

CORRELATED FORWARD HYPERCONTRACTIVITY WITH OUTER EXPONENT $\alpha < 1$

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ABSTRACT. Let $\xi = (\xi_1, \dots, \xi_n)$ be a Gaussian vector whose blocks ξ_j are standard Gaussian vectors, let λ be the smallest eigenvalue of its covariance, and let T_x be the Ornstein–Uhlenbeck (Mehler) operator with real parameter $x \in [-1, 1]$. Ivanisvili and Kalantzopoulos proved that the forward inequality $\|\prod_j (T_{r_j} f_j(\xi_j))^{p_j}\|_\alpha \leq \|\prod_j f_j(\xi_j)^{p_j}\|_1$ holds for all positive f_j if and only if an explicit covariance condition holds, for $\alpha \in (-\infty, 0) \cup [1, \infty)$. Their flow method does not apply to $\alpha \in (0, 1)$. We show that for equal parameters $r_j = x$ and equal exponents $p_j = p \geq 1$ their condition remains necessary and sufficient for every $\alpha \in (0, 1)$ and every covariance. Equivalently, for $p \geq 1$ and $0 < r < p$,

$$\left\| \prod_j T_x f_j(\xi_j) \right\|_{L^r} \leq \left\| \prod_j f_j(\xi_j) \right\|_{L^p} \text{ for all } f_j \geq 0 \iff (1 - x^2) + r\lambda x^2 \leq p\lambda.$$

When $p\lambda < 1$ this excludes a neighbourhood of $x = 0$; exponentials along the bottom eigenvector are extremal. For $0 < r < p < 1$ the inequality fails for every $|x| < 1$, so for $0 < p < 1$ (and $\lambda > 0$) the covariance condition is not sufficient; for $p < 0$ (and non-singular covariance) the forward inequality always holds. The range $p\lambda \geq 1$ follows from their theorem at $\alpha = 1$; the new range is $p\lambda < 1$. For two functions with unequal parameters $r_1 \neq r_2$ the condition is not sufficient in general, for any $\alpha \in (0, 1)$: for every $\alpha p > 2$ there are parameters where it holds while second-order perturbations in the second Wiener chaos violate the inequality, and certified examples are given. It is sufficient when $\alpha p = 1$, and when $\alpha p \geq 1$ and one of the two parameters vanishes; for $\alpha p \leq 1$ no second-order perturbation of the constants detects anything beyond it. For $\alpha p > 1$, positive correlation and positive parameters, a flow with separate noise parameters gives a sufficient condition under which the covariance condition is sharp. It is a one-dimensional differential inequality that reduces to two closed-form inequalities. It holds exactly when some path of this flow reaches the threshold, and where the inequalities fail strictly no path comes near it; it is not known to be necessary for the inequality itself. It is verified in ball arithmetic at explicit points with $1 < \alpha p \leq 2$, and numerically it holds at about 92% of the sampled points of that range not already covered by the case $\alpha = 1$. The proof combines a reverse correlated log-Sobolev inequality, which bounds $\text{Ent}(\prod_j \Psi_j(\xi_j))$ from below by $-\lambda \sum_{i < j} \mathbb{E}[(\nabla \Psi_i(\xi_i), B_{ij} \nabla \Psi_j(\xi_j)) \prod_{l \neq i, j} \Psi_l(\xi_l)]$ for every symmetric $B \geq -\text{Id}$ with zero diagonal blocks, with a flow in three parameters (exponent, correlation and noise) along which $(q(1 - \theta) - 1)s^2$ is the controlling quantity.

1. INTRODUCTION AND RESULTS

Let γ_k be the standard Gaussian measure on $\mathbb{R}^k$. For $x \in [-1, 1]$ and measurable $f : \mathbb{R}^k \rightarrow [0, \infty]$ put

$$T_x f(y) = \mathbb{E} f(xy + \sqrt{1 - x^2} G) \in [0, \infty], \quad G \sim \gamma_k.$$

This is the Ornstein–Uhlenbeck (Mehler) operator; on polynomials $T_x H_\beta = x^{|\beta|} H_\beta$. Products of values in $[0, \infty]$ are taken with $0 \cdot \infty = 0$, and negative powers with $\infty^{-a} = 0$. Let $\xi = (\xi_1, \dots, \xi_n)$

be a centred Gaussian vector in $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$ with $\xi_j \sim \gamma_{k_j}$, and let $\Sigma = \text{cov}(\xi)$; its diagonal blocks are identities. Ivanisvili and Kalantzopoulos [6, Thm. 9] considered, for $r_j \in [-1, 1]$ and $p_j \neq 0$, the forward inequality

$$(1) \quad \left\| \prod_{j=1}^n (T_{r_j} f_j(\xi_j))^{p_j} \right\|_{\alpha} \leq \left\| \prod_{j=1}^n f_j(\xi_j)^{p_j} \right\|_1, \quad \|W\|_{\alpha} = (\mathbb{E}|W|^{\alpha})^{1/\alpha},$$

for all measurable $f_j : \mathbb{R}^{k_j} \rightarrow (0, \infty)$. They proved that for $\alpha \in (-\infty, 0) \cup [1, \infty)$ it holds if and only if

$$(2) \quad \text{diag}\left(\frac{1-r_j^2}{p_j} \text{Id}_{k_j}\right)_{j=1}^n \leq ((1-\alpha r_u r_v) \text{cov}(\xi_u, \xi_v))_{u,v=1}^n$$

([6, (2.16), (2.17)]). The range $\alpha \in (0, 1)$ is excluded: their FB theorem needs $(t, y) \mapsto F''(t)y^2/|F'(t)|$ to be convex [6, (2.30)], and for $F(t) = t^{\alpha}$ this function is $(\alpha-1)\frac{y^2}{t}$ for $\alpha > 0$, which is concave when $0 < \alpha < 1$. For the reverse inequality the range $\alpha \in (0, 1]$ is covered [6, Thm. 9(ii)].

We treat the forward inequality for $\alpha \in (0, 1)$ with equal parameters and equal exponents. Write $r = \alpha p$; then (1) with $r_j = x$ and $p_j = p$ is the p -th power of

$$(3) \quad \left\| \prod_{j=1}^n T_x f_j(\xi_j) \right\|_{L^r} \leq \left\| \prod_{j=1}^n f_j(\xi_j) \right\|_{L^p},$$

where $\|W\|_{L^r} = (\mathbb{E}|W|^r)^{1/r}$ also for $r < 1$. Let $\lambda = \lambda_{\min}(\Sigma)$. Since the diagonal blocks of Σ are identities, its eigenvalues average to 1, so $\lambda \in [0, 1]$, and $\lambda = 1$ only if $\Sigma = \text{Id}$.

Theorem 1.1. *Let ξ be as above, $x \in [-1, 1]$, $p \in [1, \infty)$ and $r \in (0, p)$. Then (3) holds for all measurable $f_j : \mathbb{R}^{k_j} \rightarrow [0, \infty]$ if and only if*

$$(4) \quad (1-x^2) + r\lambda x^2 \leq p\lambda.$$

If (4) fails, then (3) fails for $f_j(y) = e^{\langle a_j, y \rangle}$, for every $a = (a_1, \dots, a_n) \neq 0$ with $\Sigma a = \lambda a$.

Proposition 1.2. *Let ξ be as above, $0 < r < p < 1$ and $|x| < 1$. Then (3) fails for some f_1 with values in $[\frac{1}{2}, \frac{3}{2}]$ and $f_2 = \cdots = f_n = 1$.*

Proposition 1.3. *Let ξ be as above with Σ non-singular, $x \in [-1, 1]$, $p < 0$ and $\alpha \in (0, 1)$. Then (1) with $r_j = x$ and $p_j = p$ holds for all measurable $f_j : \mathbb{R}^{k_j} \rightarrow (0, \infty)$. (For $p < 0$ the left side of (2) is ≤ 0 , so (2) holds as well.)*

Corollary 1.4. *Let ξ be as above with Σ non-singular, $x \in [-1, 1]$, $p \neq 0$ and $\alpha \in (0, 1)$. Then (1) with $r_j = x$ and $p_j = p$ holds for all measurable $f_j : \mathbb{R}^{k_j} \rightarrow (0, \infty)$ if and only if $|x| = 1$, or $p \in (-\infty, 0) \cup [1, \infty)$ and (2) holds. So for equal parameters $r_j = x$ and exponents $p_j = p$ with $p \in (-\infty, 0) \cup [1, \infty)$ the characterisation of [6, Thm. 9(i)] extends to $\alpha \in (0, 1)$; for unequal parameters it does not (Proposition 6.5). For $0 < p < 1$ it does not: (2) holds for x close to ± 1 , while (1) fails for every $|x| < 1$.*

Proof. With $r_j = x$ the right side of (2) is $(1-\alpha x^2)\Sigma$, and $1-\alpha x^2 > 0$. So (2) says $\frac{1-x^2}{p} \leq (1-\alpha x^2)\lambda$. For $p > 0$ this is (4) with $r = \alpha p$; for $p < 0$ it holds automatically. Moreover, (1) is the p -th power of (3) when $p > 0$. For $|x| = 1$ and $p > 0$, (3) is Lyapunov's inequality. For $p \geq 1$ apply Theorem 1.1, whose witnesses are positive, for $0 < p < 1$ apply Proposition 1.2, and for $p < 0$ apply Proposition 1.3. If $0 < p < 1$ and $x^2 = 1$, then (4) reads $r\lambda \leq p\lambda$, which holds strictly since $\lambda > 0$; so it holds for x close to ± 1 . $\square$

Remark 1.5. (a) *Examples.* If $\xi_j \in \mathbb{R}^k$ and $\text{cov}(\xi_i, \xi_j) = \rho \text{Id}_k$ for $i \neq j$, then $\lambda = 1 - \rho$ for $\rho \in [0, 1)$ and $\lambda = 1 + (n - 1)\rho$ for $\rho \in [-\frac{1}{n-1}, 0)$; for $n = 2$, $\lambda = 1 - |\rho|$. For $n = 1$, $\lambda = 1$.

(b) If $p\lambda \geq 1$, then (4) holds for every $x \in [-1, 1]$, since its left side is a convex combination of 1 and $r\lambda < p\lambda$. This case is not new. At $\alpha = 1$, (2) with $r_j = x$, $p_j = p$ reads $\frac{1-x^2}{p} \leq (1-x^2)\lambda$, which holds when $p\lambda \geq 1$; so [6, Thm. 9(i)] gives $\|\prod_j T_x f_j(\xi_j)\|_p \leq \|\prod_j f_j(\xi_j)\|_p$, and Lyapunov's inequality gives (3). For two ρ -correlated functions, [11, Prop. 1.4(a)] is proved in the same way, with [6, Thm. 1] at $\alpha = 1$ applied at an intermediate exponent $s \in [\max\{r, (1 - |\rho|)^{-1}\}, p]$ instead of p . At $x = 0$, (3) reads $\prod_j \mathbb{E} f_j \leq \|\prod_j f_j(\xi_j)\|_p$, which is [6, Theorem B] (Chen–Dafnis–Paouris) with $p_j = p$.

(c) If $p\lambda < 1$, then $r\lambda < 1$, and (4) reads $x^2 \geq \frac{1-p\lambda}{1-r\lambda} \in (0, 1]$. So (3) fails near $x = 0$. For two ρ -correlated functions at $x = 0$, for instance, $T_0 f = \mathbb{E} f$, and $f_1 = e^{y_1}$, $f_2 = e^{-y_1}$ give $\mathbb{E} f_1 \mathbb{E} f_2 = e > e^{p(1-\rho)} = \|f_1(\xi_1) f_2(\xi_2)\|_p$ when $p(1 - \rho) < 1$.

(d) Since $|T_x f| \leq T_x |f|$ for complex-valued f , (3) under (4) also holds for complex-valued f_j , in particular for polynomials and for bounded functions. Necessity holds within each of these classes too. Truncations $\min\{e^{\langle a, y \rangle}, M\}$ and the partial sums of the exponential series converge to $e^{\langle a, y \rangle}$ in every $L^s(\gamma_k)$, and so do their images under T_x . Moreover, $(f_j)_j \mapsto \prod_j f_j(\xi_j)$ is continuous from $\prod_j L^{ns}(\gamma_{k_j})$ to L^s by Hölder's inequality. For $n = 2$, (3) is the real case $z = x$ of the two-function inequality studied in [11, 7].

(e) *Comparison with* [6, Thm. 10]. For $q \geq p \geq 1/\lambda$, Ivanisvili and Kalantzopoulos show that $\|\prod_j T_x f_j(\xi_j)\|_q \leq \|\prod_j f_j(\xi_j)\|_p$ for all $f_j \geq 0$ iff $x^2 \leq \frac{p\lambda-1}{q\lambda-1}$. Theorem 1.1 (with $r = q < p$) says that for $p \geq 1$ and $p\lambda < 1$ it holds iff $x^2 \geq \frac{p\lambda-1}{q\lambda-1}$: the same threshold, with the inequality reversed. If $p\lambda \geq 1$ and $q < p$, every x is admissible.

(f) For $n = 1$ and $p \geq 1$, (3) holds for every x , as $\|T_x f\|_r \leq \|T_x f\|_{\max\{r, 1\}} \leq \|f\|_{\max\{r, 1\}} \leq \|f\|_p$. By Proposition 1.2 the restriction $p \geq 1$ in Theorem 1.1 cannot be dropped. For $n = 1$, expanding at $1 + \varepsilon h$ with h in the m -th Wiener chaos gives the necessary condition $(1 - r)x^{2m} \geq 1 - p$; for $m = 1$ (and $\lambda = 1$) this is (4), for $p \geq 1$ these conditions are empty, and for $p < 1$ and large m they fail at every $|x| < 1$.

The main tool is the following lower bound for the entropy $\text{Ent}(W) = \mathbb{E}[W \log W] - \mathbb{E}W \log \mathbb{E}W$ of a product.

Lemma 1.6 (Reverse correlated log-Sobolev inequality). *Let $\xi = (\xi_1, \dots, \xi_n)$ be a centred Gaussian vector in $\mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_n}$ with $\text{cov}(\xi) \geq \lambda \text{Id}$, $\lambda \geq 0$. Let $B = (B_{ij})$ be a symmetric matrix with blocks $B_{ij} \in \mathbb{R}^{k_i \times k_j}$, $B_{jj} = 0$ and $B \geq -\text{Id}$. Let $\Psi_j : \mathbb{R}^{k_j} \rightarrow (0, \infty)$ be C^1 , with Ψ_j and $|\nabla \Psi_j|$ bounded and $\inf \Psi_j > 0$. Then*

$$(5) \quad \text{Ent} \left(\prod_{j=1}^n \Psi_j(\xi_j) \right) \geq -\lambda \sum_{i < j} \mathbb{E} \left[\langle \nabla \Psi_i(\xi_i), B_{ij} \nabla \Psi_j(\xi_j) \rangle \prod_{l \neq i, j} \Psi_l(\xi_l) \right].$$

The basic case is $B_{ij} = \text{Id}$ for $i \neq j$ (blocks of equal size), which satisfies $B \geq -\text{Id}$; then (5) involves $\sum_{i < j} \mathbb{E}[\langle \nabla \Psi_i, \nabla \Psi_j \rangle \prod_{l \neq i, j} \Psi_l]$. If $\Psi_j = e^{\langle a_j, \cdot \rangle}$, $Ba = -a$ and $\text{cov}(\xi)a = \lambda a$, both sides of (5) equal $\frac{\lambda}{2} |a|^2 \mathbb{E} \prod_j \Psi_j$ (these Ψ_j are unbounded, but the computation is explicit). Ivanisvili and Kalantzopoulos proved a correlated log-Sobolev inequality, which bounds the same entropy from above [6, Thm. 11]. Lemma 1.6 is a lower bound of the opposite type.

2. PRELIMINARIES

2.1. The operator T_x . For measurable $f \geq 0$, $T_1 f = f$, $T_0 f = \mathbb{E} f$, $T_x T_y f = T_{xy} f$, and $T_{-x} f = T_x(f(-\cdot))$, since $-G \sim G$. Let $\mathcal{S}$ be the set of C^∞ functions f on some $\mathbb{R}^k$ with

$\inf f > 0$ such that f and all its derivatives are bounded. For $f \in \mathcal{S}$ and $s \in (0, 1]$ we have $T_s f \in \mathcal{S}$, $\nabla T_s f = s T_s \nabla f$, and

$$(6) \quad \partial_s T_s f = -\frac{1}{s} L T_s f, \quad L = \Delta - \langle y, \nabla \rangle.$$

Indeed, with $c = \sqrt{1-s^2}$, $\partial_s T_s f(y) = \mathbb{E} \langle \nabla f(sy + cG), y - \frac{s}{c} G \rangle$ for $s < 1$, and Gaussian integration by parts gives $\mathbb{E} \langle \nabla f(sy + cG), G \rangle = c \mathbb{E} \Delta f(sy + cG)$. So $\partial_s T_s f = \langle y, T_s \nabla f \rangle - s T_s \Delta f = -\frac{1}{s} (\Delta T_s f - \langle y, \nabla T_s f \rangle)$, which extends continuously to $s = 1$. If f is bounded and $s < 1$, then $T_s f$ is C^∞ with bounded derivatives: differentiate the Gaussian density in $T_s f(y) = \int f(w) \phi_c(w - sy) dw$. If moreover $\inf f > 0$, then $T_s f \in \mathcal{S}$.

2.2. Gaussian calculus. Let ζ be a centred Gaussian vector in $\mathbb{R}^K$ with covariance C . For C^1 functions h which, with their first derivatives, have polynomial growth, Gaussian integration by parts reads

$$(7) \quad \mathbb{E}[\zeta_a h(\zeta)] = \sum_b C_{ab} \mathbb{E}[\partial_b h(\zeta)].$$

If $C = C(\theta)$ depends smoothly on θ and stays positive definite, then for C^2 functions h which, with their derivatives up to order two, have polynomial growth,

$$(8) \quad \frac{d}{d\theta} \mathbb{E}_{C(\theta)}[h(\zeta)] = \frac{1}{2} \sum_{a,b} \dot{C}_{ab} \mathbb{E}_{C(\theta)}[\partial_a \partial_b h(\zeta)].$$

This follows from the heat equation $\partial_\theta \phi_{C(\theta)} = \frac{1}{2} \sum_{a,b} \dot{C}_{ab} \partial_a \partial_b \phi_{C(\theta)}$ for the Gaussian density (compare Fourier transforms, $\partial_\theta e^{-\omega^T C \omega / 2} = -\frac{1}{2} \omega^T \dot{C} \omega e^{-\omega^T C \omega / 2}$), and two integrations by parts. If $\dot{C}$ has zero diagonal blocks and $h(\zeta) = \prod_j G_j(\zeta_j)$, then

$$(9) \quad \frac{d}{d\theta} \mathbb{E}_{C(\theta)} \left[\prod_j G_j(\zeta_j) \right] = \sum_{i < j} \mathbb{E}_{C(\theta)} \left[\langle \nabla G_i(\zeta_i), \dot{C}_{ij} \nabla G_j(\zeta_j) \rangle \prod_{l \neq i,j} G_l(\zeta_l) \right].$$

2.3. The path of covariances and the case $q = 1$. Assume $0 < \lambda < 1$ and put $\beta = 1 - \lambda$, $\widehat{B} = (\Sigma - \text{Id})/\beta$ and

$$\Sigma_\theta = \text{Id} + \theta \widehat{B}, \quad \theta \in [0, 1).$$

Then $\widehat{B}$ has zero diagonal blocks and $\lambda_{\min}(\widehat{B}) = -1$. So $\widehat{B} \geq -\text{Id}$, $\lambda_{\min}(\Sigma_\theta) = 1 - \theta > 0$, and $\Sigma_\beta = \Sigma$. We write $\mathbb{E}_\theta$ for expectations with respect to $\xi \sim N(0, \Sigma_\theta)$. For measurable $f_j \geq 0$ and $s \in [0, 1]$, $\prod_j T_s f_j(\xi_j) = \mathbb{E}[\prod_j f_j(s\xi_j + \sqrt{1-s^2}A_j) \mid \xi]$, with $A_j \sim \gamma_{k_j}$ independent of ξ and of each other. The vector $(s\xi_j + \sqrt{1-s^2}A_j)_j$ has covariance $s^2 \Sigma_\theta + (1-s^2)\text{Id} = \Sigma_{s^2\theta}$. Hence, by Tonelli,

$$(10) \quad \mathbb{E}_\theta \left[\prod_j T_s f_j(\xi_j) \right] = \mathbb{E}_{s^2\theta} \left[\prod_j f_j(\xi_j) \right].$$

3. PROOF OF LEMMA 1.6

One Gaussian. Let $\nu = N(m, \sigma^2 \text{Id}_k)$ and let $\Psi > 0$ be C^1 with Ψ , $|\nabla \Psi|$ bounded. For every h with $\mathbb{E}_\nu e^h < \infty$, the entropy inequality $\mathbb{E}_\nu[\Psi h] \leq \text{Ent}_\nu(\Psi) + \mathbb{E}_\nu \Psi \log \mathbb{E}_\nu e^h$ holds. With $h(w) = \langle \lambda', w - m \rangle$ we have $\mathbb{E}_\nu e^h = e^{\sigma^2 |\lambda'|^2 / 2}$ and, by Gaussian integration by parts, $\mathbb{E}_\nu[\Psi(w)(w - m)] = \sigma^2 \mathbb{E}_\nu \nabla \Psi$. Choosing $\lambda' = \mathbb{E}_\nu \nabla \Psi / \mathbb{E}_\nu \Psi$ gives the following bound, which is the reverse local log-Sobolev inequality for the heat semigroup at time σ^2 (compare [1, Ch. 5]):

$$(11) \quad \text{Ent}_\nu(\Psi) \geq \frac{\sigma^2 |\mathbb{E}_\nu \nabla \Psi|^2}{2 \mathbb{E}_\nu \Psi}.$$

Conditioning. If $\lambda = 0$, the right side of (5) vanishes. Let $\lambda > 0$ and write $\xi = \sqrt{\lambda}A + \eta$ with $A \sim \gamma_K$ ($K = \sum_j k_j$) independent of $\eta \sim N(0, \text{cov}(\xi) - \lambda \text{Id})$. Given η , the blocks $\xi_j = \eta_j + \sqrt{\lambda}A_j$ are independent with laws $N(\eta_j, \lambda \text{Id}_{k_j})$. Put $u_j(\eta_j) = \mathbb{E}[\Psi_j(\xi_j) | \eta]$; then u_j is C^1 , $\nabla u_j(\eta_j) = \mathbb{E}[\nabla \Psi_j(\xi_j) | \eta]$, and $u_j \geq \inf \Psi_j > 0$. Put $b_j = \nabla u_j / u_j$, $b = (b_1, \dots, b_n)$ and $U = \prod_j u_j(\eta_j)$. The chain rule for entropy and its exact tensorisation for independent factors give

$$\text{Ent} \left(\prod_j \Psi_j(\xi_j) \right) = \mathbb{E} \left[\text{Ent} \left(\prod_j \Psi_j(\xi_j) \mid \eta \right) \right] + \text{Ent}(U) \geq \mathbb{E} \left[\sum_j \text{Ent}(\Psi_j(\xi_j) \mid \eta) \prod_{l \neq j} u_l \right].$$

By (11), $\text{Ent}(\Psi_j(\xi_j) \mid \eta) \geq \frac{\lambda}{2} |\nabla u_j|^2 / u_j$, so the right side is at least $\frac{\lambda}{2} \mathbb{E}[U|b|^2]$. On the other hand, by conditional independence, and since B has zero diagonal blocks,

$$\sum_{i < j} \mathbb{E} \left[\langle \nabla \Psi_i(\xi_i), B_{ij} \nabla \Psi_j(\xi_j) \rangle \prod_{l \neq i, j} \Psi_l(\xi_l) \right] = \sum_{i < j} \mathbb{E}[U \langle b_i, B_{ij} b_j \rangle] = \frac{1}{2} \mathbb{E}[U b^T B b].$$

Since $B \geq -\text{Id}$, $b^T B b \geq -|b|^2$, and so $\frac{\lambda}{2} \mathbb{E}[U|b|^2] \geq -\frac{\lambda}{2} \mathbb{E}[U b^T B b]$, which is (5). $\square$

4. A FLOW IN THREE PARAMETERS

Assume $0 < \lambda < 1$ and use $\Sigma_\theta, \widehat{B}$ from Section 2.3. Fix $f_j \in \mathcal{S}$ on $\mathbb{R}^{k_j}$. For $q > 0$, $\theta \in (0, 1)$ and $s \in (0, 1]$ put $F_j = T_s f_j$,

$$W = \prod_j F_j(\xi_j)^q, \quad \Phi(q, \theta, s) = \frac{1}{q} \log \mathbb{E}_\theta W, \quad a_j = \frac{\nabla F_j}{F_j}(\xi_j), \quad S = \sum_{i < j} \langle a_i, \widehat{B}_{ij} a_j \rangle.$$

Since $F_j \in \mathcal{S}$ uniformly for s in compact subsets of $(0, 1]$, all integrands below are bounded by polynomials in $|\xi|$, and differentiation under the expectation is justified by dominated convergence. Note the pointwise inequality

$$(12) \quad \sum_j |a_j|^2 + 2S = |a|^2 + a^T \widehat{B} a \geq 0.$$

Lemma 4.1. Φ is C^1 (one-sided in s at $s = 1$), and

$$\begin{aligned} q^2 \mathbb{E}_\theta W \partial_q \Phi &= \text{Ent}(W), & q^2 \mathbb{E}_\theta W \partial_\theta \Phi &= q^3 \mathbb{E}_\theta [W S], \\ q^2 \mathbb{E}_\theta W \partial_s \Phi &= \frac{q^2}{s} \mathbb{E}_\theta \left[W \left((q-1) \sum_j |a_j|^2 + 2\theta q S \right) \right]. \end{aligned}$$

Proof. First, $\partial_q \Phi = -\frac{1}{q^2} \log \mathbb{E} W + \frac{1}{q} \frac{\mathbb{E}[W \log \prod_j F_j]}{\mathbb{E} W} = \frac{\text{Ent}(W)}{q^2 \mathbb{E} W}$. Second, apply (9) with $C(\theta) = \Sigma_\theta$, $\dot{C} = \widehat{B}$ and $G_j = F_j^q$, for which $\nabla G_j = q F_j^{q-1} \nabla F_j$. Third, by (6), $\partial_s \Phi = \mathbb{E}[W \sum_j \partial_s F_j / F_j] / \mathbb{E} W = -\frac{1}{s} \sum_j \mathbb{E}[\Psi^{(j)} L F_j(\xi_j)] / \mathbb{E} W$ with $\Psi^{(j)} = W / F_j(\xi_j)$. The off-diagonal blocks of Σ_θ are $\theta \widehat{B}_{jl}$. By (7) applied to $h = \Psi^{(j)} \partial_c F_j(\xi_j)$ and summed over c ,

$$\mathbb{E}[\Psi^{(j)} L F_j(\xi_j)] = -\mathbb{E} \left[\langle \nabla_j \Psi^{(j)}, \nabla F_j(\xi_j) \rangle + \theta \sum_{l \neq j} \langle \nabla F_j(\xi_j), \widehat{B}_{jl} \nabla_l \Psi^{(j)} \rangle \right].$$

Here $\nabla_j \Psi^{(j)} = (q-1) \Psi^{(j)} a_j$ and $\nabla_l \Psi^{(j)} = q \Psi^{(j)} a_l$ for $l \neq j$, and $\Psi^{(j)} \nabla F_j(\xi_j) = W a_j$. So $\mathbb{E}[\Psi^{(j)} L F_j] = -\mathbb{E}[W((q-1)|a_j|^2 + \theta q \sum_{l \neq j} \langle a_j, \widehat{B}_{jl} a_l \rangle)]$. Summing over j , and using $\widehat{B}_{lj} = \widehat{B}_{jl}^T$, gives the third formula. $\square$

Put $I(q, \theta, s) = (q(1 - \theta) - 1)s^2$. Call a continuous, piecewise C^1 path $t \mapsto (q(t), \theta(t), s(t)) \in (0, \infty) \times (0, 1) \times (0, 1]$ *admissible* if, on each piece, $\dot{q} \geq 0$, $\dot{\theta} \geq 0$, $\dot{s} \geq 0$, $q \geq 1$ wherever $\dot{s} > 0$, and $\frac{d}{dt}I \geq 0$.

Lemma 4.2. *Along an admissible path, $t \mapsto \Phi(q(t), \theta(t), s(t))$ is non-decreasing.*

Proof. By Lemma 4.1, $\Lambda := q^2 \mathbb{E}W \frac{d}{dt}\Phi$ equals

$$\Lambda = \dot{q} \text{Ent}(W) + q^3 \dot{\theta} \mathbb{E}[WS] + \frac{q^2 \dot{s}}{s} \mathbb{E} \left[W \left((q-1) \sum_j |a_j|^2 + 2\theta q S \right) \right].$$

If $\mathbb{E}[WS] \geq 0$, each term is non-negative; for the last one, $\dot{s} > 0$ forces $q \geq 1$. Let $\mathbb{E}[WS] < 0$. By Lemma 1.6 with $\Psi_j = F_j^q$, $B = \widehat{B}$ and $\lambda = 1 - \theta$ (as $\Sigma_\theta \geq (1 - \theta)\text{Id}$), $\text{Ent}(W) \geq -(1 - \theta)q^2 \mathbb{E}[WS] = (1 - \theta)q^2 |\mathbb{E}[WS]|$. If $\dot{s} > 0$, then $q \geq 1$, and (12) gives $(q-1) \sum_j |a_j|^2 + 2\theta q S \geq -2(q(1 - \theta) - 1)S$. Hence, in all cases,

$$\Lambda \geq q^2 |\mathbb{E}[WS]| \left(\dot{q}(1 - \theta) - q\dot{\theta} + \frac{2\dot{s}}{s} (q(1 - \theta) - 1) \right) = \frac{q^2 |\mathbb{E}[WS]|}{s^2} \frac{d}{dt}I \geq 0. \quad \square$$

5. PROOF OF THEOREM 1.1

5.1. Reductions. By $T_{-x}f = T_x(f(-\cdot))$ and $-\xi \sim \xi$ we may assume $x \geq 0$. If $x = 1$, (3) is Lyapunov's inequality. If $\lambda = 1$, then $\Sigma = \text{Id}$, the ξ_j are independent, and $\|T_x f_j\|_r \leq \|T_x f_j\|_{\max\{r, 1\}} \leq \|f_j\|_{\max\{r, 1\}} \leq \|f_j\|_p$, while (4) holds. If $\lambda = 0$, (4) forces $x = 1$. If $x = 0$, (4) says $p\lambda \geq 1$, so by Remark 1.5(b) it holds for every x , and the case $x = 0$ follows from $x \in (0, 1)$ by letting $x \downarrow 0$ (for $f_j \in \mathcal{S}$, $T_x f_j \rightarrow \mathbb{E}f_j$ boundedly; then approximate as in Section 5.3). So let $0 < \lambda < 1$ and $x \in (0, 1)$, and put $\beta = 1 - \lambda$.

5.2. Sufficiency for $f_j \in \mathcal{S}$. We have $\Phi(r, \beta, x) = \log \|\prod_j T_x f_j(\xi_j)\|_{L^r}$ and $\Phi(p, \beta, 1) = \log \|\prod_j f_j(\xi_j)\|_{L^p}$. Put $I_0 = I(r, \beta, x) = (r\lambda - 1)x^2$ and $I_1 = I(p, \beta, 1) = p\lambda - 1$; then (4) is $I_0 \leq I_1$. We join the two points by admissible paths, using (10) once when $r < 1$. Each path below is admissible: the monotonicity of q, θ, s is clear from the formulas, θ stays in $(0, 1)$, and $\frac{d}{dt}I$ is stated. Condition (4) is used only in the case $r \geq 1$, $r\lambda < 1$ and in the subcase $\theta_1 x^2 \leq \beta$, and there it also forces $p \geq 1$. In the other two cases (4) holds automatically (in the last one because $I_0 = -\theta_1 x^2 < -\beta \leq p\lambda - 1$ when $p \geq 1$), and $p \geq 1$ is used only in the last leg of the subcase $\theta_1 x^2 > \beta$, which is where Proposition 1.2 shows the argument must break.

Case $r \geq 1$, $r\lambda < 1$. First $s \mapsto (q(s), \beta, s)$, $s \in [x, 1]$, with $q(s) = (1 + I_0 s^{-2})/\lambda$. Here $q(x) = r$, q increases (as $I_0 < 0$), $q \geq 1$, and $I \equiv I_0$. Then $(q, \beta, 1)$ for q from $q(1) = (1 + I_0)/\lambda$ to p , with $\frac{d}{dt}I = \dot{q}\lambda \geq 0$. This is possible since $q(1) \leq p$ is $I_0 \leq I_1$.

Case $r \geq 1$, $r\lambda \geq 1$. First (r, β, s) , $s \in [x, 1]$, with $\frac{d}{dt}I = 2(r\lambda - 1)s\dot{s} \geq 0$; then $(q, \beta, 1)$, $q \in [r, p]$.

Case $r < 1$. Then $r\lambda < 1$. Put $\theta_1 = 1 - r\lambda \in (\beta, 1)$, so that $I_0 = -\theta_1 x^2$.

Subcase $\theta_1 x^2 \leq \beta$. First $(q, 1 - r\lambda/q, x)$ for $q \in [r, 1]$: θ increases from β to θ_1 , s is constant, and $I \equiv I_0$. By (10), $\Phi(1, \theta_1, x) = \Phi(1, \theta_1 x^2, 1)$. Then $(q, 1 - (1 - \theta_1 x^2)/q, 1)$ for $q \in [1, q_3]$, $q_3 = (1 - \theta_1 x^2)/\lambda \geq 1$: θ increases from $\theta_1 x^2 > 0$ to β , and $I \equiv -\theta_1 x^2 = I_0$. Finally $(q, \beta, 1)$ for $q \in [q_3, p]$; here $q_3 \leq p$ is $I_0 \leq I_1$.

Subcase $\theta_1 x^2 > \beta$. Put $\theta_2 = \beta/x^2 \in (\beta, \theta_1)$. First $(q, 1 - r\lambda/q, x)$ for $q \in [r, q_4]$, $q_4 = r\lambda/(1 - \theta_2) < 1$, with I constant and θ increasing from β to θ_2 . Then (q, θ_2, x) for $q \in [q_4, 1]$, with $\frac{d}{dt}I = \dot{q}(1 - \theta_2)x^2 \geq 0$. By (10), $\Phi(1, \theta_2, x) = \Phi(1, \beta, 1)$. Finally $(q, \beta, 1)$ for $q \in [1, p]$.

By Lemma 4.2 and (10), $\Phi(r, \beta, x) \leq \Phi(p, \beta, 1)$ in every case, which is (3).

5.3. Approximation. Let $f_j : \mathbb{R}^{k_j} \rightarrow [\varepsilon, M]$ be measurable, $0 < \varepsilon < M < \infty$. For $\delta \in (0, 1)$, $f_j^\delta := T_{1-\delta} f_j \in \mathcal{S}$, and $T_x f_j^\delta = T_{x(1-\delta)} f_j$. As $\delta \rightarrow 0$, $T_{x(1-\delta)} f_j \rightarrow T_x f_j$ pointwise (continuity of the Mehler kernel in its parameter, for $x < 1$, and dominated convergence), and $f_j^\delta \rightarrow f_j$ in $L^2(\gamma_{k_j})$, hence in γ_{k_j} -probability. Since each $\xi_j \sim \gamma_{k_j}$ and all functions take values in $[\varepsilon, M]$, dominated convergence carries (3) from f_j^δ to f_j . For general measurable $f_j \geq 0$ apply this to $\max\{\varepsilon, \min\{f_j, M\}\}$, let $\varepsilon \downarrow 0$ (dominated convergence, all values in $[0, M+1]$) and then $M \uparrow \infty$ (monotone convergence on both sides).

5.4. Necessity. Let $\Sigma a = \lambda a$, $a \neq 0$, and $f_j(y) = e^{\langle a_j, y \rangle}$. Then $T_x f_j(y) = e^{x\langle a_j, y \rangle + (1-x^2)|a_j|^2/2}$, and $V = \sum_j \langle a_j, \xi_j \rangle$ is centred Gaussian with variance $a^T \Sigma a = \lambda |a|^2$. Hence

$$\log \left\| \prod_j T_x f_j(\xi_j) \right\|_{L^r} = \frac{(1-x^2)|a|^2}{2} + \frac{r\lambda x^2 |a|^2}{2}, \quad \log \left\| \prod_j f_j(\xi_j) \right\|_{L^p} = \frac{p\lambda |a|^2}{2}.$$

If (4) fails, the first exceeds the second. $\square$

5.5. Proof of Proposition 1.2. With $f_2 = \dots = f_n = 1$, (3) reads $\|T_x f_1\|_{L^r(\gamma_{k_1})} \leq \|f_1\|_{L^p(\gamma_{k_1})}$. Fix $m \geq 1$ with $(1-r)x^{2m} < 1-p$. Let H_m be the m -th Hermite polynomial in the first coordinate, so $\mathbb{E}H_m = 0$, $\mathbb{E}H_m^2 = m!$ and $T_x H_m = x^m H_m$. For $M > 0$ let $h_M = \varphi_M(H_m) - \mathbb{E}\varphi_M(H_m)$, with $\varphi_M(t) = \max\{-M, \min\{t, M\}\}$. As $M \rightarrow \infty$, $h_M \rightarrow H_m$ in $L^2(\gamma_{k_1})$, so $\mathbb{E}h_M^2 \rightarrow m!$ and, T_x being a contraction of L^2 , $\mathbb{E}(T_x h_M)^2 \rightarrow x^{2m}m!$. Fix M with $(1-r)\mathbb{E}(T_x h)^2 < (1-p)\mathbb{E}h^2$, where $h = h_M$. Then h is bounded, $\mathbb{E}h = 0$, and $T_x h$ is bounded by $\|h\|_\infty$ with $\mathbb{E}T_x h = 0$.

For $s > 0$ and measurable g with $\mathbb{E}g = 0$ and $|g| \leq 1/2$, Taylor's formula for $(1+t)^s$ on $[-\frac{1}{2}, \frac{1}{2}]$ gives $\mathbb{E}(1+g)^s = 1 + \frac{s(s-1)}{2}\mathbb{E}g^2 + R$ with $|R| \leq C_s \mathbb{E}|g|^3$. Hence $\|1+g\|_s = 1 + \frac{s-1}{2}\mathbb{E}g^2 + O(\mathbb{E}|g|^3 + (\mathbb{E}g^2)^2)$. Apply this with $g = \varepsilon T_x h$, $s = r$, and with $g = \varepsilon h$, $s = p$, for $0 < \varepsilon \leq 1/(2\|h\|_\infty)$, and put $f_1 = 1 + \varepsilon h \in [\frac{1}{2}, \frac{3}{2}]$. Then $T_x f_1 = 1 + \varepsilon T_x h$, and

$$\|T_x f_1\|_r - \|f_1\|_p = \frac{\varepsilon^2}{2} \left((1-p)\mathbb{E}h^2 - (1-r)\mathbb{E}(T_x h)^2 \right) + O(\varepsilon^3) > 0$$

for ε small. $\square$

5.6. Proof of Proposition 1.3. Taking logarithms in (1) and multiplying by $1/p < 0$ shows that (1) is equivalent to $\Phi(\alpha p, \beta, x) \geq \Phi(p, \beta, 1)$, with $\Phi(q, \theta, s) = \frac{1}{q} \log \mathbb{E}_\theta \prod_j (T_s f_j)(\xi_j)^q$ as in Section 4 and $\beta = 1 - \lambda$, so that $\Sigma_\beta = \Sigma$. If $\lambda = 1$ we take $\hat{B} = 0$. Section 4 assumed $q > 0$ and $0 < \lambda < 1$; here $q < 0$, $\lambda \in (0, 1]$ and $\theta = \beta$ stays fixed, and the proof of Lemma 4.1 applies verbatim. By symmetry we may assume $x \geq 0$, and $x = 0$ follows from $x > 0$ by letting $x \downarrow 0$. Let first $f_j \in \mathcal{S}$. Along the path $t \mapsto (q, \theta, s) = (p + t(\alpha - 1)p, \beta, 1 - t(1 - x))$, $t \in [0, 1]$, we have $\dot{q} = (\alpha - 1)p > 0$, $\dot{\theta} = 0$, $\dot{s} \leq 0$ and $q < 0$. By Lemma 4.1,

$$q^2 \mathbb{E} W \frac{d}{dt} \Phi = \dot{q} \text{Ent}(W) + \frac{q^2 \dot{s}}{s} \mathbb{E}[W a^T (q\Sigma - \text{Id}) a] \geq 0,$$

since $(q-1) \sum_j |a_j|^2 + 2\beta q S = a^T ((q-1)\text{Id} + q(\Sigma - \text{Id}))a$ and $q\Sigma - \text{Id}$ is negative definite. Hence $\Phi(p, \beta, 1) \leq \Phi(\alpha p, \beta, x)$. For measurable f_j with values in $[\varepsilon, M]$ we approximate by $T_{1-\delta} f_j$ as in Section 5.3. All functions involved take values in $[\varepsilon, M]$, so both sides converge by dominated convergence.

For general $f_j > 0$ we use two monotone limits. First let $f_j \geq 1/M$ and apply the bounded case to $\min\{f_j, K\}$. As $K \rightarrow \infty$, $T_x \min\{f_j, K\} \uparrow T_x f_j$. The negative powers are bounded by M^{-p} and $M^{-\alpha p}$, so dominated convergence applies to both sides. Then apply this to $\max\{f_j, 1/M\}$

and let $M \rightarrow \infty$. Now $\max\{f_j, 1/M\}^p \uparrow f_j^p$, and $(T_x \max\{f_j, 1/M\})^{\alpha p} \uparrow (T_x f_j)^{\alpha p}$ pointwise (with $\infty^{\alpha p} = 0$), so monotone convergence applies to both sides. $\square$

6. UNEQUAL PARAMETERS

In this section $n = 2$, (X, Y) is a pair of ρ -correlated standard Gaussian vectors in $\mathbb{R}^k$ (that is, $\text{cov}(X, Y) = \rho \text{Id}_k$), $\rho \in [0, 1]$, $p \geq 1$, $0 < r < p$ and $x_1, x_2 \in [-1, 1]$, and we consider

$$(13) \quad \|T_{x_1} f(X) T_{x_2} g(Y)\|_{L^r} \leq \|f(X) g(Y)\|_{L^p},$$

which is (1) with $r_j = x_j$, $p_j = p$ and $\alpha = r/p$, raised to the power $1/p$. For $m \geq 1$ put

$$M_m = \begin{pmatrix} p-1-(r-1)x_1^{2m} & (p-r(x_1x_2)^m)\rho^m \\ (p-r(x_1x_2)^m)\rho^m & p-1-(r-1)x_2^{2m} \end{pmatrix}, \quad M = M_1.$$

For $k = 1$, $f = e^{a \cdot}$ and $g = e^{b \cdot}$ give the log-ratio $-\frac{1}{2}(a, b)M(a, b)^T$, and (2) is $M \geq 0$ (multiply (2) by p). For $k > 1$ the matrices are $M \otimes \text{Id}_k$, and the statements below hold with $k = 1$ replaced by any k .

Proposition 6.1. *Let $r = 1 < p$. Then (13) holds for all measurable $f, g \geq 0$ if and only if $1 - x_1x_2\rho \leq p(1 - \rho)$, which is (2).*

Proof. By the argument of (10), $(x_1X + \sqrt{1-x_1^2}A_1, x_2Y + \sqrt{1-x_2^2}A_2)$, with $A_1, A_2 \sim \gamma_k$ independent of each other and of (X, Y) , is a ρ' -correlated pair with $\rho' = x_1x_2\rho$, so $\mathbb{E}[T_{x_1}f(X)T_{x_2}g(Y)] = \mathbb{E}_{\rho'}[f(X)g(Y)]$, where $\mathbb{E}_{\rho'}$ refers to a ρ' -correlated pair. Necessity is the exponential computation above.

If $x_1x_2 \geq 0$, then $\mathbb{E}_{\rho'}[f(X)g(Y)] = \mathbb{E}[T_x f(X)T_x g(Y)]$ with $x = \sqrt{x_1x_2}$, and (13) with $r = 1$ is the case $r = 1$, $n = 2$, $\lambda = 1 - \rho$ of Theorem 1.1 with parameter x ; there (4) reads $(1 - x_1x_2) + x_1x_2(1 - \rho) \leq p(1 - \rho)$.

If $x_1x_2 < 0$, put $s = 1 + |\rho'| = 1 - x_1x_2\rho$. Nelson's theorem in two-function form gives $\mathbb{E}_{\rho'}[f(X)g(Y)] \leq \|f\|_{L^s} \|g\|_{L^s}$: indeed $\mathbb{E}_{\rho'}[F(X)G(Y)] = \mathbb{E}[F T_{\rho'} G]$ and $\|T_{\rho'} G\|_{L^{s'}} \leq \|G\|_{L^s}$, since $(s' - 1)\rho'^2 = |\rho'| = s - 1$. For bounded positive f, g , [6, Theorem B] with $F_1 = f^s$, $F_2 = g^s$ and exponents p/s gives $\|f\|_{L^s}^p \|g\|_{L^s}^p \leq \mathbb{E}[f(X)^p g(Y)^p]$, because its covariance condition $\text{diag}(\frac{s}{p}\text{Id}, \frac{s}{p}\text{Id}) \leq \text{cov}(X, Y)$ is $1 - \frac{s}{p} \geq \rho$, i.e. $1 - x_1x_2\rho \leq p(1 - \rho)$. For bounded $f, g \geq 0$ apply this to $f + \varepsilon, g + \varepsilon$ and let $\varepsilon \rightarrow 0$; for general $f, g \geq 0$ use $\min\{f, K\}, \min\{g, K\}$ and monotone convergence. $\square$

So for two functions and $r = 1$ condition (2) is sharp, and for $x_1x_2 \geq 0$ unequal parameters reduce to equal ones. For $r \neq 1$ this reduction is not available, and for $r > 2$ condition (2) is not sufficient (Propositions 6.3 and 6.5). The mechanism is the following family of necessary conditions.

Proposition 6.2. *Let $k = 1$ and $m \geq 1$. If (13) holds for all measurable f, g with values in $[\frac{1}{2}, \frac{3}{2}]$, then $M_m \geq 0$.*

Proof. Let H_m be the m -th Hermite polynomial and, as in the proof of Proposition 1.2, $h_K = \varphi_K(H_m) - \mathbb{E}\varphi_K(H_m)$ with $\varphi_K(t) = \max\{-K, \min\{t, K\}\}$. As $K \rightarrow \infty$, $h_K \rightarrow H_m$ in $L^2 = L^2(\mathbb{R}, e^{-y^2/2}dy/\sqrt{2\pi})$. The quantities $\mathbb{E}h^2$, $\mathbb{E}[h(X)h(Y)]$, $\mathbb{E}(T_x h)^2$ and $\mathbb{E}[T_{x_1}h(X)T_{x_2}h(Y)]$ are continuous in $h \in L^2$: T_x is a contraction of L^2 , and $\mathbb{E}[u(X)v(Y)] = \mathbb{E}[u T_{\rho} v]$. Since $T_x H_m = x^m H_m$ and $\mathbb{E}[H_m(X)H_m(Y)] = \rho^m m!$, they tend to $m!$, $\rho^m m!$, $x^{2m} m!$ and $(x_1x_2)^m \rho^m m!$.

Fix $(a, b) \in [-1, 1]^2$ and K , put $h = h_K$, and let $0 < \varepsilon \leq 1/(2\|h\|_{\infty})$, $f = 1 + \varepsilon ah$ and $g = 1 + \varepsilon bh$; these take values in $[\frac{1}{2}, \frac{3}{2}]$. For $s > 0$ and $|u|, |v| \leq \frac{1}{2}$, Taylor's formula gives $(1+u)^s(1+v)^s = 1 + s(u+v) + \frac{s(s-1)}{2}(u^2+v^2) + s^2 uv + R$ with $|R| \leq C_s(|u|^3 + |v|^3)$. Apply it

with $s = r$, $u = \varepsilon a T_{x_1} h(X)$, $v = \varepsilon b T_{x_2} h(Y)$, and with $s = p$, $u = \varepsilon a h(X)$, $v = \varepsilon b h(Y)$. All these variables have mean zero. Taking logarithms we get

$$\log \|f(X)g(Y)\|_{L^p} - \log \|T_{x_1} f(X) T_{x_2} g(Y)\|_{L^r} = \frac{\varepsilon^2}{2} Q_K(a, b) + O(\varepsilon^3),$$

where

$$\begin{aligned} Q_K(a, b) &= (p-1)(a^2 + b^2)\mathbb{E}h^2 + 2pab\mathbb{E}[h(X)h(Y)] \\ &\quad - (r-1)(a^2\mathbb{E}(T_{x_1}h)^2 + b^2\mathbb{E}(T_{x_2}h)^2) - 2rab\mathbb{E}[T_{x_1}h(X)T_{x_2}h(Y)]. \end{aligned}$$

As $K \rightarrow \infty$, $Q_K(a, b) \rightarrow m!(a, b)M_m(a, b)^T$. If M_m has a negative eigenvalue with unit eigenvector (a, b) , choose K with $Q_K(a, b) < 0$ and then ε small: (13) fails. $\square$

For $m = 1$ this is $M \geq 0$. For $x_1 = x_2$, Theorem 1.1 shows that $M \geq 0$ implies $M_m \geq 0$ for all m . For unequal parameters this fails in general (Propositions 6.3 and 6.5), although it holds for $r \leq 1$ (Proposition 6.6).

Proposition 6.3. *Let $k = 1$.*

(a) *In each of the cases*

$$(E1) (\rho, r, x_1, x_2, p) = \left(\frac{97}{100}, 5, 1, \frac{4}{5}, \frac{29}{2}\right), \quad (E2) (\rho, r, x_1, x_2, p) = \left(\frac{49}{50}, 8, \frac{49}{50}, \frac{4}{5}, 25\right),$$

the matrix M is positive definite, so that (2) holds, but $\det M_2 < 0$. Consequently (13) fails for some f, g with values in $[\frac{1}{2}, \frac{3}{2}]$, i.e. (1) with $r_j = x_j$, $p_j = p$ and $\alpha = r/p (= \frac{10}{29}$, resp. $\frac{8}{25})$ fails although (2) holds. Explicitly, $f(y) = e^{-c_1 y^2/2}$ and $g(y) = e^{-c_2 y^2/2}$ with $(c_1, c_2) = (\frac{49}{100}, -\frac{31}{100})$ in (E1) and $(\frac{1}{5}, -\frac{3}{20})$ in (E2) give

$$\log \|T_{x_1} f(X) T_{x_2} g(Y)\|_{L^r} - \log \|f(X) g(Y)\|_{L^p} = 3.6672 \dots \cdot 10^{-4}, \text{ resp. } 2.0526 \dots \cdot 10^{-4}.$$

(b) *In the case (E3) $(\rho, r, x_1, x_2, p) = (\frac{97}{100}, 8, \frac{19}{20}, \frac{1}{2}, 47)$ every M_m , $m \geq 1$, is positive definite, but $(c_1, c_2) = (-\frac{653}{2500}, \frac{4373}{5000})$ gives the log-ratio $0.20181 \dots > 0$. So even all second-order conditions at $f = g = 1$ together do not imply (13).*

Proof. In (E1), $\det M = \frac{7831}{40000}$ with $M_{11} = \frac{19}{2}$, and $\det M_2 = -0.3578 \dots < 0$; in (E2), $\det M = \frac{15710319}{39062500}$ with $M_{11} > 0$, and $\det M_2 = -1.2650 \dots < 0$. These are rational numbers, and their signs were checked in exact arithmetic (`numerics/certify_unequal_counterexample.py`). Proposition 6.2 with $m = 2$ gives the first claim of (a). For the Gaussians, $T_x f(y) = D^{-1/2} e^{-by^2/2}$ with $D = 1 + c(1 - x^2)$ and $b = cx^2/D$, when $D > 0$. For $W = (X, Y)$ with covariance $S = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and a diagonal matrix A with $S^{-1} + A$ positive definite, $\mathbb{E} e^{-W^T A W/2} = \det(\text{Id} + SA)^{-1/2}$. Hence the log-ratio equals

$$-\frac{1}{2} \log D_1 - \frac{1}{2} \log D_2 - \frac{1}{2r} \log \det(\text{Id} + S \text{diag}(rb_1, rb_2)) + \frac{1}{2p} \log \det(\text{Id} + S \text{diag}(pc_1, pc_2)).$$

The script checks the conditions $D_j > 0$, $S^{-1} + \text{diag}(rb_1, rb_2) > 0$ and $S^{-1} + \text{diag}(pc_1, pc_2) > 0$ exactly, and evaluates the expression in Arb ball arithmetic (256 bits), with the values stated. In (E1) and (E2) direct Gauss–Hermite quadrature (160×160 nodes) agrees to eleven digits. In (E3) the Gaussians are close to the boundary of integrability. There M_m is checked to be positive definite exactly for $m \leq 3$; for $m \geq 4$, $(M_m)_{jj} \geq p - 1 - (r - 1)x_j^8$ (as $r > 1$) and $|(M_m)_{12}| \leq pp^4$ (as $0 < r(x_1 x_2)^m < p$), and $(46 - 7 \cdot 0.95^8)(46 - 7 \cdot 0.5^8) > (47 \cdot 0.97^4)^2$. $\square$

The Gaussian g in (a) grows. Replacing it by $\min\{g, K\}$ gives bounded counterexamples, since both sides converge monotonically as $K \rightarrow \infty$.

On the faces where one parameter vanishes, the condition is sharp when $r \geq 1$.

Proposition 6.4. *Let $r \geq 1$ and $x_2 = 0$. Then (13) holds for all measurable $f, g : \mathbb{R}^k \rightarrow (0, \infty)$ if and only if $M \geq 0$, i.e. $p \geq s := 1 + (r-1)x_1^2$ and $(p-s)(p-1) \geq p^2\rho^2$. The same holds with the roles of x_1 and x_2 exchanged.*

Proof. Necessity is the exponential computation above. Let $M \geq 0$. Then $1 \leq s \leq r$, $T_0g = \mathbb{E}g$, and the left side of (13) is $\|T_{x_1}f\|_{L^r} \mathbb{E}g$. By Nelson's hypercontractivity theorem, $\|T_{x_1}f\|_{L^r} \leq \|f\|_{L^s}$; for $s > 1$ this is because $x_1^2 = \frac{s-1}{r-1}$, and for $s = 1$ it is trivial. By [6, Theorem B] applied to $F_1 = f^s$ with exponent p/s and $F_2 = g$ with exponent p ,

$$\|f\|_{L^s}^p (\mathbb{E}g)^p = (\mathbb{E}F_1)^{p/s} (\mathbb{E}F_2)^p \leq \mathbb{E}[F_1(X)^{p/s} F_2(Y)^p] = \mathbb{E}[f(X)^p g(Y)^p],$$

because the covariance condition of that theorem, $\text{diag}(\frac{s}{p}\text{Id}, \frac{1}{p}\text{Id}) \leq \text{cov}(X, Y)$, is $(1 - \frac{s}{p})(1 - \frac{1}{p}) \geq \rho^2$ with $s \leq p$, which is $M \geq 0$. Combining the two bounds gives (13). $\square$

The examples of Proposition 6.3(a) are not isolated: sufficiency fails for every $r > 2$.

Proposition 6.5. *Let $k = 1$, $r > 2$ and $p > \max\{r, 2 + \frac{2}{r-2}\}$. Then there are $\rho, x_2 \in (0, 1)$ such that, with $x_1 = 1$, $M \geq 0$ and $\det M_2 < 0$. Hence (2) holds while (13) fails for some f, g with values in $[\frac{1}{2}, \frac{3}{2}]$. After a small increase of p and a small decrease of x_1 , M is positive definite and still $\det M_2 < 0$. In particular, for every $\alpha \in (0, 1)$ condition (2) is not sufficient for (1) with unequal parameters: take $r \geq 2 + \sqrt{2}$ and $p = r/\alpha$, so that $p > r \geq 2 + \frac{2}{r-2}$.*

Proof. Put $A = p - r > 0$ and $\Gamma = 2 - (p-2)(r-2) < 0$. For $\sigma > 0$ and $\kappa > 0$ let $\rho = e^{-\sigma}$ and $x_2 = e^{-\kappa\sigma}$. With $x_1 = 1$ the entry $(M_m)_{11} = A$ does not depend on m , and

$$D_m(\sigma, \kappa) := \det M_m = A(A + (r-1)(1 - e^{-2m\kappa\sigma})) - (A + r(1 - e^{-m\kappa\sigma}))^2 e^{-2m\sigma}.$$

This is an entire function of (σ, κ) with $D_m(0, \kappa) = 0$. Inserting $1 - e^{-2m\kappa\sigma} = 2m\kappa\sigma - 2m^2\kappa^2\sigma^2 + O(\sigma^3)$, $1 - e^{-m\kappa\sigma} = m\kappa\sigma - \frac{1}{2}m^2\kappa^2\sigma^2 + O(\sigma^3)$ and $e^{-2m\sigma} = 1 - 2m\sigma + 2m^2\sigma^2 + O(\sigma^3)$ gives

$$D_m(\sigma, \kappa) = 2mA(A - \kappa)\sigma + m^2q(\kappa)\sigma^2 + O(\sigma^3), \quad q(\kappa) = (2A - Ar - r^2)\kappa^2 + 4Ar\kappa - 2A^2,$$

uniformly for κ in compact sets. The function $G = D_1/\sigma$ extends analytically to $\sigma = 0$, with $G(0, \kappa) = 2A(A - \kappa)$ and $\partial_\kappa G(0, A) = -2A \neq 0$. By the implicit function theorem there is an analytic $\kappa(\sigma)$ for small σ with $\kappa(0) = A$ and $D_1(\sigma, \kappa(\sigma)) = 0$, and comparing coefficients gives $\kappa(\sigma) = A + \frac{q(A)}{2A}\sigma + O(\sigma^2)$. Therefore

$$D_2(\sigma, \kappa(\sigma)) = 4A(A - \kappa(\sigma))\sigma + 4q(\kappa(\sigma))\sigma^2 + O(\sigma^3) = 2q(A)\sigma^2 + O(\sigma^3),$$

and $q(A) = A^2(2A - Ar - r^2 + 4r - 2) = A^2\Gamma < 0$. So for small $\sigma > 0$ we have $\rho, x_2 \in (0, 1)$, $\det M = D_1 = 0$ and $M_{11} = A > 0$, hence $M \geq 0$, while $\det M_2 < 0$. Proposition 6.2 with $m = 2$ gives the failure. Increasing p adds a positive multiple of $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ to M , which makes it positive definite. A small change of p and x_1 keeps $\det M_2 < 0$ by continuity. For the last sentence, $r \geq 2 + \sqrt{2}$ gives $2 + \frac{2}{r-2} \leq 2 + \sqrt{2} \leq r < p$. $\square$

In the regime of this proof the sign of $\det M_2$ is that of $\Gamma = 2 - (p-2)(r-2)$, which is positive for all $r \leq 2$ and $p \geq 1$ (it is at least r). So this mechanism needs $r > 2$ (`numerics/forward_unequal_asymptotics.py` checks the expansion and the limit $2A^2\Gamma$ in 50-digit arithmetic).

For $r \leq 1$ the chaos conditions give nothing beyond (2).

Proposition 6.6. *Let $0 < r \leq 1$, $p > r$, $p \geq 1$, $\rho \in [0, 1]$ and $x_1, x_2 \in [-1, 1]$. If $M \geq 0$, then $M_m \geq 0$ for every $m \geq 1$.*

Proof. Put $P_0 = p - 1$ and $c = r - 1 \leq 0$. The diagonal entries $P_0 - cx_j^{2m} \geq P_0 \geq 0$ of M_m are non-negative, so it suffices to show $\det M_m \geq 0$. Replacing p by $p + \varepsilon$ adds $\varepsilon \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ to M and makes it positive definite, an open condition; so, by continuity of M_m in all parameters, we may assume $\rho, |x_1|, |x_2| \in (0, 1)$. Write $\rho = e^{-\sigma}$, $x_1^2 = e^{-u}$, $x_2^2 = e^{-v}$ with $\sigma, u, v > 0$, and put

$$\phi(s) = \log(P_0 - ce^{-s}), \quad \chi(s) = \log(p - re^{-s}), \quad \tilde{\chi}(s) = \log(p + re^{-s}) \quad (s \geq 0).$$

Here χ and $\tilde{\chi}$ are well defined since $p > r$, and ϕ since $P_0 \geq 0 \geq c$ with $(P_0, c) \neq (0, 0)$. The diagonal of M_m is $(e^{\phi(mu)}, e^{\phi(mv)})$, and its off-diagonal entry satisfies $|(M_m)_{12}| \leq e^{\tilde{\chi}(m(u+v)/2) - m\sigma}$, with $\hat{\chi} = \chi$ if $x_1x_2 \geq 0$ and $\hat{\chi} = \tilde{\chi}$ if $x_1x_2 < 0$, with equality for $m = 1$. So $\det M \geq 0$ means $\sigma \geq \hat{\sigma}(u, v)$, and $\det M_m \geq 0$ follows from $m\sigma \geq \hat{\sigma}(mu, mv)$, where $\hat{\sigma}(u, v) = \hat{\chi}(\frac{u+v}{2}) - \frac{1}{2}\phi(u) - \frac{1}{2}\phi(v)$.

First let $\hat{\chi} = \chi$ and $\sigma^* := \hat{\sigma}$. We have $\chi'' < 0$ and, because $c \leq 0$, $\phi'' \geq 0$. Hence $\ell(\tau) = \sigma^*(\tau u, \tau v)$ is concave on $[0, \infty)$, and $\ell(0) = 0$ since $\phi(0) = \chi(0) = \log(p - r)$. A concave function ℓ on $[0, \infty)$ with $\ell(0) = 0$ satisfies $\ell(m) \leq m\ell(1)$ for $m \geq 1$. So $\sigma^*(mu, mv) \leq m\sigma^*(u, v) \leq m\sigma$. If $x_1x_2 < 0$, then $\hat{\sigma} = \sigma^* + \psi(\frac{u+v}{2})$ with $\psi = \tilde{\chi} - \chi = \log \frac{p+re^{-s}}{p-re^{-s}}$, which is non-negative and decreasing. Hence $\hat{\sigma}(mu, mv) \leq m\sigma^*(u, v) + \psi(\frac{u+v}{2}) \leq m\hat{\sigma}(u, v) \leq m\sigma$. $\square$

For $r \geq 1$ and $x_1x_2 \leq 0$ the implication $M \geq 0 \Rightarrow M_m \geq 0$ is immediate: the diagonal entries of M_m do not decrease with m , and $|(M_m)_{12}| \leq (p + r|x_1x_2|^m)\rho^m \leq |M_{12}|$. So every failure of the chaos conditions has $r > 1$ and $x_1x_2 > 0$. For $r \leq 1$ no perturbation of the constants $f = g = 1$ at second order detects anything beyond (2), while for every $r > 2$ the second chaos does (Proposition 6.5). For $1 < r \leq 2$ the same reduction applies, but ϕ is concave and the concavity argument fails. Numerically $\sigma^*(mu, mv) \leq m\sigma^*(u, v)$ still holds there: the derivative of $\det M$ along the dilation $(\sigma, u, v) \mapsto t(\sigma, u, v)$ is non-negative at every boundary point sampled for $r < 2$ and in a targeted minimisation at $r = 2$ (values below rounding level are positive in 50-digit arithmetic), and it is negative at many sampled points for $r > 2$ (`numerics/forward_unequal_barrier.py`).

Remark 6.7. (a) *Where the conditions fail.* In our scans (`numerics/forward_unequal_chaos_scan.py`, floating point, $x_j \in (0, 1)$), at the threshold $p = p^*$ of $M \geq 0$ some M_m with $m \leq 60$ is indefinite only for $r > 2$. None of the 80000 random cases with $r < 2$ fails (for $r \leq 1$ this is Proposition 6.6). The 40000 random cases with $2 \leq r < 2.5$ find nothing either, although Proposition 6.5 gives failures there, with ρ and x_1 extremely close to 1 and p large. A targeted minimisation finds nothing below rounding level (about $-2 \cdot 10^{-14}$) for $r \in \{0.5, 0.9, 1.5, 2\}$, and negative values from $r = 2.2$ on. The smallest values of $\lambda_{\min}(M_m)/p^*$ found are $-9 \cdot 10^{-6}$ at $r = 2.2$, $-2 \cdot 10^{-4}$ at $r = 2.5$, $-1.6 \cdot 10^{-3}$ at $r = 3$ and $-1.9 \cdot 10^{-2}$ at $r = 5$. They need ρ close to 1 ($\rho \geq 0.9$ in all cases found) and one x_j close to 1; the other can be as small as about 0.6. In all 18012 failing random cases the level $m = 2$ fails first. Proposition 6.3(b) shows that centred Gaussians, which see the second chaos non-perturbatively, can violate (13) even where every M_m is positive definite. So for every $r > 2$ condition (2) is not sufficient in general, although it is sufficient on the faces $x_1x_2 = 0$ (Proposition 6.4) and wherever Theorem 6.9 below applies. We do not know whether it is sufficient for all $r \in (0, 2] \setminus \{1\}$, nor what the exact condition is for $r > 2$. The natural candidate is that all Gaussian test functions satisfy (13). A random search over centred Gaussians at $p = p^*$ found no violation for $r \in \{0.3, 0.7, 1.3, 1.7, 2, 2.2, 2.5\}$ (`numerics/forward_unequal_gaussian_search.py`). Non-Gaussian test functions $f, g = \exp(\sum_{k \leq 6} a_k T_k(s))$ and $\exp(\sum_{k \leq 6} a_k \text{He}_k(s))$, $s = t/\sqrt{16 + t^2}$ resp. $t/\sqrt{1 + t^2/16}$, with bounded coefficients, optimised at $p = p^*$ at five points with $r \in \{1.5, 1.8, 2\}$ and at four points with $r \in \{0.5, 0.8\}$, and re-evaluated on finer quadrature grids, found no violation either (best log-ratios between $-4 \cdot 10^{-7}$ and $-6 \cdot 10^{-10}$;

`numerics/forward_unequal_flex_attack.py` and `forward_unequal_flex_attack_he.py`). In the related comparison setting of [4], however, Gaussian functions are not extremal [4, Prop. 1.10], so Gaussian tests alone are weak evidence here.

(b) *What the method gives.* For $r < 1$ and $x_1 x_2 > 0$, raising q to 1 at fixed noise and then using (10) yields sufficiency under $(1 - r(1 - \rho))x_1 x_2 \geq 1 - p(1 - \rho)$. We only sketch this; when $(1 - r(1 - \rho))x_1 x_2 > \rho$ the path has to stop at $\theta = \rho/(x_1 x_2)$, as in the second subcase of Section 5.2. This condition implies $M \geq 0$, and is strictly stronger for some parameters with $x_1 \neq x_2$. Here the flow method cannot be sharp for the following structural reason. Consider flows of the type of Lemma 4.2, with a common exponent q , the correlation θ and separate noise parameters s_1, s_2 . For exponential test functions the flow evaluates the Gaussian log-ratio matrix $N(q, \theta, s_1, s_2)$ with diagonal entries $1 + (q - 1)s_j^2$ and off-diagonal entry $q\theta s_1 s_2$. A flow that is monotone for all exponentials has $N(t)$ non-decreasing between its initial and final values, whose difference is $M = M(p)$. When the path crosses $q = 1$, $P = N(t_1) - N(0)$ satisfies $0 \leq P \leq M(p)$ and has the fixed diagonal $((1 - r)x_1^2, (1 - r)x_2^2)$. At $p = p^*$ the matrix M has rank one, which forces $P = \tau M$ and then $(1 - r)(p - 1)(x_1^2 - x_2^2) = 0$; for $r < 1$, $p > 1$ and $x_j \geq 0$ this means $x_1 = x_2$. The set of such P is compact and, for $x_1 \neq x_2$, empty at p^* , so it is also empty for p near p^* .

For $r \geq 1$ no such crossing is needed. With q allowed to pause and $s_j \leq 1$ enforced, candidate lossless paths reach p^* in at least 1405 of 1707 random cases with 2001 steps, and 1409 with 8001 steps ($r \in [1, 4]$, $\rho \leq 0.95$; `numerics/forward_unequal_flow_obstruction.py`). For $r > 1$, Corollary 6.10 and Proposition 6.12 below show that a path of the flow of Lemma 6.8 reaching p^* exists exactly when a closed-form criterion holds; the criterion holds at 1410 of these cases, including all 1405 found by the search (`numerics/forward_unequal_flow_obstruction_vs_criterion.py`). Turning such paths into a proof requires a version of Lemma 4.2 with separate noise parameters and conditioning on a non-isotropic Gaussian component, along which θ may decrease. Lemma 6.8 below is such a version, and Theorem 6.9 turns it into a sufficient condition under which (2) is sharp.

(c) *Earlier tests.* At the test points of `numerics/test_forward_unequal.py` and `numerics/test_forward_unequal_rge1.py` with $M \geq 0$ ($r \leq 3$, $\rho \leq 0.8$), optimised test functions found no violation; at the control points with $M \not\geq 0$ they did. The closed-form Gaussian scan `numerics/forward_unequal_gaussians.py` ($p \leq 3$, $\rho \leq 0.9$) found no violation either; its parameter box contains no failure of any M_m .

6.1. A flow with separate noise parameters. Let (X, Y) be a θ -correlated pair of standard Gaussian vectors in $\mathbb{R}^k$, $|\theta| < 1$, and let $f, g \in \mathcal{S}$. For $q \geq 1$ and $s_1, s_2 \in (0, 1]$ put $F = T_{s_1} f$, $G = T_{s_2} g$,

$$W = F(X)^q G(Y)^q, \quad \Phi(q, \theta, s_1, s_2) = \frac{1}{q} \log \mathbb{E}_\theta W, \quad a_1 = \frac{\nabla F}{F}(X), \quad a_2 = \frac{\nabla G}{G}(Y).$$

The proof of Lemma 4.1, applied to each noise parameter separately (there $\widehat{B}_{12} = \text{Id}$), gives

$$\begin{aligned} q^2 \mathbb{E} W \partial_q \Phi &= \text{Ent}(W), & q^2 \mathbb{E} W \partial_\theta \Phi &= q^3 \mathbb{E}[W \langle a_1, a_2 \rangle], \\ q^2 \mathbb{E} W \partial_{s_1} \Phi &= \frac{q^2}{s_1} \mathbb{E}[W((q - 1)|a_1|^2 + \theta q \langle a_1, a_2 \rangle)], \end{aligned}$$

and the same for s_2 with a_1, a_2 exchanged.

Lemma 6.8. *Let $t \mapsto (q, \theta, s_1, s_2)(t)$ be continuous on $[0, 1]$ and piecewise C^1 with finitely many pieces, with $q \geq 1$, $|\theta| < 1$ and $s_j \in (0, 1]$. Suppose that on each piece $\dot{q} \geq 0$ and $\dot{s}_j \geq 0$, and*

that for every t in the interior of a piece there are $d_1, d_2 \geq 0$ with

$$(14) \quad \begin{pmatrix} 1-d_1 & \theta \\ \theta & 1-d_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \dot{q}d_1 + 2(q-1)\frac{\dot{s}_1}{s_1} & q(\dot{\theta} + \theta(\frac{\dot{s}_1}{s_1} + \frac{\dot{s}_2}{s_2})) \\ q(\dot{\theta} + \theta(\frac{\dot{s}_1}{s_1} + \frac{\dot{s}_2}{s_2})) & \dot{q}d_2 + 2(q-1)\frac{\dot{s}_2}{s_2} \end{pmatrix} \geq 0.$$

Then $t \mapsto \Phi(q(t), \theta(t), s_1(t), s_2(t))$ is non-decreasing.

Here $d_j = 0$ is allowed. Where $\dot{q} = 0$ one may take $d_1 = d_2 = 0$; the first condition is then $|\theta| \leq 1$, and only the second one remains. The dimension k plays no role, since all matrices act through $\otimes \text{Id}_k$.

Proof. Write L_0 for the second matrix in (14) without the terms $\dot{q}d_j$. By the formulas above, $\Lambda := q^2 \mathbb{E} W \frac{d}{dt} \Phi = \dot{q} \text{Ent}(W) + \frac{q^2}{2} \mathbb{E}[W a^T (L_0 \otimes \text{Id}_k) a]$, $a = (a_1, a_2)$. By the first condition in (14) we may write $(X, Y) = \eta + (\sqrt{d_1} A_1, \sqrt{d_2} A_2)$ with $A_1, A_2 \sim \gamma_k$ and η centred Gaussian, all independent. Given η , X and Y are independent with laws $N(\eta_1, d_1 \text{Id})$ and $N(\eta_2, d_2 \text{Id})$. Put $\Psi_1 = F^q$, $\Psi_2 = G^q$, $u_j = \mathbb{E}[\Psi_j | \eta]$, $b_j = \mathbb{E}[\nabla \Psi_j | \eta]/u_j$ and $U = u_1 u_2$. As in the proof of Lemma 1.6, by (11), $\text{Ent}(W) \geq \frac{1}{2} \mathbb{E}[U(d_1 |b_1|^2 + d_2 |b_2|^2)]$. Since $\nabla \Psi_j = q \Psi_j a_j$, conditional independence gives $\mathbb{E}[W \langle a_1, a_2 \rangle] = q^{-2} \mathbb{E}[U \langle b_1, b_2 \rangle]$, and the conditional Cauchy-Schwarz inequality $\mathbb{E}[|\nabla \Psi_j|^2 | \eta] \geq |\mathbb{E}[\nabla \Psi_j | \eta]|^2 / u_j$ gives $\mathbb{E}[W |a_j|^2] \geq q^{-2} \mathbb{E}[U |b_j|^2]$. The diagonal entries $2(q-1)\dot{s}_j/s_j$ of L_0 are non-negative, so

$$\Lambda \geq \frac{1}{2} \mathbb{E} \left[U b^T \left((\dot{q} \text{diag}(d_1, d_2) + L_0) \otimes \text{Id}_k \right) b \right] \geq 0.$$

Differentiation under the expectation is justified as in Lemma 4.1, and Φ is continuous along the path. $\square$

The same proof works for n functions and a general path of covariances. Let $t \mapsto (C(t), q(t), s_1(t), \dots, s_n(t))$ be continuous on $[0, 1]$ and piecewise C^1 , with $C(t)$ positive definite with identity diagonal blocks, $q > 0$ and $s_j \in (0, 1]$. Let $\xi \sim N(0, C(t))$, $\Phi = \frac{1}{q} \log \mathbb{E} \prod_j (T_{s_j} f_j(\xi_j))^q$ and $\sigma_j = \dot{s}_j/s_j$, and suppose that on each piece $\dot{q} \geq 0$, $\sigma_j \geq 0$ and $(q-1)\sigma_j \geq 0$. If for every t in the interior of a piece some block-diagonal D with $0 \leq D \leq C(t)$ satisfies $\dot{q}D + L \geq 0$, where L has the diagonal blocks $2(q-1)\sigma_j \text{Id}$ and the off-diagonal blocks $q(\dot{C}_{jl} + (\sigma_j + \sigma_l)C_{jl})$, then Φ is non-decreasing. For non-scalar blocks D_j the proof uses (11) in the matrix form $\text{Ent}_\nu \Psi \geq \langle A \mathbb{E}_\nu \nabla \Psi, \mathbb{E}_\nu \nabla \Psi \rangle / (2 \mathbb{E}_\nu \Psi)$ for $\nu = N(m, A)$, which is (11) with $\sigma = 1$ applied to $z \mapsto \Psi(m + A^{1/2} z)$. With all $s_j \equiv 1$ and $q = p^t$ this is the ungrouped case of [4, Thm. 4.1], which also allows a singular starting covariance and grouped blocks. With equal s_j , $C(t) = \Sigma_{\theta(t)}$ and $D = (1-\theta) \text{Id}$ it gives Lemma 4.2: as $\hat{B} \geq -\text{Id}$ and $\dot{\theta} + 2\theta\sigma \geq 0$, the condition $\dot{q}D + L \geq 0$ reduces to its value on the bottom eigenvectors of $\hat{B}$, which is $s^{-2} \frac{d}{dt} I \geq 0$.

For two functions with $\rho \in (0, 1)$, $x_1, x_2 \in (0, 1]$ and $r > 1$ this lemma yields a sufficient condition under which (2) is sharp. Put $\mu_j = 1 + (r-1)x_j^2$ and $\mu_0 = rx_1x_2$, so that $M_{jj} = p - \mu_j$ and $M_{12} = \rho(p - \mu_0)$. The quadratic $\det M(p) = (p - \mu_1)(p - \mu_2) - \rho^2(p - \mu_0)^2$ has leading coefficient $1 - \rho^2 > 0$ and is ≤ 0 at $p = \mu_1, \mu_2$, so it has real roots $p_- \leq p^*$ with $p_- \leq \mu_1, \mu_2 \leq p^*$. Hence $M(p) \geq 0$ for $p \geq p^*$, $\det M(p) < 0$ for $p_- < p < p^*$, and the diagonal of $M(p)$ is negative for $p < p_-$; if $p_- < p^*$, then $M(p_-) \geq 0$ would force $p_- = \mu_1 = \mu_2$ and $\rho(p_- - \mu_0) = 0$, and then p_- would be a double root. So $\{p : M(p) \geq 0\} = [p^*, \infty)$: p^* is the threshold of (2). Let $u = (1, -\kappa)$ be the kernel vector of $M(p^*)$, $\kappa = M_{11}(p^*)/M_{12}(p^*)$; for $r < p^*$ and $x_j \in (0, 1]$ one has $p^* > r \geq \mu_j, \mu_0$, so $\kappa > 0$. Put

$$c_1 = 1 + (r-1)x_1^2 - \kappa r \rho x_1 x_2, \quad c_2 = r \rho x_1 x_2 - \kappa(1 + (r-1)x_2^2),$$

$\gamma_1(\varphi) = c_1 - 1 + \kappa\varphi$, $\gamma_2(\varphi) = \varphi - c_2 - \kappa$, and $\varphi_{\min} = \max\{(1 - c_1)/\kappa, c_2 + \kappa\}$, so that $\gamma_1, \gamma_2 \geq 0$ exactly for $\varphi \geq \varphi_{\min}$.

Theorem 6.9. *Let $k \geq 1$, $\rho \in (0, 1)$, $x_1, x_2 \in (0, 1]$ and $1 < r < p^*$. Suppose there is a continuous, piecewise C^1 , non-decreasing $\varphi : [r, p^*] \rightarrow \mathbb{R}$ with*

- (a) $\varphi(r) \geq \max\{\rho x_1 x_2, \varphi_{\min}\}$ and $\gamma_1(\varphi(r)) + \gamma_2(\varphi(r)) > 0$;
- (b) $\varphi'(q) \geq \max\left\{\frac{\gamma_1(\varphi(q))}{\kappa q(q-1)}, \frac{\gamma_2(\varphi(q))}{q(q-1)}\right\}$ on each piece;
- (c) $\varphi(p^*) \leq \rho$.

Then (13) holds for $p = p^$, hence for every $p \geq p^*$, and for all measurable $f, g \geq 0$. So at this point (2) is necessary and sufficient.*

Proof. For $q \in [r, p^*]$ and $\varphi \in \mathbb{R}$ put

$$w_1(q, \varphi) = \frac{c_1 - 1 + \kappa q \varphi}{q - 1}, \quad w_2(q, \varphi) = \frac{q \varphi - c_2 - \kappa}{\kappa(q - 1)},$$

$$N(q, \varphi) = \begin{pmatrix} 1 + (q-1)w_1 & q\varphi \\ q\varphi & 1 + (q-1)w_2 \end{pmatrix}.$$

A direct computation gives $N(q, \varphi)u = (c_1, c_2)$ for all q, φ ; N is the Gaussian log-ratio matrix of the flow for exponential test functions, with $w_j = s_j^2$ and $\varphi = \theta s_1 s_2$. Moreover $w_j(r, \rho x_1 x_2) = x_j^2$, and the two rows of $M(p^*)u = 0$ give $w_1(p^*, \rho) = w_2(p^*, \rho) = 1$. Consider the path made of three stretches: at $q = r$, φ rises from $\rho x_1 x_2$ to $\varphi(r)$; then q runs from r to p^* with $\varphi = \varphi(q)$; at $q = p^*$, φ rises from $\varphi(p^*)$ to ρ . Along it put $s_j = w_j^{1/2}$ and $\theta = \varphi/(s_1 s_2)$. At the start $(q, \theta, s_1, s_2) = (r, \rho, x_1, x_2)$ and at the end $(p^*, \rho, 1, 1)$, so by Lemma 6.8 it suffices to check its hypotheses, for $f, g \in \mathcal{S}$; Section 5.3 then gives all measurable $f, g \geq 0$.

Conjugating the matrices in (14) by $\text{diag}(s_1, s_2)$ and putting $e_j = w_j d_j$, the conditions become

$$\begin{pmatrix} w_1 - e_1 & \varphi \\ \varphi & w_2 - e_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \dot{q}e_1 + (q-1)\dot{w}_1 & q\dot{\varphi} \\ q\dot{\varphi} & \dot{q}e_2 + (q-1)\dot{w}_2 \end{pmatrix} \geq 0,$$

together with $e_j \geq 0$, $\dot{q} \geq 0$ and $\dot{w}_j \geq 0$. On the first and third stretches $\dot{q} = 0$ and we take $e_1 = e_2 = 0$, so that the first matrix is $\begin{pmatrix} w_1 & \varphi \\ \varphi & w_2 \end{pmatrix}$, which is positive semi-definite because $|\theta| < 1$ (checked below). There φ is the parameter, $\partial_\varphi w_1 = \kappa q/(q-1) > 0$, $\partial_\varphi w_2 = q/(\kappa(q-1)) > 0$, and the second matrix is $q \begin{pmatrix} \kappa & 1 \\ 1 & 1/\kappa \end{pmatrix} \geq 0$. On the second stretch take q as the parameter and $e_1 = \gamma_1(\varphi)/(q-1)$, $e_2 = \gamma_2(\varphi)/(\kappa(q-1))$, which are non-negative by (a), since φ is non-decreasing. Then $w_1 - e_1 = \kappa\varphi$ and $w_2 - e_2 = \varphi/\kappa$, so the first matrix is $\varphi \begin{pmatrix} \kappa & 1 \\ 1 & 1/\kappa \end{pmatrix} \geq 0$. The two matrices add up to $\frac{d}{dq}N(q, \varphi(q))$, which annihilates u ; the first does too, hence so does the second, which is therefore positive semi-definite as soon as its trace $e_1 + e_2 + (q-1)(w'_1 + w'_2)$ is non-negative. Finally $w'_1 \geq 0$ and $w'_2 \geq 0$ are equivalent to the two lower bounds in (b).

It remains to check $s_j \in (0, 1]$ and $|\theta| < 1$. Along the whole path w_1 and w_2 are non-decreasing, and they end at $w_j(p^*, \rho) = 1$, so $w_j \leq 1$. Moreover $w_j \geq x_j^2 > 0$ on the first stretch, $w_1 \geq \kappa\varphi > 0$ and $w_2 \geq \varphi/\kappa > 0$ on the second, and on the third w_j increases from its positive value at the end of the second. On the second stretch $w_1 w_2 - \varphi^2 = e_1(w_2 - e_2) + e_2 w_1 > 0$, because $e_1 + e_2 > 0$ there by (a). On the first and third stretches $Q(\varphi) = w_1 w_2 - \varphi^2$ satisfies $Q' = \frac{q}{q-1}(\kappa w_2 + w_1/\kappa) - 2\varphi > 2(\sqrt{w_1 w_2} - \varphi)$, which is positive wherever $Q \geq 0$; since $Q > 0$ at the start of each of these stretches, $Q > 0$ throughout. So $\theta^2 = \varphi^2/(w_1 w_2) < 1$. $\square$

The flow is lossless on the exponential test functions in the direction u : for $f = e^{a\cdot}$, $g = e^{b\cdot}$ ($k = 1$) one has $\Phi = \frac{1}{2}(a, b)N(q, \varphi)(a, b)^T$, and $N(q, \varphi)u$ does not depend on (q, φ) , so Φ is constant along the path when $(a, b) \parallel u$. This is why the exponent p^* is reached. The hypothesis is not known to be necessary for the sufficiency of (2) (see the end of this section). By the following corollary it can be checked in closed form.

Corollary 6.10. *In the setting of Theorem 6.9 put $\varphi_s = \max\{\rho x_1 x_2, \varphi_{\min}\}$. If $\gamma_1(\varphi_s) + \gamma_2(\varphi_s) > 0$ and $w_j(r, \varphi_s) \leq 1$ for $j = 1, 2$, then (13) holds for $p = p^*$, so (2) is necessary and sufficient. This is the case, in particular, whenever $\rho x_1/x_2 \leq \kappa \leq x_1/(\rho x_2)$; for $x_1 = x_2$ this always holds. Conversely, if the hypothesis of Theorem 6.9 holds, then $w_j(r, \varphi_s) \leq 1$ for $j = 1, 2$. So, if $\gamma_1(\varphi_s) + \gamma_2(\varphi_s) > 0$, the hypothesis of Theorem 6.9 holds if and only if $w_1(r, \varphi_s) \leq 1$ and $w_2(r, \varphi_s) \leq 1$.*

Proof. Put $K_0 = 1 - c_1 - \kappa c_2 - \kappa^2$. Then $\gamma_1(\varphi)/\kappa - \gamma_2(\varphi) = -K_0/\kappa$ for all φ , so the first lower bound in (b) is the larger one exactly when $K_0 \leq 0$. In the proof of Theorem 6.9, $\partial_q w_1 = -\gamma_1/(q-1)^2$ and $\partial_\varphi w_1 = \kappa q/(q-1)$, so w_1 is constant along $q \mapsto (q, \varphi(q))$ iff $\varphi' = \gamma_1(\varphi)/(\kappa q(q-1))$; similarly w_2 is constant iff $\varphi' = \gamma_2(\varphi)/(q(q-1))$. Let $K_0 \leq 0$, $W_1 = w_1(r, \varphi_s)$, and define φ by $w_1(q, \varphi(q)) = W_1$, i.e. $\varphi(q) = ((q-1)W_1 + 1 - c_1)/(\kappa q)$. Then $\varphi(r) = \varphi_s$, which gives (a); φ' equals the larger lower bound in (b) as long as $\gamma_1(\varphi) \geq 0$, and is then non-negative, so φ is non-decreasing and $\gamma_1, \gamma_2 \geq 0$ persist; and $w_1(p^*, \varphi(p^*)) = W_1 \leq 1 = w_1(p^*, \rho)$ gives $\varphi(p^*) \leq \rho$, since $w_1(p^*, \cdot)$ is increasing. This is (c). For $K_0 \geq 0$ argue in the same way with w_2 . For the last statement, $\gamma_1(\rho x_1 x_2) = (r-1)x_1(x_1 - \kappa \rho x_2)$ and $\gamma_2(\rho x_1 x_2) = (r-1)x_2(\kappa x_2 - \rho x_1)$, so under the stated inequalities $\varphi_s = \rho x_1 x_2$ and $w_j(r, \varphi_s) = x_j^2 \leq 1$. Both $\gamma_j(\rho x_1 x_2)$ vanish only if $x_1 = \kappa \rho x_2$ and $\kappa x_2 = \rho x_1$, which would give $x_1 = \rho^2 x_1$; so their sum is positive. For $x_1 = x_2$ one has $\kappa = 1$.

For the converse, let φ be as in Theorem 6.9. By (b), $q \mapsto w_j(q, \varphi(q))$ is non-decreasing (see the end of the proof of Theorem 6.9), and $w_j(r, \cdot)$, $w_j(p^*, \cdot)$ are increasing. With (a) and (c), $w_j(r, \varphi_s) \leq w_j(r, \varphi(r)) \leq w_j(p^*, \varphi(p^*)) \leq w_j(p^*, \rho) = 1$. $\square$

The four points of the following proposition all need the initial pause ($\varphi_s > \rho x_1 x_2$).

Proposition 6.11. *At $(\rho, r, x_1, x_2) = (\frac{3}{5}, \frac{3}{2}, \frac{9}{10}, \frac{1}{5})$, $(\frac{1}{2}, \frac{6}{5}, \frac{4}{5}, \frac{3}{10})$, $(\frac{7}{10}, 2, \frac{7}{10}, \frac{2}{5})$ and $(\frac{4}{5}, \frac{9}{5}, \frac{3}{5}, \frac{3}{4})$, inequality (13) holds for all measurable $f, g \geq 0$ if and only if $p \geq p^*$, that is, iff (2) holds.*

Proof (computer-assisted, ball arithmetic). Necessity is the exponential computation. For sufficiency, `numerics/certify_forward_noise_flow.py` computes p^* (a root of a quadratic) and $\kappa, c_1, c_2, \varphi_{\min}$ in ball arithmetic and checks the hypothesis of Corollary 6.10. As an independent check of Theorem 6.9 it also builds a polygonal φ with 400 pieces of equal length on $[r, p^*]$. On a piece $[q_k, q_{k+1}]$ the right side of (b) is increasing in φ and decreasing in q (as $\gamma_j \geq 0$), so it is bounded by its value at $(q_k, \varphi(q_{k+1}))$; the slope on the piece is chosen above this bound, and the bound is re-checked in ball arithmetic. Conditions (a) and (c) are checked the same way; the end values $\varphi(p^*)$ of the polygon are 0.535, 0.343, 0.388 and 0.489, against ρ . For the exact path of the proof of Corollary 6.10 they are 0.5328, 0.3413, 0.3875 and 0.4886 (`numerics/certify_forward_noise_flow_remarks.py`). Theorem 6.9 applies. $\square$

Numerically (`numerics/forward_noise_flow_scan.py`) the hypothesis of Theorem 6.9 holds at 2731 of 2976 sampled points with $1 < r \leq 2$ and $p^* > r$ (about 92%; r, ρ, x_1, x_2 independent and uniform on $(1, 2)$, $(0.05, 0.95)$, $(0.05, 1)$, $(0.05, 1)$, points with $|x_1 - x_2| < 0.02$ dropped), and at 638 of 1222 sampled points with $2 < r \leq 8$ and $p^* > r$ (r uniform on $(2, 8)$, ρ on $(0.05, 0.99)$, x_j as before, no points dropped). Points with $p^* \leq r$ are left out: there (13) holds for every $p \geq r$ by [6, Thm. 9(i)] at $\alpha = 1$ and Lyapunov's inequality, as in Remark 1.5(b). Condition (a) by itself can always be met by taking $\varphi(r)$ large. At every observed failure the pause forced by (a) already pushes some $w_j(r, \varphi_s)$ above 1, as it must by the converse part of Corollary 6.10. For the same reason, the agreement of that criterion with a direct integration of the extremal differential equation at all 8505 random points tested is a consistency check of the script, not independent evidence. The hypothesis fails, as it must, at the counterexamples (E1)–(E3) and at all 400 random points with $r > 2$ at which some M_m is indefinite at p^* .

A direct quadrature check of Lemma 6.8 along the path of Theorem 6.9 at the first point of Proposition 6.11 finds Φ increasing for random non-Gaussian f, g , and constant up to $2 \cdot 10^{-15}$ for the extremal exponentials (`numerics/forward_noise_flow_check.py`). Among the points with $p^* > r$ in a sample of 300000 (r, ρ, x_1, x_2 uniform on $(1, 2), (0, 1), (0, 1), (0, 1)$; same script as the scan above), the criterion of Corollary 6.10 failed only when $\max\{x_1, x_2\} > 0.82$ (the smallest value observed was 0.83), at a rate rising with ρ from 2% to 43% among the points with $\max\{x_1, x_2\} \geq 0.8$. This cut-off is a property of the sample: at $(\rho, r, x_1, x_2) = (\frac{6249}{6250}, 2, \frac{163}{400}, \frac{4083}{5000})$, where $\max\{x_1, x_2\} = 0.8166$ and $p^* \approx 4694$, the criterion fails, with $w_2(r, \varphi_s) = 1.000145\dots$ (ball arithmetic, `numerics/certify_forward_noise_flow_remarks.py`).

We do not know whether (2) is sufficient where the criterion of Corollary 6.10 fails, for $1 < r \leq 2$. The evidence suggests that the hypothesis of Theorem 6.9 is not necessary. At $(\rho, x_1, x_2) = (\frac{3}{5}, 1, \frac{3}{10})$ it fails at $r = 1 + \frac{j}{100}$, $j = 1, \dots, 100$, and at $r = 1 + 10^{-k}$, $k = 3, \dots, 8$, with $w_1(r, \varphi_s) > 1.09$ throughout and $w_1(r, \varphi_s) \rightarrow 1.09$ numerically as $r \downarrow 1$ (ball arithmetic, same script). Yet at $r = 1$ condition (2) is sufficient (Proposition 6.1). At $(\rho, r, x_1, x_2) = (\frac{3}{5}, \frac{11}{10}, \frac{99}{100}, \frac{3}{10})$ the hypothesis fails ($w_1(r, \varphi_s) = 1.0770\dots$). There an optimised attack on (13) at $p = p^*$, with non-Gaussian test functions in a 14-parameter family for each of f and g and a finer re-evaluation, found no violation (best log-ratio $-3 \cdot 10^{-10}$); the same attack finds the violation at (E1) (log-ratio $+1.2 \cdot 10^{-2}$; `numerics/forward_noise_flow_nonhyp_attack.py`). It found no violation at $p = p^*$ either at $(\rho, r, x_1, x_2) = (\frac{4}{5}, \frac{3}{2}, \frac{19}{20}, \frac{1}{2}), (\frac{4}{5}, 2, \frac{19}{20}, \frac{1}{2})$ and $(\frac{3}{5}, 2, 1, \frac{3}{10})$, where $w_1(r, \varphi_s) \geq 1.10$ (ball arithmetic); the best log-ratios lie between $-1.2 \cdot 10^{-9}$ and $-5 \cdot 10^{-10}$ (`numerics/out_forward_noise_flow_nonhyp_attack_more.txt`). If (2) is sufficient at such points, the proof needs a different argument: by the next proposition, no path of Lemma 6.8 from the starting point comes near the threshold there.

Proposition 6.12. *Let $\rho \in (0, 1)$, $x_1, x_2 \in (0, 1]$ and $1 < r < p^*$.*

- (a) *If a path as in Lemma 6.8 joins $(q, \theta, s_1, s_2) = (r, \rho, x_1, x_2)$ to $(p^*, \rho, 1, 1)$, then $w_1(r, \varphi_s) \leq 1$ and $w_2(r, \varphi_s) \leq 1$, with strict inequalities if $\gamma_1(\varphi_s) + \gamma_2(\varphi_s) = 0$.*
- (b) *If for every n a path as in Lemma 6.8 joins (r, ρ, x_1, x_2) to a point P_n , and $P_n \rightarrow (p^*, \rho, 1, 1)$, then $w_1(r, \varphi_s) \leq 1$ and $w_2(r, \varphi_s) \leq 1$.*

So if $\max_j w_j(r, \varphi_s) > 1$, some neighbourhood of $(p^, \rho, 1, 1)$ contains no endpoint of a path of Lemma 6.8 starting at (r, ρ, x_1, x_2) .*

Proof. (a) Put $w_j = s_j^2$, $\varphi = \theta s_1 s_2$ and $N(t) = \begin{pmatrix} 1+(q-1)w_1 & q\varphi \\ q\varphi & 1+(q-1)w_2 \end{pmatrix}$, all continuous and piecewise C^1 in t . In the interior of a piece let A and B be the two matrices of (14) conjugated by $\text{diag}(s_1, s_2)$, with $e_j = w_j d_j$, as in the proof of Theorem 6.9; then $A, B \geq 0$, and a direct computation gives $\dot{N} = \dot{q}A + B$. At the endpoints $N(1) - N(0) = M(p^*)$, so $\int_0^1 u^T \dot{N} u dt = u^T M(p^*) u = 0$. The integrand is non-negative and continuous on each piece, hence $u^T \dot{N} u = 0$ in the interior of every piece. Since $\dot{N} \geq 0$, this gives $\dot{N} u = 0$; and since $\dot{q} u^T A u$ and $u^T B u$ are non-negative with sum 0, it gives $Au = 0$ wherever $\dot{q} > 0$.

From $\dot{N} u = 0$, $N(t)u = N(0)u = (c_1, c_2)$ for all t . As $q \geq r > 1$, this says $w_j(t) = w_j(q(t), \varphi(t))$, with the functions $w_j(q, \varphi)$ of the proof of Theorem 6.9. Where $\dot{q} > 0$, the two rows of $Au = 0$ give $e_1 = w_1 - \kappa\varphi = \gamma_1(\varphi)/(q-1)$ and $e_2 = w_2 - \varphi/\kappa = \gamma_2(\varphi)/(\kappa(q-1))$, so $\gamma_1(\varphi), \gamma_2(\varphi) \geq 0$ there.

Let $t_0 = \max\{t : q(t) = r\}$; then $t_0 < 1$ and $q = r$ on $[0, t_0]$. For every $t > t_0$, $q(t) - r = \int_{t_0}^t \dot{q} > 0$, so (t_0, t) contains interior points of pieces with $\dot{q} > 0$, where $\gamma_j(\varphi) \geq 0$. By continuity $\gamma_j(\varphi(t_0)) \geq 0$ for $j = 1, 2$, that is $\varphi(t_0) \geq \varphi_{\min}$. Since $w_j(t) = s_j(t)^2$ is non-decreasing, $w_1(r, \varphi(t_0)) = w_1(t_0) \geq w_1(0) = x_1^2 = w_1(r, \rho x_1 x_2)$, and $w_1(r, \cdot)$ is increasing; so $\varphi(t_0) \geq \rho x_1 x_2$,

hence $\varphi(t_0) \geq \varphi_s$, and

$$w_j(r, \varphi_s) \leq w_j(r, \varphi(t_0)) = w_j(t_0) \leq w_j(1) = 1.$$

If $\gamma_1(\varphi_s) + \gamma_2(\varphi_s) = 0$, then $\gamma_1(\varphi_s) = \gamma_2(\varphi_s) = 0$. Suppose that $\varphi(t_0) = \varphi_s$. Along points $t \downarrow t_0$ with $\dot{q}(t) > 0$, $e_1 = \gamma_1(\varphi)/(q-1) \rightarrow 0$ and $e_2 \rightarrow 0$, so $Au = 0$ gives in the limit $w_1(t_0) = \kappa\varphi(t_0)$ and $w_2(t_0) = \varphi(t_0)/\kappa$, with $\varphi(t_0) > 0$ as $w_1(t_0) \geq x_1^2$. Then $\theta(t_0)^2 = \varphi^2/(w_1 w_2) = 1$, which is impossible. So $\varphi(t_0) > \varphi_s$, and the first inequality in the display is strict.

(b) Let $N_n(t)$ be the matrix $N(t)$ for the n -th path and $\Delta_n = N_n(1) - N_n(0)$. As in (a), $\dot{N}_n \geq 0$, so $0 \leq N_n(t) - N_n(0) \leq \Delta_n$ for all t . Here $N_n(0)$ does not depend on n , and $\Delta_n \rightarrow M(p^*)$, so $\varepsilon_n := u^T \Delta_n u \rightarrow 0$. For a positive semi-definite P and any vector v , $|Pv|^2 \leq \lambda_{\max}(P) v^T P v$. With $P = N_n(t) - N_n(0)$ and $v = u$ this gives $|N_n(t)u - (c_1, c_2)| \leq \delta_n := (\varepsilon_n \operatorname{tr} \Delta_n)^{1/2}$, and hence $|w_j(t) - w_j(q(t), \varphi(t))| \leq C\delta_n$ for all t and j , with $C = \max\{1, \kappa^{-1}\}/(r-1)$.

Fix $\eta \in (0, p^* - r)$. For large n the n -th path has $q(1) > r + \eta$; let $t_0 = \max\{t : q(t) = r\}$ and $t_1 = \min\{t : q(t) = r + \eta\}$. Then $\int_{t_0}^{t_1} \dot{q} = \eta$ and $\int_{t_0}^{t_1} u^T \dot{N}_n u \leq \varepsilon_n$. So some interior point $t^* \in (t_0, t_1)$ of a piece has $\dot{q} > 0$ and $u^T \dot{N}_n u \leq 2\varepsilon_n \dot{q}/\eta$ there: otherwise integration would give $\int_{t_0}^{t_1} u^T \dot{N}_n u > 2\varepsilon_n$. As $u^T \dot{N}_n u \geq \dot{q} u^T A u$, at t^* we get $u^T A u \leq 2\varepsilon_n/\eta$; since $A \geq 0$ and $\operatorname{tr} A \leq w_1 + w_2 \leq 2$, also $|Au| \leq 2(\varepsilon_n/\eta)^{1/2}$. The rows of Au are $w_1 - e_1 - \kappa\varphi$ and $\varphi - \kappa(w_2 - e_2)$. With $e_j \geq 0$ and the bound on $w_j - w_j(q, \varphi)$ this gives, at t^* ,

$$\frac{\gamma_1(\varphi)}{q-1} = w_1(q, \varphi) - \kappa\varphi \geq -|Au| - C\delta_n, \quad \frac{\gamma_2(\varphi)}{\kappa(q-1)} = w_2(q, \varphi) - \frac{\varphi}{\kappa} \geq -\frac{|Au|}{\kappa} - C\delta_n.$$

Moreover $x_j^2 = w_j(0) \leq w_j(t^*) \leq 1$, $q(t^*) \in (r, r + \eta]$ and $|\varphi(t^*)| \leq 1$. Letting first $n \rightarrow \infty$ and then $\eta \rightarrow 0$, along subsequences, we obtain $\bar{\varphi}$ with $\gamma_j(\bar{\varphi}) \geq 0$ and $x_j^2 \leq w_j(r, \bar{\varphi}) \leq 1$ for $j = 1, 2$. As in (a), this gives $\bar{\varphi} \geq \varphi_s$ and $w_j(r, \varphi_s) \leq w_j(r, \bar{\varphi}) \leq 1$. $\square$

Together with Corollary 6.10 and the construction in its proof, part (a) shows that the hypothesis of Theorem 6.9 holds if and only if some path of Lemma 6.8 joins (r, ρ, x_1, x_2) to $(p^*, \rho, 1, 1)$, and if and only if $w_1(r, \varphi_s) \leq 1$ and $w_2(r, \varphi_s) \leq 1$, with strict inequalities when $\gamma_1(\varphi_s) + \gamma_2(\varphi_s) = 0$; in that case the construction of Corollary 6.10 is started at a value $\varphi(r) > \varphi_s$ with $w_j(r, \varphi(r)) \leq 1$. At $(\rho, r, x_1, x_2) = (\frac{3}{5}, \frac{11}{10}, \frac{99}{100}, \frac{3}{10})$, for instance, no path of Lemma 6.8 starting at (r, ρ, x_1, x_2) ends near $(p^*, \rho, 1, 1)$.

7. NUMERICAL CHECKS

`numerics/check_forward_real.py` (floating point, equicorrelated case, nested Gauss-Hermite quadrature over the common factor; evidence for the algebra, not part of the proof) checks: (A) inequality (5) with $B_{ij} = \operatorname{Id}$ for 500 random pairs (in the stronger form with $|\mathbb{E}[\Psi'_1 \Psi'_2]|$) and 300 random triples of smooth positive functions with logarithms of at most linear growth (minimal ratio of the two sides 1.03 and 1.02), and equality for anti-aligned exponentials; these functions are not bounded, so the test goes slightly beyond the hypotheses of Lemma 1.6; (B) the formulas of Lemma 4.1 against finite differences in random directions of (q, θ, s) (relative errors $\leq 10^{-9}$); (C) inequality (3) for $n = 2$ at $x^2 = 1.02 x_*^2$, where $x_*^2 = \frac{1-p\lambda}{1-r\lambda}$, for six parameter sets with $p\lambda < 1$ (including $r < 1$ and $r > 1$), with random functions and a local optimiser over a seven-parameter family (no violation; the supremum of the ratio is 1, attained by constants); (D) the closed form of the necessity computation, and the violation at $x^2 = 0.98 x_*^2$; (E) for $n = 3$, 60 random functions at $x^2 = 1.03 x_*^2$ (no violation) and exponential witnesses at $x^2 = 0.97 x_*^2$ (violation), for three parameter sets. `numerics/check_forward_general.py` checks Lemma 1.6 and inequality (3) for $n = 3$ with random non-equicorrelated covariances

(for $n = 2$ with scalar blocks every covariance is equicorrelated): random pairings B , random test functions above the threshold (largest log-ratio between -0.32 and -0.07 , so this is not a test of sharpness), and exponential witnesses along the bottom eigenvector below it. `numerics/check_forward_negative_p.py` tests (1) for $p < 0$ (Proposition 1.3) at 400 random equicorrelated cases ($n \in \{2, 3\}$, $\rho \in [0, 0.9)$, $p \in (-3, -0.2)$, $\alpha \in (0.05, 0.95)$, $x \in (-1, 1)$, random smooth positive f_j): the largest value of $\log(\text{left side}) - \log(\text{right side})$ is $-1.5 \cdot 10^{-3}$.

8. NOVELTY AND LIMITATIONS

Novelty. The forward inequality (1) with $\alpha \in (0, 1)$ is excluded from [6, Thm. 9(i)]. The range $p\lambda \geq 1$, where the condition is void, follows from [6, Thm. 9(i)] at $\alpha = 1$ and Lyapunov's inequality (Remark 1.5(b)); for two functions this is [11, Prop. 1.4(a)]. As far as we know, Theorem 1.1 is the first characterisation in the range $p\lambda < 1$, where the inequality fails near $x = 0$. It shows that the covariance condition (2) stays exact for $\alpha \in (0, 1)$ in this case, although the FB method does not apply. Propositions 6.3 and 6.5 show that this exactness does not extend to unequal parameters: for every $\alpha \in (0, 1)$ condition (2) is not sufficient in general, even for $p \geq 1$, where for equal parameters it is exact; this is in contrast with the ranges covered by [6, Thm. 9(i)]. The inequality is of Brascamp–Lieb type, with a correlated L^p norm on the right and degenerate Gaussian (exponential) extremisers. Gaussian saturation is known for many inequalities of this kind: Lieb [10], Barthe–Wolff [2] (inverse Brascamp–Lieb), Liu–Courtade–Cuff–Verdú and Courtade–Liu [8, 5] (forward–reverse Brascamp–Lieb), Nakamura–Tsuji [12], and the semigroup approach of Ledoux [9]. We are not aware of a result among these that covers $r < p$ with a correlated L^p norm on the right, but we did not search the literature systematically beyond [6], its references and these works. Lemma 1.6 is a reverse counterpart of the correlated log-Sobolev inequality [6, Thm. 11]. Its one-function ingredient (11) is classical; we did not find the correlated version. The quantity Φ of Lemma 6.8 is monotone along a heat-type flow, in the tradition surveyed by Ledoux [9]; Bennett and Bez [3] give a general method for generating such quantities. We did not find the flow with separate noise parameters, or a one-dimensional criterion like Theorem 6.9, in these works. Novelty is claimed to the best of this search.

Limitations. (1) Only equal parameters $r_j = x$ and equal exponents $p_j = p$ are treated in general. For two functions and $r = 1$, condition (2) is sharp (Proposition 6.1); the companion note [4] extends this to general blocks and cross-covariance, and to any number of scalar functions with sign-balanced correlations and $x_i x_j \geq 0$ on the non-zero correlations [4, Cor. 1.5]. For unequal parameters it is not sufficient in general: sufficiency fails for every $r > 2$ (Propositions 6.3 and 6.5), but it holds when $r \geq 1$ and $x_1 x_2 = 0$ (Proposition 6.4), and wherever the one-dimensional condition of Theorem 6.9 holds ($r > 1$ and $\rho, x_1, x_2 > 0$; verified at explicit points, Proposition 6.11, and numerically at about 92% of the sampled points with $1 < r \leq 2$ and $p^* > r$). That condition is not known to be necessary, and the evidence at the end of Section 6.1 suggests that it is not. Where its closed-form inequalities fail strictly, no path of the flow of Lemma 6.8 comes near the threshold (Proposition 6.12), so those cases need a different argument. Whether (2) is sufficient for all $r \in (0, 2] \setminus \{1\}$, and what the exact condition is for $r > 2$, is open (Remark 6.7). (2) Unequal negative exponents are not treated; for equal exponents $p < 0$ see Proposition 1.3. (3) Negative α and the reverse inequality are covered by [6], and are not revisited.

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