blocks the flow is rigid at $p = p^*$: if some v with $\Sigma_0 v = p^*\Sigma_1 v$ has no zero entries, then Lemma 5.1 leaves only the candidate $D = \mathrm{diag}(\Sigma_0 v/v)$. (2) Theorems 1.1 and 4.1 are not sharp in forward situations (Remarks 4.2(c) and 5.3), and Proposition 1.3 is not sharp on the segment of Corollary 1.4(a); we have no method that interpolates between the two. (3) Only the outer exponent $\alpha = 1/p$ of (5) is covered. For other $\alpha \in (0, 1)$ and two functions with unequal parameters, condition (6) is not sufficient in general when $r = \alpha p > 2$ [5, Props. 6.3, 6.5]. For $r > 1$ and $\rho, x_1, x_2 > 0$ a flow with separate noise parameters gives a sufficient condition under which it is sharp, verified at explicit points with $1 < r \le 2$ [5, Thm. 6.9, Prop. 6.11]; it is also sufficient when $r \ge 1$ and $x_1 x_2 = 0$ [5, Prop. 6.4]. For $r < 1$ the analogous flows cannot be sharp [5, Remark 6.7(b)].

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