

No 421 equiangular lines in $\mathbb{R}^{44}$, no 94 in $\mathbb{R}^{20}$, no 74 in $\mathbb{R}^{19}$ and no 59 in $\mathbb{R}^{18}$

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Abstract

Let $N(d)$ be the maximum number of equiangular lines in $\mathbb{R}^d$. We prove that there are no 421 equiangular lines in $\mathbb{R}^{44}$, no 94 in $\mathbb{R}^{20}$, no 74 in $\mathbb{R}^{19}$ and no 59 in $\mathbb{R}^{18}$, so

$$344 \leq N(44) \leq 420, \quad 90 \leq N(20) \leq 93, \quad 72 \leq N(19) \leq 73, \quad 57 \leq N(18) \leq 58.$$

The published upper bounds are 422 [3, Table 3], [12, Thm. 8], 94 and 74 [9, Thms. 5.5, 5.2], and 59 [7]. All four proofs follow the method of Greaves, Syatriadi and Yatsyna.

For $d = 44$ the 2-adic structure of Seidel characteristic polynomials forces $\chi_S(x) = (x + 7)^{377}(x - 59)^{16}(x - 61)^{15}Q(x - 59)$ with Q of degree 13. There are exactly 82 possibilities for Q . Each one is excluded by an exact Farkas certificate showing that the interlacing conditions on the 421 vertex-deleted submatrices are infeasible. Two independently written programs (Python/FLINT and PARI/GP) carried out the computation.

For $d = 20$ the reduction of [9] leaves 169 candidate spectra. Interlacing with the 2-adic class condition of [10] leaves 20. An analysis of pairs of vertices (compatibility at the simple eigenvalues, Jacobi's identity for the adjugate, and linear systems around one vertex) leaves only $(x - 21)(x - 19)^{16}(x - 15)^3(x + 5)^{74}$. For this spectrum the components of the lines in the eigenspaces of 21 and 15 are rigid. The question becomes a finite problem about vectors in the 16-dimensional eigenspace of 19, which we decide by exhaustive search. Two programs without shared code exclude every case. One of them produces an exact integer Farkas certificate for every leaf of its search, and all certificates are re-checked without an LP solver.

For $d = 19$ the same method reduces 74 lines to the spectrum $(x - 17)(x - 15)^{15}(x - 11)^3(x + 5)^{55}$, and both programs exclude every case; again every certificate of the certificate-producing program is re-checked.

For $d = 18$, 59 lines force $\chi_S = (x + 5)^{41}(x - 11)\psi(x - 11)$ with ψ of degree 17. Two programs enumerate the ψ that satisfy the 2-adic conditions of Greaves and Yatsyna, on prefixes of the coefficient vector, and find the same 1,214 candidates. Interlacing with these conditions, the pair analysis at odd order, and a third level (triangle types of three vertices, linear systems for each edge type, and a relaxed vertex-star test for large cliques) exclude all but one. The last one, $(x + 5)^{41}(x - 11)^{14}(x - 15)^2(x^2 - 21x + 94)$, forces a rigid structure on the 40 vertices of one deck member, and a finite computation excludes it. Both programs, written independently, exclude every candidate. As a by-product we record a parity sharpening of the relative bound. It comes from known tools.

1 Results

A set of n equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$ is given by unit vectors $v_1, \dots, v_n$ with Gram matrix $G = I + \alpha S$, where S is a *Seidel matrix*: symmetric, with zero diagonal and

off-diagonal entries ± 1 . Since $G \succeq 0$ and $\text{rank } G = \dim \text{span}\{v_i\} \leq d$, the smallest eigenvalue of S is at least $-1/\alpha$, with multiplicity $n - \text{rank } G$. $N_\alpha(d)$ denotes the maximum for a fixed α , and $\chi_M(x) = \det(xI - M)$.

Theorem 1. *There is no set of 421 equiangular lines in $\mathbb{R}^{44}$. Consequently $N(44) \leq 420$.*

Theorem 2. *There is no set of 94 equiangular lines in $\mathbb{R}^{20}$. Consequently $N(20) \leq 93$.*

Theorem 3. *There is no set of 74 equiangular lines in $\mathbb{R}^{19}$. Consequently $N(19) \leq 73$.*

Theorem 4. *There is no set of 59 equiangular lines in $\mathbb{R}^{18}$. Consequently $57 \leq N(18) \leq 58$.*

Theorems 2, 3 and 4 are computer-assisted. Theorems 2 and 3 share the chain of reductions of §§7–9, and the final search of §10 is done by two independently written programs, one of which writes an exact certificate for every leaf; Section 11 lists the computations for both theorems. Theorem 4 (§12) runs the chain at the odd order 59, with a third level and a structural argument for one spectrum; every step is done by two independently written programs.

The bound $N(44) \leq 421$ is the starting point. It is not stated in the literature, but it follows directly from [5, Thm. 5.21 and Cor. 3.6], or from [9, Lemma 3.1] with Cauchy–Schwarz, once $N(43) = 344$ forces full rank. For completeness, §3 gives a proof in the notation used here.

Previous bounds. $N(43) = 344$, by Taylor’s unitary two-graph [16] and the relative bound. For $d = 44$ the relative bound gives $\lfloor 48 \cdot 44/5 \rfloor = 422$. The same value is the best upper bound in [3, Table 3] (SDP bounds) and [12, Thm. 8]; we found no published bound below 422. The lower bound $N(44) \geq 344$ is only the 43-dimensional system. For $d = 19$ and 20 the published ranges are $72 \leq N(19) \leq 74$ and $90 \leq N(20) \leq 94$ [9, Thms. 5.2, 5.5], [7, Table 1]. The lower bounds come from Asche’s 72 lines in $\mathbb{R}^{19}$ [14] and Taylor’s 90 lines in $\mathbb{R}^{20}$ [15]. For $d = 18$ the published range is $57 \leq N(18) \leq 59$: the lower bound is the lattice construction of [10], and the upper bound is [7], which excludes 60 lines. Greaves and Syatriadi [7] note that $N(18)$, $N(19)$ and $N(20)$ are the only unknown values for $d \leq 41$.

By-product. Proposition 12 (§6) combines [9, Lemma 3.1] with Cauchy–Schwarz [5, Thm. 5.6]. Applied at the top value $n_0 = \lfloor R \rfloor$ of a non-integral relative bound R , it gives $N_{1/a}(d) \leq \lfloor R \rfloor - 1$ for many (a, d) , including the infinite family $N_{1/a}(a^2 - 5) \leq \lfloor (a^2 - 1)(a^2 - 5)/5 \rfloor - 1$ for $a \equiv \pm 3 \pmod{10}$, $a \geq 7$. These exclusions are not new in substance. On all 140 cases we tabulated, they coincide exactly with what the even-centre inequality of [5, Thm. 5.21] gives, and the family also follows from that theorem. We have not seen the family stated.

2 Tools

2.1 Known results used

- (K1) (Neumann, [13, Thm. 3.4]; see also [3, Thm. 1.2]) If $n > 2d$ lines are equiangular in $\mathbb{R}^d$ with angle $\arccos \alpha$, then $1/\alpha$ is an odd integer.
- (K2) (Lemmens–Seidel [13, Thm. 4.5]; see also [12]) $N_{1/3}(d) = 2(d - 1)$ for $d \geq 15$.
- (K3) $N_{1/5}(d) \leq 276$ for $23 \leq d \leq 59$ [18, Thm. 1.1] (and for $d \leq 185$, [4]).
- (K4) (Relative bound [17, 13]) If $a^2 > d$ then $N_{1/a}(d) \leq (a^2 - 1)d/(a^2 - d) = d + d(d - 1)/(a^2 - d)$.
- (K5) (Cauchy interlacing) If $S[i]$ is S with row and column i deleted, the eigenvalues of $S[i]$ interlace those of S ; and $\sum_{i=1}^n \chi_{S[i]}(x) = \chi'_S(x)$.

2.2 The 2-adic structure of χ_S

Write $S = J - I - 2A$, where A is the adjacency matrix of a graph on n vertices. The facts below are due to Greaves and Yatsyna [11, §3] and Greaves, Syatriadi and Yatsyna [9, Thm. 2.5, Lemmas 2.7, 2.8]. We include short proofs because we use a parity refinement for an arbitrary odd shift (Lemma 7) and its transfer to factors (Lemma 8(2)).

Definition 5 ([9, Def. 2.6]). A monic $P = \sum_i a_i x^{N-i} \in \mathbb{Z}[x]$ of degree N is *type 2* if $2^i \mid a_i$ for all i , equivalently $P(2y)/2^N \in \mathbb{Z}[y]$; it is *weakly type 2* if $2^{i-1} \mid a_i$ for all $i \geq 1$, equivalently $P(2y)/2^{N-1} \in \mathbb{Z}[y]$.

Lemma 6. For a graph adjacency matrix A of order n , $\det(zI + A) \equiv \sum_{j \geq 0} m_j z^{n-2j} \pmod{2}$, where m_j is the number of j -edge matchings.

Proof. Modulo 2 signs disappear, so $\det(zI + A) \equiv \sum_{\sigma} \prod_i (zI + A)_{i\sigma(i)}$. The terms of σ and σ^{-1} coincide because $zI + A$ is symmetric, so all $\sigma \neq \sigma^{-1}$ cancel in pairs. An involution with j transpositions contributes $z^{n-2j} \prod A_{uv}$ over its transpositions (uv) . This is z^{n-2j} if they form a matching of the graph and 0 otherwise. $\square$

Lemma 7 (cf. [9, Thm. 2.5, Lemma 2.7]). Let S be a Seidel matrix of order n , κ odd, $P(x) = \chi_{S-\kappa I}(x) = \chi_S(x + \kappa)$ and $\tilde{P}(y) = P(2y)/2^{n-1}$. Then $\tilde{P} \in \mathbb{Z}[y]$. If n is even, $\tilde{P} \equiv 0 \pmod{2}$, i.e. P is type 2. If n is odd, $\tilde{P} \pmod{2}$ is a polynomial in y^2 .

Proof. Put $k = (\kappa + 1)/2 \in \mathbb{Z}$ and $B = A + kI$, so that $S - \kappa I = J - 2B$. By the matrix determinant lemma,

$$P(2y) = \det(2(yI + B) - \mathbf{1}\mathbf{1}^T) = 2^n D(y) - 2^{n-1} \mathbf{1}^T \text{adj}(yI + B) \mathbf{1}, \quad D(y) = \det(yI + B).$$

Hence $\tilde{P} = 2D - \mathbf{1}^T \text{adj}(yI + B) \mathbf{1} \in \mathbb{Z}[y]$. Since $\text{adj}(yI + B)$ is symmetric, the off-diagonal entries contribute an even amount, so $\tilde{P} \equiv \text{tr adj}(yI + B) = D'(y) \pmod{2}$. By Lemma 6, $D(y) = \det((y + k)I + A) \equiv \sum_j m_j (y + k)^{n-2j}$. If n is even, all exponents are even, so $D' \equiv 0$. If n is odd, $D'(y) \equiv \sum_j m_j (y + k)^{n-2j-1}$, a polynomial in $(y + k)^2 \equiv y^2 + k^2$. $\square$

Lemma 8 (cf. [9, Lemma 2.8]). Let $P = FG$ with $P, F, G \in \mathbb{Z}[x]$ monic of degrees N, f, g .

1. A monic P is type 2 iff $\rho/2$ is an algebraic integer for every root ρ . Hence every monic integer factor of a type-2 polynomial is type 2.
2. If F is type 2 and P is weakly type 2, then G is weakly type 2. If moreover $\tilde{P} := P(2y)/2^{N-1} \pmod{2}$ and $\hat{F} := F(2y)/2^f \pmod{2}$ are polynomials in y^2 , then so is $\tilde{G} := G(2y)/2^{g-1} \pmod{2}$.

Proof. (1) $P(2y)/2^N = \prod (y - \rho/2)$, and its coefficients are symmetric functions of the $\rho/2$. (2) $\tilde{P} = \hat{F} \tilde{G}$ with $\hat{F} \in \mathbb{Z}[y]$ monic and $\tilde{P} \in \mathbb{Z}[y]$, so long division by a monic integer polynomial gives $\tilde{G} \in \mathbb{Z}[y]$. Write $\tilde{G} \equiv G_0(y^2) + yG_1(y^2)$ and $\hat{F} \equiv H(y^2)$ in $\mathbb{F}_2[y]$, with $H \neq 0$ because $\hat{F}$ is monic. Then $\tilde{P} \equiv HG_0(y^2) + yHG_1(y^2)$ is a polynomial in y^2 only if $HG_1 = 0$, i.e. $G_1 = 0$. $\square$

Remark 9. The hypothesis on $\hat{F}$ in Lemma 8(2) cannot be dropped. For $P = (x - 2)(x^2 - x + 2) = x^3 - 3x^2 + 4x - 4$ one has $\tilde{P} = 2y^3 - 3y^2 + 2y - 1 \equiv y^2 + 1$, but for $G = x^2 - x + 2$ one has $\tilde{G} = 2y^2 - y + 1 \equiv y + 1 \pmod{2}$; here $\hat{F} = y - 1$.

Lemma 10 (forcing; the step in the proof of [9, Lemma 3.1]). *Let S be a Seidel matrix of order n , $\gamma = n \bmod 2$, and κ odd. Split the eigenvalues (with multiplicity) as $\Lambda = \Lambda_0 \sqcup E$, where every element of Λ_0 is an integer $\equiv \kappa \pmod{2}$. Let $\eta = |E| \geq 1$ and $T = \sum_{\lambda \in E} (\lambda - \kappa)^2$. If $\kappa \notin E$, then $\eta^\eta 4^{\eta-\gamma} \leq T^\eta$. In particular, if $T^\eta < \eta^\eta 4^{\eta-\gamma}$ then $\kappa \in E$.*

Proof. $\chi_S(x + \kappa) = F(x)q(x)$ with $F = \prod_{\lambda \in \Lambda_0} (x - (\lambda - \kappa))$, which is type 2 because its roots are even integers, and $q = \prod_{\lambda \in E} (x - (\lambda - \kappa))$. By Lemmas 7 and 8, q is weakly type 2, and type 2 if n is even. So its constant term $\pm \prod_E (\lambda - \kappa) \neq 0$ is divisible by $2^{\eta-\gamma}$. By AM–GM, $(T/\eta)^\eta \geq \prod_E (\lambda - \kappa)^2 \geq 4^{\eta-\gamma}$. $\square$

3 No 422 lines (known tools)

Proposition 11 ([5, Thm. 5.21, Cor. 3.6], [9, Lemma 3.1]). *There is no set of 422 equiangular lines in $\mathbb{R}^{44}$.*

Proof. Reduction. Suppose there are $n \geq 422$ lines in $\mathbb{R}^{44}$; keep 422 of them. By (K1) ($422 > 88$) the angle is $\arccos \frac{1}{a}$ with a odd. By (K2) ($2 \cdot 43 = 86$), by (K3), and by (K4) for $a \geq 9$ ($R(a, 44) \leq R(9, 44) = 3520/37 < 96$), we get $a = 7$. If the lines spanned at most 43 dimensions, (K4) would give $422 \leq R(7, 43) = 344$. So S (order 422) has smallest eigenvalue -7 of multiplicity exactly 378, and its other eigenvalues $\lambda_1, \dots, \lambda_{44}$ satisfy $\sum \lambda_i = 7 \cdot 378 = 2646$ and $\sum \lambda_i^2 = 422 \cdot 421 - 49 \cdot 378 = 159140$ (from $\text{tr } S = 0$, $\text{tr } S^2 = n(n-1)$).

Exclusion. Put $\kappa = 61$, $x_i = \lambda_i - 61$: $\sum x_i = 2646 - 44 \cdot 61 = -38$ and $\sum x_i^2 = 159140 - 122 \cdot 2646 + 44 \cdot 61^2 = 52$. Let E be the set of i with $x_i \neq 0$ and $\eta = |E|$. Lemma 10 with Λ_0 the 378 eigenvalues -7 and the eigenvalues equal to 61 (all odd), and $\gamma = 0$, gives $4\eta \leq 52$, so $\eta \leq 13$. By Cauchy–Schwarz, $38^2 \leq \eta \cdot 52 \leq 676$, which is false. $\square$

The same reduction applies to every n with $344 < n \leq 422$: the angle is $\arccos \frac{1}{7}$ and -7 has multiplicity exactly $n - 44$.

4 Proof of Theorem 1

Suppose there are 421 equiangular lines in $\mathbb{R}^{44}$. As above, S has eigenvalue -7 of multiplicity exactly 377, and the other 44 eigenvalues satisfy $\sum \lambda = 2639$ and $\sum \lambda^2 = 158347$.

Step 1: stripping. Apply Lemma 10 repeatedly. At each step Λ_0 consists of the 377 eigenvalues -7 and the eigenvalues already identified, and E of the r remaining ones; $\sum_E \lambda$ and $\sum_E \lambda^2$ are known at every stage. Sixteen steps with $\kappa = 59$, followed by fifteen with $\kappa = 61$, each satisfy $T^r < r^r 4^{r-1}$ ($\gamma = 1$). The first step has $r = 44$, $T = \sum (\lambda - 59)^2 = 109$, and $109^{44} < 44^{44} 4^{43}$. Removing an eigenvalue equal to κ leaves T unchanged; after the $\kappa = 59$ steps, $r = 28$ and $\sum_E (\lambda - 61)^2 = 49$; the last step is $r = 14$, where $49^{14} < 14^{14} 4^{13}$, while at $r = 13$ the test fails. All 31 inequalities are checked in both programs. Hence

$$\chi_S(x) = (x + 7)^{377} (x - 59)^{16} (x - 61)^{15} Q(x - 59),$$

where Q is monic of degree 13 and real-rooted, and its roots $x_j = \lambda_j - 59$ satisfy $\sum x_j = 13$ and $\sum x_j^2 = 49$. So $Q = x^{13} - 13x^{12} + 60x^{11} + \dots$.

Step 2: the lattice for Q . Apply Lemmas 7 and 8 with $\kappa = 59$. Here $P = \chi_S(x + 59) = FQ$ with $F = (x + 66)^{377}x^{16}(x - 2)^{15}$, which is type 2, and

$$\widehat{F} = (y + 33)^{377}y^{16}(y - 1)^{15} \equiv (y + 1)^{392}y^{16} = ((y + 1)^{196}y^8)^2 \pmod{2},$$

a polynomial in y^2 . Since $n = 421$ is odd, $\tilde{P} \pmod{2}$ is a polynomial in y^2 . Hence $\tilde{Q} = Q(2y)/2^{12} = \sum_i a_i 2^{1-i} y^{13-i} \in \mathbb{Z}[y]$, and its coefficients at odd powers of y are even. In terms of $Q = \sum_i a_i x^{13-i}$:

$$2^{i-1} \mid a_i \text{ (} i \text{ odd),} \quad 2^i \mid a_i \text{ (} i \text{ even).}$$

The first condition is the weakly-type-2 property [9, Lemma 2.7]. The second comes from [11, Lemma 3.8] ($2^r \mid a_r$ for even r ; for odd order this is the parity refinement of Lemma 7), transferred to Q by Lemma 8(2). The proof does not depend on the second condition; see the paragraph on robustness in §5.

Step 3: enumeration. There are exactly 82 monic real-rooted $Q \in \mathbb{Z}[x]$ of degree 13 with $a_1 = -13$, $a_2 = 60$ and these divisibilities (§5). Table 1 (Appendix A) lists their factorisations. For each candidate $p = \chi_S$, the number δ of distinct eigenvalues satisfies $5 \leq \delta \leq 8$.

Step 4: interlacing characteristic polynomials. This is the method of [10, §5]; the term *deck* is from [7, Def. 2.5], and [8] treats the enumeration of interlacing polynomials by linear programming. Fix a candidate $p = \chi_S$. Let Min_p be the product of the distinct monic irreducible factors of p , $\delta = \deg \text{Min}_p$, $\mu = p/\text{Min}_p$ and $h = p'/\mu$. For each vertex i , (K5) gives $\mu \mid \chi_{S[i]}$: a root of multiplicity e of p has multiplicity at least $e - 1$ in $\chi_{S[i]}$, and the same holds for all its conjugates. So $\chi_{S[i]} = \mu g_i$, where $g_i \in \mathbb{Z}[x]$ is monic of degree $\delta - 1$ and lies in the *deck* of p , the set of all g with:

- (D1) $g = x^{\delta-1} + b_1 x^{\delta-2} + b_2 x^{\delta-3} + \dots$ with $b_1 = c_1$ and $b_2 = c_2 + n - 1$, where $\text{Min}_p = x^\delta + c_1 x^{\delta-1} + c_2 x^{\delta-2} + \dots$. This follows from $\text{tr } S[i] = 0$ and $\text{tr } S[i]^2 = (n - 1)(n - 2)$.
- (D2) $g(\theta_j)/\text{Min}'_p(\theta_j) \geq 0$ for every root θ_j of Min_p , i.e. g interlaces Min_p (g has one root in each $[\theta_k, \theta_{k+1}]$). Indeed $(xI - S)_{ii}^{-1} = \chi_{S[i]}/\chi_S = g/\text{Min}_p = \sum_j w_j/(x - \theta_j)$ with $w_j = \|E_j e_i\|^2 \geq 0$, where E_j is the orthogonal projection onto the θ_j -eigenspace, and the residue at θ_j is $w_j = g(\theta_j)/\text{Min}'_p(\theta_j)$.
- (D3) $g(x - 1)$ is type 2. Indeed $\chi_{S[i]}(x - 1) = \chi_{S[i+I]}(x)$ is type 2 by Lemma 7 ($n - 1 = 420$ is even, $\kappa = -1$), and $g(x - 1)$ is a monic integer factor of it (Lemma 8(1)).
- (D6) $g(-7) \neq 0$. By interlacing, $S[i]$ has -7 with multiplicity at least 376. It has multiplicity at most 376, since otherwise the 420 remaining lines would span at most 43 dimensions, contradicting $N_{1/7}(43) \leq 344$. Since $(x + 7)^{376}$ divides μ , $g(-7) \neq 0$.
- (D7) Every root of g lies in $[2166/44, 3098/44]$. By (D6) the roots of g are among the 44 eigenvalues of $S[i]$ other than -7 . Their sum is $7 \cdot 376 = 2632$ and their sum of squares is $420 \cdot 419 - 49 \cdot 376 = 157556$, so their variance sum is $5040/44$. By Cauchy–Schwarz each of them lies within $\sqrt{43 \cdot 5040/44} < 466/44$ of the mean $2632/44$.

Moreover, $\sum_i \chi_{S[i]} = p'$ (K5) gives $\sum_{i=1}^{421} g_i = h$, where $h = \sum_k e_k q'_k \text{Min}_p / q_k$ if $p = \prod_k q_k^{e_k}$ with distinct irreducible q_k . Conditions (D1)–(D3) are those of [10, §5]; condition (iv) there, on $\mathcal{P}_{n-1,7}$, is only used when $n - 1$ is odd. Conditions (D6) and (D7) are specific to this case.

Step 5: Farkas certificates. For each of the 82 candidates the deck was computed exhaustively (at most 73 members), and an integer vector Y was found with $\text{Coeff}(g) \cdot Y \geq 0$ for every deck member g and $\text{Coeff}(h) \cdot Y < 0$, where Coeff is the coefficient vector (a certificate of infeasibility in the sense of [10, §5]). Then $0 \leq \sum_i \text{Coeff}(g_i) \cdot Y = \text{Coeff}(h) \cdot Y < 0$, a contradiction. So no candidate is the characteristic polynomial of a Seidel matrix, and there are no 421 equiangular lines in $\mathbb{R}^{44}$. Since every set of more than 421 lines contains 421 of them, $N(44) \leq 420$. $\square$

5 Computation

Enumeration principle (McKee–Smyth, as in [9, §2.3]). If $f = \sum_{j=0}^t c_j x^{t-j}$ is real-rooted, so is every derivative $D_i = f^{(t-i)}$ (of degree i), and $D'_i = D_{i-1}$. Given $c_0, \dots, c_{i-1}$, write $D_i = P_i + c_i(t-i)!$. If $\mu_1 \leq \dots \leq \mu_{i-1}$ are the roots of D_{i-1} , then D_i is real-rooted only if $(-1)^{i-k} D_i(\mu_k) \geq 0$ for all k , which confines c_i to an interval. All c_i in that interval and in the required residue class are tried, and each D_i is then tested exactly. For decks, the k -th root of D_i must in addition lie in $[\theta_k, \theta_{k+t-i+1}] \cap [2166/44, 3098/44]$, which follows from (D2), (D7) and Rolle’s theorem.

Program A (Python 3 with `python-flint` 0.9 and `SciPy/HiGHS`). The code is in `src/n421_enum.py`, `src/n421_decks.py` and `src/eqlines/`. Interval endpoints use ball arithmetic. Real-rootedness and the root windows are decided by FLINT’s certified real root isolation, and signs at algebraic eigenvalues by exact divisibility or refined balls. Certificates are found by HiGHS and made exact on the tight set. One candidate (no. 72) needed an exact search over vertices of the dual polyhedron (`find_farkas_vertex`). Every certificate is verified in exact integer arithmetic.

Program B (PARI/GP 2.15, `verify/n421.gp`), written separately and sharing no code with Program A. From Program A it reads only the stripping chain and the certificates Y (`data/n421/export.gp`). It re-checks the 31 stripping inequalities and re-enumerates Q with exact Sturm sequences and exact root counting with multiplicity. It rebuilds each p , Min_p and h from its own factorisation, re-enumerates each deck with exact window and box tests, and decides (D2) by exact sign evaluation on isolating rational intervals. Finally it checks each certificate against its *own* deck. Floating point is used only to locate candidate ranges for coefficients, which are then widened; every accept/reject decision is exact.

Results. Both programs find the same 82 polynomials Q (13,784 search nodes) and decks of identical sizes (120,527 deck search nodes in Program B). All 82 certificates are valid in both. The largest certificate entry has 35 decimal digits. Run times: Program A about 20 s on 6 cores, Program B 77 s on one core.

Robustness. The parity refinement of Step 2 ($2^i \mid a_i$ for even i) is not needed. Requiring only $2^{i-1} \mid a_i$ for all $i \geq 1$ (that is, Q weakly type 2), both programs find the same 190 polynomials Q (53,127 search nodes) and decks of identical sizes (615,463 deck search nodes, at most 80 members). Program A finds a certificate for each of the 190 candidates, and Program B confirms every one on its own decks (`--weak` options of Program A, `verify/n421_weak.gp`; Program B takes 395 s on one core).

Negative tests. The same pipeline (`src/remainder_case.py`, conditions (D1)–(D7) with the parameters of each case) was run on two line systems that exist. For 343 lines in $\mathbb{R}^{43}$ (the 344 unitary lines minus one) it finds 704 remainder polynomials. Nine are not excluded, and among them is

the true spectrum $\{(-7)^{300}, 49^{42}, 42\}$. For 125 lines in $\mathbb{R}^{21}$ (the 126 unitary lines for $q = 5$ minus one) it finds 14 remainders. Two are not excluded, including the true spectrum $\{(-5)^{104}, 25^{20}, 20\}$. The moment test of Proposition 12 was run on 15,565 parameter pairs coming from all subsets of size $> N(d-1)$ of the systems with 28, 126, 176, 276 and 344 lines, and on actual Seidel matrices obtained by deleting vertices from the 126- and 344-line systems. No existing system was excluded (`verify/negative_tests.py`).

6 By-product: a parity sharpening of the relative bound

Proposition 12 ([9, Lemma 3.1] with [5, Thm. 5.6]). *Let $a \geq 3$ be odd and let S be a Seidel matrix of order n whose smallest eigenvalue $-a$ has multiplicity exactly $n-d$, with other eigenvalues $\lambda_1, \dots, \lambda_d$. For an odd integer κ put*

$$s = \sum_{i=1}^d (\lambda_i - \kappa) = a(n-d) - d\kappa, \quad T = \sum_{i=1}^d (\lambda_i - \kappa)^2 = n(n-1) - a^2(n-d) - 2\kappa a(n-d) + d\kappa^2,$$

$\gamma = n \bmod 2$ and $\eta^*(T) = \max\{\eta \in \{0, \dots, d\} : \eta = 0 \text{ or } \eta 4^{\eta-\gamma} \leq T^\eta\}$. Then $s^2 \leq \eta^*(T) T$.

Proof. Let $E = \{i : \lambda_i \neq \kappa\}$ and $\eta = |E|$. The remaining eigenvalues are $-a$ (multiplicity $n-d$) and copies of κ , all odd integers. If $\eta = 0$ then $s = 0$. Otherwise Lemma 10 gives $\eta \leq \eta^*(T)$, and Cauchy–Schwarz gives $s^2 = (\sum_{i \in E} (\lambda_i - \kappa))^2 \leq \eta \sum_{i \in E} (\lambda_i - \kappa)^2 = \eta T$. $\square$

Remark 13. For $\gamma = 0$, $\eta^*(T) = \min(d, \lfloor T/4 \rfloor)$. For $\gamma = 1$, $\eta 4^{(\eta-1)/\eta} \geq 4\eta - 4 \ln 4$ gives $\eta^*(T) \leq T/4 + \ln 4$. Let $\bar{\mu} = a(n-d)/d$, let $\delta = \text{dist}(\bar{\mu}, 2\mathbb{Z} + 1)$ with κ the nearest odd integer, and let $V = \sum (\lambda_i - \bar{\mu})^2 = n(a^2 - d)(R - n)/d$ with $R = (a^2 - 1)d/(a^2 - d)$. Then $|s| = d\delta$ and $T = V + d\delta^2$, so for even n the criterion reads (up to the floor) $V < d\delta(2 - \delta)$. The even-centre inequality of [5, Thm. 5.21] at the even integer μ nearest to $\bar{\mu}$, with $m = \dim \ker(S - \mu I) = 0$ for even n [5, Cor. 3.6] and $m \leq 1$ for odd n ($S - \mu I \equiv J - I$ has $\mathbb{F}_2$ -rank $n-1$), reads $V + d(1 - \delta)^2 < d - m$, i.e. $V < d\delta(2 - \delta) - m$. For even n the two tests agree up to the floor in η^* ; for odd n they differ by lower-order terms. Since $V \approx (a^2 - 1)(R - n)$ and $d < a^2$, either test can only remove the top value $\lfloor R \rfloor$.

Corollary 14. *For every odd $a \geq 7$ with $a \equiv \pm 3 \pmod{10}$, $N_{1/a}(a^2 - 5) \leq \left\lfloor \frac{(a^2-1)(a^2-5)}{5} \right\rfloor - 1$.*

Proof. Let $d = a^2 - 5$. As $a^2 \equiv 4 \pmod{5}$, $(a^2 - 1)(a^2 - 5) \equiv 2 \pmod{5}$, so $R = n_0 + \frac{2}{5}$ with $n_0 \in \mathbb{Z}$. By (K4), $N_{1/a}(d) \leq n_0$. Also $R(a, d-1) = (a^2 - 1)(a^2 - 6)/6 = R - a^2(a^2 - 1)/30 < n_0 - 1$, so it suffices to exclude n_0 lines spanning $\mathbb{R}^d$, i.e. a Seidel matrix of order n_0 with $-a$ of multiplicity exactly $n_0 - d$. Here $V = n_0(a^2 - d)(R - n_0)/d = 2n_0/d < \frac{2}{5}(a^2 - 1)$ and $\bar{\mu} = a(a^2 - 6)/5 - 2a/(5(a^2 - 5))$. If $a \equiv 3 \pmod{10}$ then $a(a^2 - 6) \equiv 9 \pmod{10}$, so $\delta = \frac{4}{5} - \frac{2a}{5(a^2 - 5)} \in [0.768, 0.8]$ for $a \geq 13$. If $a \equiv 7 \pmod{10}$ then $a(a^2 - 6) \equiv 1 \pmod{10}$ and $\delta = \frac{4}{5} + \frac{2a}{5(a^2 - 5)} \in (0.8, 0.864]$ for $a \geq 7$. Then $T = V + d\delta^2 < 0.4a^2 + 0.7465a^2 < 1.1465a^2$ and $\eta^* \leq T/4 + \ln 4 < 0.2867a^2 + 1.39$, so

$$\eta^* T < 1.147a^2(0.2867a^2 + 1.39) < 0.329a^4 + 1.6a^2 < 0.5898(a^2 - 5)^2 \leq d^2\delta^2 = s^2$$

for $a \geq 7$, and Proposition 12 excludes n_0 . $\square$

The family also follows from [5, Thm. 5.21] at the even centre (checked exactly for $7 \leq a \leq 2001$ by `verify/compare_gkms.py`). Table 2 (Appendix B) lists all (a, d) with $a \leq 13$ odd, $d < a^2 - 2$, non-integral R and $\lfloor R \rfloor > 2d$ for which Proposition 12 excludes $n_0 = \lfloor R \rfloor$: 72 of the 140 such pairs.

In each row also $\lfloor R(a, d-1) \rfloor \leq n_0 - 1$, so $N_{1/a}(d) \leq n_0 - 1$. The even-centre test of [5, Thm. 5.21] excludes exactly the same 72. Only the row $d = 44$ ($a = 7$) changes a bound on $N(d)$ itself; in every other row the known bound for another angle is larger.

7 94 lines in $\mathbb{R}^{20}$ and 74 lines in $\mathbb{R}^{19}$: the candidate spectra

Throughout §§7–11, (n, d, κ) is $(94, 20, 19)$ or $(74, 19, 15)$, and S is a Seidel matrix of order n that comes from n equiangular lines in $\mathbb{R}^d$.

Lemma 15. *The angle is $\arccos \frac{1}{5}$, and*

$$\chi_S(x) = (x+5)^{n-d}(x-\kappa)^{d-13} 2^{13} \psi\left(\frac{x-\kappa}{2}\right),$$

where $\psi \in \mathbb{Z}[y]$ is monic and real-rooted of degree 13 with $\psi = y^{13} + 5y^{12} + 6y^{11} + \dots$. There are exactly 169 such polynomials ψ .

Proof. Since $n > 2d$, (K1) gives the angle $\arccos \frac{1}{a}$ with a odd. For $a = 3$, (K2) allows at most $2(d-1) < n$ lines. For $a \geq 7$, (K4) allows at most $R(7, d) \leq R(7, 20) = 960/29 < 34$. So $a = 5$. If the lines spanned at most $d-1$ dimensions, (K4) would give $n \leq R(5, d-1)$; but $R(5, 19) = 76 < 94$ and $R(5, 18) = 432/7 < 74$. So -5 has multiplicity exactly $n-d$. For the other d eigenvalues, $\text{tr } S = 0$ and $\text{tr } S^2 = n(n-1)$ give $\sum(\lambda - \kappa) = 5(n-d) - d\kappa = -10$ and $T = \sum(\lambda - \kappa)^2 = n(n-1) - 25(n-d) - 10\kappa(n-d) + d\kappa^2 = 52$ in both cases. Lemma 10 ($\gamma = 0$, Λ_0 the eigenvalues -5 and those already found equal to κ) applies while $52 < 4\eta$, i.e. for $\eta = d, d-1, \dots, 14$. This gives $d-13$ eigenvalues κ and leaves 13, say $\kappa + 2y_j$. Then $\sum y_j = -5$ and $\sum y_j^2 = 13$, which is the stated form of $\psi = \prod(y - y_j)$. By Lemma 7 (n even), $\chi_S(x + \kappa)$ is type 2. So is its monic factor $2^{13}\psi(x/2)$ (Lemma 8(1)), i.e. $\psi \in \mathbb{Z}[y]$. The enumeration principle of §5 gives exactly 169 polynomials ψ (data/psi_deg13_p1m5_p2_13.json). Two independent checks obtained the same list with their own programs. $\square$

We number the candidates $0, \dots, 168$ as in that file. The two that matter below are

$$\#3: \psi = y^{11}(y+2)(y+3), \quad \#35: \psi = y^9(y-1)(y+2)^3,$$

that is, $\chi_S = (x+5)^{74}(x-19)^{18}(x-15)(x-13)$ resp. $(x+5)^{55}(x-15)^{17}(x-11)(x-9)$ for $\#3$, and

$$\#35: \chi_S = (x-21)(x-19)^{16}(x-15)^3(x+5)^{74} \quad \text{resp.} \quad (x-17)(x-15)^{15}(x-11)^3(x+5)^{55}.$$

Level 1: one deleted vertex. For a candidate p let Min_p , δ , $\mu = p/\text{Min}_p$ and $h = p'/\mu$ be as in Step 4 of §4, so that $\chi_{S[i]} = \mu g_i$ and $\sum_i g_i = h$. The order $n-1$ is odd. The deck consists of all monic g of degree $\delta-1$ with: (D1) and (D2) as in §4; (D3) $g(x-1)$ is weakly type 2, and type 2 if $\mu(x-1)$ is not [9, Lemmas 2.7, 2.8]; (D4) $\chi_{S[i]}(x-1) = \mu(x-1)g(x-1)$ satisfies [9, Thm. 2.5]; (D6) $g(-5) \neq 0$, since otherwise $S[i]$ would have -5 with multiplicity $n-d$ and $n-1$ lines would lie in $\mathbb{R}^{d-1}$, which is impossible as above; (iv) $\mu g \bmod 2^7$ is the residue of the characteristic polynomial of some Seidel matrix of order $n-1$ [10, §5, condition (iv)]. For (iv) we use the complete set $\mathcal{P}_{n-1,7}$ of such residues. It has 2048 elements for both orders. It is complete because the necessary conditions of [11, §3] (Remark 3.3 and Lemmas 3.1, 3.8, 3.12 for $k \leq 3$) allow exactly 2048 residues, and for each of them we have an explicit witness matrix. For each candidate the LP $\sum_g n_g \text{Coeff}(g) = \text{Coeff}(h)$, $n \geq 0$, over the deck is decided, with an exact integer Farkas certificate when it is infeasible. This excludes 147 of the 169 candidates for 74 lines and 149 for 94 lines. The survivors are $\#3, 9, 20, 27, 35, 42, 50, 78, 81, 83, 85, 89, 91, 99, 100, 110, 111, 142, 144, 149, 153, 162$ for 74 lines, and the same without $\#42$ and $\#85$ for 94 lines. (An independent check reproduced every deck, verdict and certificate by two methods of its own, and a third program was also run.)

Level 2: pairs of vertices. Three exact steps, each run by our programs and by an independent program (an independent audit also reproduced all three).

1. *Lemma 3.12 of [11] and compatibility.* Each deck member must also satisfy the congruences of [11, Lemma 3.12] at the odd order $n - 1$. Let Sim be the product of $x - \lambda$ over the simple eigenvalues, $M = \text{Min}_p / \text{Sim}$ and $s = \deg \text{Sim}$. Let $i \neq j$ be vertices with members $g = g_i$ and $g' = g_j$ (possibly $g = g'$). At a simple eigenvalue λ , $E_\lambda = u_\lambda u_\lambda^\top$ with $u_\lambda(i)^2 = g(\lambda) / \text{Min}'_p(\lambda)$ [10, Lemma 4.4] and $\text{Min}'_p(\lambda) = \text{Sim}'(\lambda)M(\lambda)$. Since $x^k M$ vanishes at the multiple eigenvalues, $S^k M(S) = \sum_{\lambda \text{ simple}} \lambda^k M(\lambda) E_\lambda$. Hence, for some signs d_λ independent of k ,

$$N_k := \sum_{\lambda \text{ simple}} d_\lambda \lambda^k \frac{\sqrt{g(\lambda)g'(\lambda)}}{\text{Sim}'(\lambda)} = (S^k M(S))_{ij} \in \mathbb{Z} \quad (0 \leq k < s)$$

(for $k = 0$ this is [10, Lemma 4.7] with $\xi = \text{Sim}$), and $N_k \equiv (x^k M)'(1) \pmod{2}$. The parity condition is our addition: [10] impose one only for odd order. Proof: n is even, so $S \equiv I + J$ and $J^2 = nJ \equiv 0 \pmod{2}$. Hence $S^k \equiv I + kJ$, and every off-diagonal entry of $F(S)$ is $\equiv F'(1)$ for $F \in \mathbb{Z}[x]$. We call g, g' *compatible* if some signs d_λ satisfy both conditions, and say that g has a *loop* if g, g are compatible. The members used by S form a clique of the compatibility graph, and a member without a loop is used at most once. The LP of level 1 restricted to each maximal clique leaves #3, 20, 35, 50, 89, 91, 99, 111, 142, 153 (and also #162 for 94 lines).

2. *Jacobi pair polynomials.* For $i \neq j$ put $\varphi = \text{adj}(xI - S)_{ij}$. Then $\varphi = \mu \psi_{ij}$ with $\psi_{ij} \in \mathbb{Z}[x]$ of degree $\delta - 2$ and leading coefficient S_{ij} , and Jacobi's identity $\chi_{S[i]}\chi_{S[j]} - \chi_S \chi_{S[ij]} = \varphi^2$ (used in this way in [7], after de Caen) gives $\chi_{S[ij]} = \mu_2 P_{ij}$ with $P_{ij} = (g_i g_j - \psi_{ij}^2) / \text{Sim}$ and $\mu_2 = \mu / M$. We impose: (J1) the coefficients $(S^k)_{ij}$ of $\varphi/p = \sum_k (S^k)_{ij} x^{-k-1}$ satisfy $(S^k)_{ij} \equiv k \pmod{2}$, by $S^k \equiv I + kJ$; (J2) $\text{Sim} \mid g_i g_j - \psi_{ij}^2$; (J3) $\psi_{ij}(\lambda)^2 \leq g_i(\lambda)g_j(\lambda)$ at the multiple eigenvalues (Cauchy–Schwarz for $E_\lambda \succeq 0$); (J4) $\chi_{S[ij]}$ is real-rooted and interlaces $\chi_{S[i]}$ and $\chi_{S[j]}$; (J5) $\chi_{S[ij]}$ has the trace, the trace of the square and the 2-adic form [9, Thm. 2.5] of a Seidel matrix of order $n - 2$. Every ψ_{ij} allowed by (J1)–(J5) is enumerated. The graph “some pair polynomial exists” and its cliques exclude #20.
3. *Vertex stars.* Since $\sum_{j \neq i} \chi_{S[ij]} = \chi'_{S[i]}$, for every vertex i we get $\sum_{j \neq i} P_{ij} = (\mu g_i)' / \mu_2$. With unknowns n_g (the number of vertices with member g) and $m_{g,P}^f$ (for a vertex with member f , the number of other vertices j with member g and pair polynomial P), this gives the linear system $\sum_g n_g g = h$, $\sum_P m_{g,P}^f = n_g - [g = f]$, $\sum_{g,P} m_{g,P}^f P = (\mu f)' / \mu_2$, for each possible support $\{g : n_g \geq 1\}$ inside an LP-feasible clique. Every system is decided by an exact rational simplex, with a Farkas vector when it is infeasible. Only #3 and #35 survive, in both cases.

Lemma 16. *Candidate #3 is impossible.*

The bordering step for one deleted point is [6, Prop. 5.3, Cor. 5.4] (numbering of arXiv:1612.03644v3), and [6, Remark 5.5] records its instances for 75 and 95 points. The rank-2 step for two deleted points uses the device of [5, Lemma 5.24, Thm. 5.25].

Proof. Let $\rho = \kappa = 19$ resp. 15 and $c = \rho - 5$, so that $\chi_S = (x + 5)^{n-d}(x - \rho)^{d-2}(x - c + 1)(x - c - 1)$. The matrix $M = -(S + 5I)(S - \rho I)$ is positive semidefinite of rank 2, with nonzero eigenvalues $(\rho - 1) \cdot 6$ and $(\rho + 1) \cdot 4$ on the $(c - 1)$ - and $(c + 1)$ -eigenspaces. Its diagonal is $5\rho - (n - 1) = 2$, and

for $i \neq j$, $M_{ij} = -(S^2)_{ij} + cS_{ij}$ is even with $|M_{ij}| \leq 2$. So $M/2$ is a positive semidefinite $\{0, \pm 1\}$ matrix with unit diagonal and rank 2. By [5, Lemma 5.24] it equals $s_A s_A^\top + s_B s_B^\top$ for a partition $A \sqcup B$ of the vertices and sign vectors s_A, s_B supported on A, B . Comparing eigenvalues gives $|A| = 3(\rho - 1)$ and $|B| = 2(\rho + 1)$, with s_A, s_B spanning the $(c - 1)$ - and $(c + 1)$ -eigenspaces. After switching, $S\mathbf{1}_A = (c - 1)\mathbf{1}_A$ and $S\mathbf{1}_B = (c + 1)\mathbf{1}_B$. Bordering S by a vertex with entries $+1$ on A and -1 on B gives a Seidel matrix of order $n + 1$: the eigenvectors of S for -5 and ρ are orthogonal to $\mathbf{1}_A, \mathbf{1}_B$ and stay eigenvectors, and on $\text{span}\{e_{\text{new}}, \mathbf{1}_A, \mathbf{1}_B\}$ the bordered matrix has characteristic polynomial $(x - c)(x - \rho)(x + 5)$, by $|A| + |B| = n$ and $|A| - |B| = c$. So its characteristic polynomial is $(x + 5)^{n+1-d}(x - c)(x - \rho)^{d-1}$, that is, $(x + 5)^{75}(x - 14)(x - 19)^{19}$ resp. $(x + 5)^{56}(x - 10)(x - 15)^{18}$. No Seidel matrix has this characteristic polynomial [9, Thm. 5.1], [6, Remark 5.5]: by [6, Cor. 5.4] its switching class would contain an $\text{srg}(95, 40, 12, 20)$ resp. $\text{srg}(75, 32, 10, 16)$, and these do not exist [2, 1]. (Equivalently, the bordered matrix would give $n + 1$ equiangular lines in $\mathbb{R}^d$, i.e. 95 in $\mathbb{R}^{20}$ resp. 75 in $\mathbb{R}^{19}$, contradicting $N(20) \leq 94$ [9, Thm. 5.5] resp. $N(19) \leq 74$ [9, Thm. 5.2].) $\square$

So χ_S is the spectrum #35. We write $\lambda_1 > \lambda_2 > \lambda_3$ for its eigenvalues other than -5 : $(21, 19, 15)$ with multiplicities $(1, 16, 3)$ for 94 lines, and $(17, 15, 11)$ with multiplicities $(1, 15, 3)$ for 74 lines. Let $m_2 = 16$ resp. 15 be the multiplicity of λ_2 .

8 The structure of the spectrum #35

Let E_λ be the spectral projections of S . Since $G = S + 5I = \sum_{\lambda \neq -5} (\lambda + 5)E_\lambda$ is the Gram matrix of the lines (up to the factor 5), each line v has components $E_\lambda e_v$. We scale them: $X_v = (\sqrt{2}r_v, \sqrt{3}z_v, p_v) \in V_1 \oplus V_2 \oplus V_3$, where $V_1 \cong \mathbb{R}$, $V_2 \cong \mathbb{R}^{m_2}$, $V_3 \cong \mathbb{R}^3$ are the eigenspaces of $\lambda_1, \lambda_2, \lambda_3$, and

$$2r_u r_v = 6(\lambda_1 + 5)(E_1)_{uv}, \quad 3\langle z_u, z_v \rangle = 6(\lambda_2 + 5)(E_2)_{uv}, \quad \langle p_u, p_v \rangle = 6(\lambda_3 + 5)(E_3)_{uv}.$$

Then $\langle X_u, X_v \rangle = 6G_{uv}$, i.e. $2r_u r_v + 3\langle z_u, z_v \rangle + \langle p_u, p_v \rangle = 6S_{uv}$ for $u \neq v$, and $2r_v^2 + 3|z_v|^2 + |p_v|^2 = 30$. Since $E_2^2 = E_2$ and E_2 has rank m_2 , the matrix Z with columns z_v satisfies the *frame identity*

$$\sum_v z_v z_v^\top = ZZ^\top = 2(\lambda_2 + 5)I_{m_2} \quad (= 40I_{15} \text{ resp. } 48I_{16}), \quad (1)$$

and in the same way $\sum_v r_v^2 = 3(\lambda_1 + 5)$ and $\sum_v p_v p_v^\top = 6(\lambda_3 + 5)I_3$ (in an orthonormal basis of V_3). Since the E_λ are mutually orthogonal, $\sum_v r_v z_v = 0$, $\sum_v r_v p_v = 0$ and $\sum_v z_v p_v^\top = 0$. Finally $SE_\lambda = \lambda E_\lambda$ gives the *eigen-equations* $\sum_{u \neq v} S_{vu}(r_u, z_u, p_u) = (\lambda_1 r_v, \lambda_2 z_v, \lambda_3 p_v)$.

Types. For every vertex v , $\chi_{S[v]} = \mu g_v$ with g_v one of the six deck members $g_0, \dots, g_5$ of #35 that survive level 2, and $(E_\lambda)_{vv} = g_v(\lambda)/\text{Min}'(\lambda)$. This gives $(r_v^2, |z_v|^2, |p_v|^2) = (0, 9, 3), (1, 8, 4), (4, 5, 7), (5, 4, 8), (8, 1, 11), (9, 0, 12)$ for types $0, \dots, 5$, in both cases. For $u \neq v$, $(E_\lambda)_{uv} = \psi_{uv}(\lambda)/\text{Min}'(\lambda)$ with the pair polynomial ψ_{uv} of level 2, so every pair of types has a finite list of tuples $(S_{uv}, r_u r_v, \langle z_u, z_v \rangle, \langle p_u, p_v \rangle)$. They are checked mechanically (`verify/c35_structure.py`, checks (1)–(3)). Types 3 and 4 have pair polynomials only with type 0 and with themselves.

Supports and counts. The vertex-star systems of level 2 with the counts n_t of the six types as unknowns are infeasible except for three supports, and each of these has a unique solution:

support	(n_0, n_1, n_2, n_5) for 74 lines	for 94 lines
$(0, 1, 5)$	$(24, 48, 0, 2)$	$(32, 60, 0, 2)$
$(0, 1, 2, 5)$	$(25, 45, 3, 1)$	$(33, 57, 3, 1)$
$(0, 1, 2)$	$(26, 42, 6, 0)$	$(34, 54, 6, 0)$

(src/star_counts_35.py; an independent program derives the same counts.)

Proposition 17 (rigidity of V_1 and V_3). *After switching, and in a suitable orthonormal frame of V_3 , every line's (r_v, p_v) depends only on its kind:*

- *type 0 (kind T0): $r = 0, p = w$; type 1: $r = 1, p = 2u_x$ for a class $x \in \{a, b, c\}$ (kinds T1a, T1b, T1c); type 5: $r = 3, p = \pm 2w$; type 2: $r = 2, p = \sigma w - 2u_x$ with $\sigma = \pm 1$;*

where $|w|^2 = 3, w \perp u_a, u_b, u_c$, and u_a, u_b, u_c are unit vectors at 120° ($u_a + u_b + u_c = 0$). The numbers $n_x = |L_x|$ of type-1 lines of class x and m_x of type-2 lines with $p = \sigma w - 2u_x$ are: $(n_x, m_x) = (16, 0)$ resp. $(20, 0)$ in $(0, 1, 5)$, $(15, 1)$ resp. $(19, 1)$ in $(0, 1, 2, 5)$, and $(14, 2)$ resp. $(18, 2)$ in $(0, 1, 2)$, for every x . In $(0, 1, 5)$ the two type-5 lines have $p = 2w$ and $p = -2w$. In $(0, 1, 2, 5)$ the type-5 line has $p = 2w$ and every type-2 line has $\sigma = -1$. In $(0, 1, 2)$ there is one type-2 line for each pair (σ, x) .

Proof. Switching a line negates X_v ; make $r_v \geq 0$ for all v , so $r = 0, 1, 2, 3$ for types 0, 1, 2, 5. The pair data (check (3)) give $\langle p, p' \rangle = \pm 3 = \pm |p||p'|$ for two type-0 lines, so all type-0 parts are parallel; switch the type-0 lines so that they are all equal to w . Type 0/type 5 gives $\langle p, p' \rangle = \pm 6 = \pm |p||p'|$, so $p = \pm 2w$ for type 5. Type 0/type 1 gives $\langle p, p' \rangle = 0$, so $p = 2u$ with $u \perp w$ a unit vector, and type 1/type 1 gives $\langle p, p' \rangle \in \{4, -2\}$: the vectors u are equal or at 120° , so they lie in a set $\{u_a, u_b, u_c\}$ of three unit vectors at 120° in the plane $w^\perp$, $u_a + u_b + u_c = 0$. Type 0/type 2 gives $\langle p, p' \rangle = \pm 3$, so $p = \sigma w + q$ with $q \perp w, |q| = 2$, and type 1/type 2 gives $\langle 2u, q \rangle \in \{-4, 2\}$ for every type-1 vector u . The vectors $q \in w^\perp$ of norm 2 with $\langle u, q \rangle \in \{-2, 1\}$ are exactly $-2u_a, -2u_b, -2u_c$, so $q = -2u_x$ for one class x . On the plane $w^\perp$ the frame $\sum_v p_v p_v^\top = 6(\lambda_3 + 5)I_3$ reads $\sum_x 4(n_x + m_x)u_x u_x^\top = cI_2$. Three unit vectors at 120° form a tight frame only with equal weights (and two or one vectors never do), so all three classes occur and $n_x + m_x$ is the same for every x : 16 resp. 20. Next $\sum_v r_v p_v = 0$. In $(0, 1, 5)$ let the two type-5 lines have $p = 2\varepsilon_1 w$ and $p = 2\varepsilon_2 w$. Only they contribute to $\langle \sum_v r_v p_v, w \rangle$, which is therefore $3 \cdot 2|w|^2(\varepsilon_1 + \varepsilon_2) = 18(\varepsilon_1 + \varepsilon_2) = 0$, so $\varepsilon_1 = -\varepsilon_2$. In $(0, 1, 2, 5)$, $z_f = 0$ for the type-5 line f forces $6S_{tf} = 2r_t r_f + \langle p_t, p_f \rangle$ for every line t ; replacing w by $-w$ (and switching the type-0 lines) we may take $p_f = 2w$, and then a type-2 line has $S_{tf} = 2 + \sigma$, so $\sigma = -1$. The component of $\sum_v r_v p_v = 0$ in $w^\perp$ reads $\sum_x (2n_x - 4m_x)u_x = 0$, so $2n_x - 4m_x$ is the same for all x ; with $n_x + m_x = 16$ (resp. 20) and $\sum m_x = n_2 = 3$ this gives $m_x = 1$. In $(0, 1, 2)$ the same argument gives $m_x = 2$. The w -component of $\sum_v r_v p_v = 0$ gives $\sum_t \sigma_t = 0$ over the type-2 lines, and the $(w, w^\perp)$ block of the frame gives $\sum_x (\sum_{t \in x} \sigma_t)u_x = 0$, so the three class sums of σ are equal, hence 0. $\square$

The problem in V_2 . Put $K(u, v) = 2r_u r_v + \langle p_u, p_v \rangle$, a function of the two kinds. Then

$$3\langle z_u, z_v \rangle = 6S_{uv} - K(u, v), \quad (2)$$

so the $z_v \in V_2$ have prescribed norms $(9, 8, 5, 0)$ for types $0, 1, 2, 5$ and inner products determined by $S_{uv} = \pm 1$, subject to Cauchy–Schwarz. They satisfy the frame identity (1), $\sum_v r_v z_v = 0$ and $\sum_v z_v p_v^\top = 0$. The V_1 - and V_3 -components of the eigen-equations, $\sum_{u \neq v} S_{vu}(r_u, p_u) = (\lambda_1 r_v, \lambda_3 p_v)$, are a 4×4 linear system in the sums of the row S_v . over the blocks T0, T1a, T1b, T1c. They fix the number of -1 entries of every row in every block, given the row's entries against the type-2

and type-5 lines (the *row counts*). The V_2 -component gives the block sums $Z_\beta = \sum_{v \in \beta} z_v$ as fixed combinations of the z of the type-2 lines. For example, in $(0, 1, 5)$ for 74 lines: two type-0 lines have $\langle z, z' \rangle \in \{1, -3\}$ (per row: 15 times 1, 8 times -3); type 0/type 1: ± 2 (8 and 8 per class); two lines of the same class: $\{0, -4\}$ (13 times 0, twice -4); two type-1 lines of different classes: ± 2 (8 and 8).

9 The base-class model

Fix a class a ; everything is symmetric in a, b, c . Let $q_1, \dots, q_{n_a}$ be the z -vectors of the lines of L_a , and call a type-2 line of class $y \neq a$ an *other-class type-2 line*.

Lemma 18 (the class graph). $\langle q_i, q_j \rangle \in \{0, -4\}$ for $i \neq j$, and $\langle z_t, q_i \rangle \in \{0, -4\}$ for an other-class type-2 line t . Every q_i has exactly two -4 neighbours among the q_j and the other-class type-2 lines. Hence the (-4) -graph on L_a is a disjoint union of paths and cycles, the ends of each path are joined to other-class type-2 lines (a single vertex to two of them), the Gram matrix of L_a is $M = 8I - 4A$ with A the adjacency matrix of this graph, and all cycles have even length ≥ 4 .

Proof. The inner products follow from (2): $K = 2 + 4 = 6$ for two class- a lines, and $K = 4 + \langle 2u_a, \sigma w - 2u_y \rangle = 6$ for an other-class type-2 line. Every other line y with $z_y \neq 0$ has $K(y, q_i) = 0$, so $\langle z_y, q_i \rangle = 2S_{y, q_i} = \pm 2$: type 0 ($K = 0$), type 1 of another class ($K = 2 - 2 = 0$), and type 2 of class a ($K = 4 - 4 = 0$). Now apply (1) to q_i : $\sum_v \langle z_v, q_i \rangle^2 = 2(\lambda_2 + 5) \cdot 8$. The left side is 64 (from $v = q_i$), plus 4 for each line y of the kinds just listed, plus 16 for each -4 neighbour. With the counts of the table and of Proposition 17, the lines of the listed kinds number $n_0 + 2n_x + m_x$, and $64 + 4(n_0 + 2n_x + m_x) = 288$ (for 74 lines) resp. 352 (for 94 lines) in all three supports, while $2(\lambda_2 + 5) \cdot 8 = 320$ resp. 384. So there are exactly two -4 neighbours. A vertex of degree two in a simple graph lies on a path or a cycle. On a cycle component C the all-ones vector spans the kernel of $M|_C$, so for every line y the vector $s_i = \langle z_y, q_i \rangle / 2$ ($i \in C$) is orthogonal to it; a type-0 line has $s_i = \pm 1$, so $|C|$ is even. $\square$

Walks and F-parts. For a line $y \notin L_a$ write $s(y) = (\langle z_y, q_i \rangle / 2)_i$ (the *steps*; each s_i takes the two values fixed by the kind of y and S_{y, q_i}). The projection ω_y of z_y onto $\text{span}(L_a)$ satisfies $\langle \omega_x, \omega_y \rangle = 4s(x)^\top M^+ s(y)$, and it exists only if $s(y)$ lies in the range of M (on each cycle: sum zero). The rest $\varphi_y = z_y - \omega_y$ lies in $F = \text{span}(L_a)^\perp$, of dimension $\dim F = m_2 - n_a + (\text{number of cycles})$; a structure with $\dim F < 0$ is impossible. By (1), $\sum_y \varphi_y \varphi_y^\top = 2(\lambda_2 + 5)I_F$; the class- a lines have $\varphi = 0$ and the type-5 lines have $z = 0$, so the φ of the type-0, type-1 (classes b, c) and type-2 lines span F .

References. Let D be the least common multiple of the denominators of $4M^+$. Since $(6S - K)/3$ is always an integer, $D\langle \varphi_x, \varphi_y \rangle \in \mathbb{Z}$ for all x, y . We choose *references* $\varphi_{j_1}, \dots, \varphi_{j_k}$ as follows. First a maximal independent set of F -parts of type-2 lines (their Gram matrix is fixed by the instance). Then repeatedly, while $k < \dim F$: *case A*, some type-0 line has φ outside the span of the references; add the type-0 line with the largest residual. *Case B*, otherwise; add the type-1 line (class b or c) with the largest residual. Every configuration is reached by this rule (ties included). The later lines must satisfy the recorded maximality conditions. With G the Gram matrix of the references (scaled by D), every line y gets integer coordinates $C_y = (D\langle \varphi_y, \varphi_{j_r} \rangle)_r$ and residual $R_y = D \det(G) |\varphi_y|^2 - C_y^\top \text{adj}(G) C_y \geq 0$, all determined by its kind, its steps and its entries S against the references. At $k = \dim F$ every $R_y = 0$ and $\langle \varphi_x, \varphi_y \rangle = C_x^\top G^{-1} C_y / D$ exactly. A *node* is a possible (kind, steps, coordinates) of a line.

Pair rule. Two nodes x, y can both occur iff $4s(x)^\top M^+ s(y) + \langle \varphi_x, \varphi_y \rangle = (6S - K(x, y))/3$ for $S = +1$ or $S = -1$, and then S is their Seidel entry. Two distinct lines never have the same node (equal z would need $S = 5$ in (2)). So a configuration is a set of pairwise compatible nodes.

Symmetry. Automorphisms of the class graph that fix the type-2 data map configurations to configurations, and the candidate set of each reference is invariant under the stabiliser of the references chosen so far. So one candidate per orbit suffices. Every automorphism used is checked to preserve M .

Leaf conditions. At a leaf (all references chosen) a configuration selects, for each block, the right number of nodes, and it must satisfy: the frame (1) in walk, walk- F and F coordinates; the block sums; the numbers of -1 entries in the columns of the type-2 lines; the row counts of every selected node; and pairwise compatibility. All of these are necessary conditions.

10 Computation for #35

Program 1 (`src/c35gen.py`, `c35k.py`, `c35t.py`, `c35u.py`; OR-Tools CP-SAT). It builds explicit coordinates for the class vectors $(2(e_k - e_{k+1}))$ along each path or cycle), computes walks and F -coordinates from them, checks the pair rule against explicit vectors (`verify/c35_relation_test.py`: no mismatch), and decides each leaf with CP-SAT.

Program 2 (`verify/c35_cert.py`), written without program 1's code. It takes the p -parts in the basis (w, u_a, u_b) , solves the row counts and block sums from the eigen-equations, uses the class graph only through M (exact Moore–Penrose inverse), enumerates structures, relation graphs and partner assignments without symmetry reduction, and reduces own type-2 lines and references by explicit automorphisms (each one checked). Every leaf is decided by an LP over 0/1 node variables: equality rows for the block sizes, the frame, the block sums, the type-2 columns and the row counts of the selected nodes, with box bounds. Infeasibility is proved by an integer Farkas vector Y , whose margin $\sum_r Y_r b_r - \max_{\ell \leq x \leq u} \sum_r Y_r a_r x$ is checked to be positive in exact arithmetic. Branch and bound over cardinality groups (child i fixes the first i options to 0 and option i to 1; the last child fixes all options to 0) and propagation (compatibility, a counting core, cardinality groups) refine the leaf; the propagation depends only on the fixings, and the checker recomputes it. A certificate file has one record per leaf (instance, reference path, references, leaf index, tree), and `--check` re-verifies a whole file without an LP solver, stopping on a missing, extra or altered record (all three tested). The relation matrix is computed in floating point with tolerance 10^{-6} ; an exactly compatible pair always passes, so the computed relation contains the true one, which only weakens the model.

Agreement. On 984 identical leaves (all supports, $\dim F \leq 2$ for 74 lines) the node sets and relation matrices of the two programs are equal (`verify/c35_cert_compare.py`). Program 2's pair rules equal those allowed by the level-2 pair data for every pair of kinds (`verify/c35_structure.py`, check (4)). A third program, written independently from its own derivation of §§8–9 (exact pair relation, a different reference rule, no shared code), excludes all 444 leaves of its own search for 94 lines with integer Farkas vectors.

Coverage. In support $(0, 1, 5)$ there are no type-2 lines, so the class graph is a union of even cycles and $\dim F = m_2 - n_a + c$. For 94 lines ($m_2 = 16$, $n_a = 20$) this needs $c \geq 4$ cycles: the structures are $(8, 4, 4, 4)$, $(6, 6, 4, 4)$ and $(4, 4, 4, 4, 4)$. For 74 lines ($m_2 = 15$, $n_a = 16$) all seven partitions of 16

into even parts ≥ 4 occur. In the other supports the structures are the unions of paths and even cycles allowed by Lemma 18.

case	support	program 2: leaves	program 1
94	(0, 1, 5)	1 + 1 + 10 for (8, 4, 4, 4), (6, 6, 4, 4), (4, 4, 4, 4, 4)	all infeasible
94	(0, 1, 2, 5)	3 (seven other instances: type-2 Gram)	all excluded
94	(0, 1, 2)	14 (others: type-2 Gram, or no own type-2 lines)	all excluded
74	(0, 1, 5)	1 + 50 + 28 + 26 + 5873 + 4648 for the structures (16), (12, 4), (10, 6), (8, 8), (8, 4, 4), (6, 6, 4)	all infeasible
74	(0, 1, 5)	68,378 + 68,209 for (4, 4, 4, 4)	all infeasible (§11)
74	(0, 1, 2, 5)	24,792	all excluded
74	(0, 1, 2)	163,015	all excluded

Every program-2 leaf in the table is infeasible. All program-2 certificate files are re-verified by `verify/c35_cert_checkall.py` with one frozen version of `c35_cert.py` (sha256 prefix 911efe268df167a3); the logs are in `logs/cert_final_check*`.

11 Proofs of Theorems 2 and 3

Proof of Theorem 2. Suppose there are 94 equiangular lines in $\mathbb{R}^{20}$. By Lemma 15, levels 1 and 2 and Lemma 16, $\chi_S = (x - 21)(x - 19)^{16}(x - 15)^3(x + 5)^{74}$. By §8 the lines have one of the three supports, with the counts of the table and the rigid (r, p) -parts of Proposition 17. By §9 every configuration of the z -vectors appears at a leaf of the base-class model for some structure. Program 2 proves every leaf infeasible with checked certificates, and program 1 independently finds every leaf infeasible. Every set of more than 94 lines contains 94, so $N(20) \leq 93$. $\square$

The computations for 74 lines. The same chain applies with the 74-line data. The largest search is support (0, 1, 5) with class structure (4, 4, 4, 4), i.e. four 4-cycles, so $\dim F = 3$. For this structure every line $y \notin L_a$ has steps ± 1 with sum 0 on each cycle, and its projection onto the span of each 4-cycle is one of six vectors (four of norm 2, two of norm 1). Of the 56 lines of type 0 and classes b, c , exactly 24 alternate on each cycle. In particular, if a is chosen as a class with the fewest cycles, all three classes have structure (4, 4, 4, 4). The computations (completed 23 October 2026, 13:38 UTC):

- Supports (0, 1, 2, 5) and (0, 1, 2), and the six other structures of (0, 1, 5): excluded by program 2 (all certificate files checked) and by program 1.
- Structure (4, 4, 4, 4), program 2: complete. The level-2 reference tree has 785 units in two shards, with $68,378 + 68,209 = 136,587$ leaves, all excluded with certificates. In 42 of them (units 217, 232 and 240) the type-0 block has hundreds of LP-feasible completions, and each completion needs a search over the type-1 lines of about 1,500 LPs, so the run’s budget of 400 LPs per leaf was not enough. Branching on the type-1 groups first (`verify/c35_cert_hard.py`; certificates in the same format, so the same check applies to any choice of branching group) closed all 42, in 1,394 to 6,579 LPs each. The merged certificate file of each shard passes `--check` with the frozen code.
- Structure (4, 4, 4, 4), program 1: all 1,490 second-level leaves are infeasible in CP-SAT, 1,258 directly (`src/c35k.py`, `src/c35k3.py`) and 232 after a split by a third reference (188,648 sub-leaves, `src/c35k3.py`). `verify/c35k3_coverage.py` re-enumerates every second-level leaf

and its third-level leaves and finds an infeasible record for each in the union of the run logs; no leaf or sub-leaf was found feasible. With three references every F -part is exact. On 92 sampled leaves the node sets and pair relations equal program 2's (`verify/c35k3_compare.py`).

Proof of Theorem 3. Suppose there are 74 equiangular lines in $\mathbb{R}^{19}$. By Lemma 15, levels 1 and 2 and Lemma 16, $\chi_S = (x - 17)(x - 15)^{15}(x - 11)^3(x + 5)^{55}$. By §8 the lines have one of the three supports, with the counts of the table and the rigid (r, p) -parts of Proposition 17, and by §9 every configuration appears at a leaf of the base-class model for some structure. By the computations above, program 2 proves every leaf infeasible with checked certificates, and program 1 independently finds every leaf infeasible. Every set of more than 74 lines contains 74, so $N(19) \leq 73$. $\square$

12 59 lines in $\mathbb{R}^{18}$

This section proves Theorem 4. The published range is $57 \leq N(18) \leq 59$ [10, 7]. In this section S is a Seidel matrix of order $n = 59$ that comes from 59 equiangular lines in $\mathbb{R}^{18}$. The chain is that of §§7–11 up to level 2, followed by a third level. Since n is odd, three things change: the parity of the powers of S , the 2-adic conditions (Lemma 7 gives the weak form at the odd orders 59, 57 and type 2 at the even orders 58, 56), and an extra rule for even eigenvalues.

12.1 The candidate spectra

Lemma 19. *The angle is $\arccos \frac{1}{5}$, and*

$$\chi_S(x) = (x + 5)^{41}(x - 11)\psi(x - 11),$$

where $\psi = \sum_{k=0}^{17} c_k y^{17-k} \in \mathbb{Z}[y]$ is monic and real-rooted with $c_1 = -7$, $c_2 = -8$ and

$$(E) \quad 2^k \mid c_k \quad (k \text{ even}), \quad 2^{k-1} \mid c_k \quad (k \text{ odd}).$$

Proof. Since $59 > 2 \cdot 18$, (K1) gives the angle $\arccos \frac{1}{a}$ with a odd. For $a = 3$, (K2) allows at most 34 lines. For $a \geq 7$, (K4) allows at most $R(7, 18) = 864/31 < 28$. So $a = 5$. By (K4), $N_{1/5}(17) \leq R(5, 17) = 51 < 59$, so the lines span $\mathbb{R}^{18}$ and -5 has multiplicity exactly 41. For the other 18 eigenvalues, $\text{tr } S = 0$ and $\text{tr } S^2 = 59 \cdot 58$ give $\sum \lambda = 205$ and $\sum \lambda^2 = 2397$, so $\sum (\lambda - 11) = 7$ and $T = \sum (\lambda - 11)^2 = 65$. Lemma 10 with $\gamma = 1$, $\kappa = 11$ and $\eta = 18$ applies because $65^{18} < 18^{18} 4^{17}$, so 11 is an eigenvalue. (At $\eta = 17$ the test fails: $65^{17} > 17^{17} 4^{16}$.) Write the other 17 eigenvalues as $11 + y_j$ and put $\psi = \prod_j (y - y_j)$. Then $\chi_S(y + 11) = (y + 16)^{41} y \psi(y)$, so $\psi \in \mathbb{Z}[y]$ (division by a monic integer polynomial). $\sum y_j = 7$ and $\sum y_j^2 = 65$ give $c_1 = -7$ and $c_2 = \frac{1}{2}(49 - 65) = -8$.

For (E) apply Lemmas 7 and 8(2) with $\kappa = 11$. Here $P = \chi_S(x + 11) = F\psi$ with $F = (x + 16)^{41}x$, which is type 2 since its roots are even integers, and $\hat{F} = (y + 8)^{41}y \equiv y^{42} \pmod{2}$ is a polynomial in y^2 . Since n is odd, $\tilde{P} \pmod{2}$ is a polynomial in y^2 . Hence $\tilde{\psi} = \psi(2y)/2^{16} = \sum_k c_k 2^{1-k} y^{17-k} \in \mathbb{Z}[y]$, and its coefficients at the odd powers y^{17-k} (that is, k even) are even. This is (E). $\square$

Two more necessary conditions on χ_S are applied during the enumeration.

- (P) $\chi_S \pmod{2^7}$ lies in $\mathcal{P}_{59,7}$, the set of residues modulo 2^7 of the characteristic polynomials of Seidel matrices of order 59 [11, §3], [10, §5]. By [11, Cor. 3.13], $|\mathcal{P}_{59,7}| \leq 2^{\binom{5}{2}+1} = 2048$. Both programs found 2048 distinct residues of explicit Seidel matrices of order 59 (`src/gen_classes.py`; `verify/classes.gp`, from 15,472 random matrices of its own), so their common list is $\mathcal{P}_{59,7}$.

- (L) The congruences of [11, Lemma 3.12] for $\chi_{S+I} = \chi_S(x-1)$ (an odd order), together with [11, Remark 3.3, Lemmas 3.1, 3.8]. The lemma is stated for $J-2A$ with A an Euler graph. χ_{S+I} is a switching invariant, and for odd n every switching class contains an Euler graph, so it holds for every S of odd order.

Both conditions can be tested on prefixes of $(c_0, c_1, \dots)$. Indeed $\chi_S = \sum_i c_i v_i$ with $v_i = (x+5)^{41}(x-11)^{18-i}$ and $\chi_{S+I} = \sum_i c_i w_i$ with $w_i = (x+4)^{41}(x-12)^{18-i}$. So the coefficients of $x^{59}, \dots, x^{59-j}$ involve only $c_0, \dots, c_j$. After c_j is chosen, the positions $0, \dots, j$ of $\chi_S \bmod 2^7$ are final, and after c_7 the whole residue is final, because $c_i \equiv 0 \pmod{2^7}$ for $i \geq 8$ by (E). Similarly, after c_{2k+1} the congruences of (L) at all levels $\leq k$ are final.

Lemma 20. *There are exactly 1,214 polynomials ψ as in Lemma 19 that satisfy (P), and (L) at the levels $k \leq 8$.*

Computation. This is the enumeration principle of §5, with the residue classes of (E). After each c_j the prefix of (P) is tested, and after each c_j with $j \geq 9$ odd, (L) at the levels $\leq (j-1)/2$. Program 2 (`verify/enum59.gp`, PARI/GP, written from these statements only) visits 42,473,735 nodes in 16 shards and finds 1,214 leaves. Program 1 (`src/enum59.py`, Python with `python-flint`, ball arithmetic) splits the tree at the same level in the same order. On each of the 16 shards the two programs visit the same number of nodes at every level and find the same leaves in the same order (`verify/enum59_summary_cmp.py`). On two test shards of 400 they also agree node for node, and `verify/enum59_1312_cmp.py` checks on three shards that the leaves found with the prefix test of (L) are exactly the leaves found without it that satisfy (L) at the levels $k \leq 8$. $\square$

12.2 Level 1 at odd order

Fix a candidate $p = \chi_S$, and let Min_p , δ , $\mu = p/\text{Min}_p$ and $h = p'/\mu$ be as in Step 4 of §4. Two rules apply to p itself.

- (F1) *An even integer λ is an eigenvalue of multiplicity at most 1.* Indeed $S - \lambda I \equiv J + I \pmod{2}$. Over $\mathbb{F}_2$, $(J+I)v = 0$ means $v = (\sum_i v_i)\mathbf{1}$, and $(J+I)\mathbf{1} = (n+1)\mathbf{1} = 0$ since n is odd. So the kernel over $\mathbb{F}_2$ is spanned by $\mathbf{1}$, $\text{rank}_{\mathbb{F}_2}(S - \lambda I) = n - 1$, and $\text{rank}_{\mathbb{Q}}(S - \lambda I) \geq n - 1$, because a minor that is odd is nonzero.

- (F2) (L) at every level $k \leq 29$.

For a vertex i , $\chi_{S[i]} = \mu g_i$ with g_i in the deck of p : (D1) and (D2) as in §4, and

- (D3) $g(x-1)$ is type 2: $\chi_{S[i]}(x-1)$ is type 2 by Lemma 7 (order 58 is even), and $g(x-1)$ is a monic integer factor of it (Lemma 8(1));
- (D6) $g(-5) \neq 0$. Otherwise $\chi_{S[i]} = \mu g$ has -5 with multiplicity at least 41, because $(x+5)^{40} \mid \mu$. Then $I + \frac{1}{5}S[i]$ has rank at most 17, and the 58 remaining lines would lie in $\mathbb{R}^{17}$, against $N_{1/5}(17) \leq 51$.

Each member is also flagged by condition (iv) of [10, §5]: $\mu g \bmod 2^7 \in \mathcal{P}_{58,7}$. This list is complete for the same reason as in (P): it has $2^{\binom{5}{2}} = 1024$ elements, the bound of [11, Cor. 3.4] for even order, all realised by explicit matrices in both programs. With $\sum_i g_i = h$, an integer vector Y with $\text{Coeff}(g) \cdot Y \geq 0$ on (A) the full deck or (B) the members that pass (iv), and $\text{Coeff}(h) \cdot Y < 0$, excludes p as in Step 5 of §4.

12.3 Level 2 at odd order

Level 2 works on the members that pass (iv), with the three steps of §7, adapted to odd order as follows.

1. *Compatibility.* $S \equiv J + I \pmod{2}$, and $J^2 = nJ \equiv J$, so $S^k \equiv I + J$ for $k \geq 1$ and $r(S) \equiv r(1)I + (r(1) - r(0))J \pmod{2}$ for $r \in \mathbb{Z}[x]$. So the integers $N_k = (S^k M(S))_{ij}$ of §7 satisfy $N_k \equiv r(1) - r(0) \pmod{2}$ with $r = x^k M$ (for $k = 0$ this is condition (2) of [10, §4] with $\xi = \text{Sim}$), and Min_p must satisfy $\text{Min}_p(0) \equiv \text{Min}_p(1) \equiv 0 \pmod{2}$. On each maximal clique of the compatibility graph, the LP $\sum_f n_f g_f = h$, $n \geq 0$, with $n_f \leq 1$ for members without a loop, is decided exactly (a box-form Farkas vector when it is infeasible).
2. *Pair polynomials.* (J1): $(S^k)_{ij}$ is odd for every $k \geq 1$. (J2)–(J4) as in §7. (J5): $S[ij]$ has odd order 57, so by Lemma 7 ($\kappa = -1$) $\chi_{S[ij]}(x-1) = \sum_i a_i x^{57-i}$ has $2^{i-1} \mid a_i$ for $i \geq 1$ and $2^i \mid a_i$ for even i [9, Thm. 2.5], besides trace 0 and trace of the square $57 \cdot 56$. (J6): $P_{ij}(-5) \neq 0$. Otherwise -5 has multiplicity at least 40 in $\chi_{S[ij]} = \mu_2 P_{ij}$, and 57 lines would lie in $\mathbb{R}^{17}$.
3. *Vertex stars* as in §7, for every nonempty support U inside an LP-feasible clique K of the pair graph with $|K| \leq 12$. Larger cliques go to stage R (§12.4). Here a floating-point LP first proposes a Farkas vector or a feasible point. It counts only after an exact check, and when neither check succeeds the exact rational simplex decides (its answer, a feasible point or a Farkas vector, is checked exactly as well).

12.4 Level 3: triangle types, edge systems, and stage R

Write $R(x) = (xI - S)^{-1}$, so that $R_{ii} = g_i / \text{Min}_p$ and $R_{ij} = \psi_{ij} / \text{Min}_p$ for $i \neq j$, and put $\hat{\psi}_{ij} = S_{ij} \psi_{ij}$, a function of the pair type of $\{i, j\}$. For distinct i, j, k let $\tau_{ijk} = S_{ij} S_{jk} S_{ki}$. Let Sim , Dbl , L_3 be the products of the distinct irreducible factors of p of multiplicity 1, 2, ≥ 3 , and $\mu_3 = \prod_{m_q \geq 3} q^{m_q-3}$.

Triangle types. By Jacobi's complementary-minor identity, $\chi_{S[ijk]} = \chi_S \det R[\{i, j, k\}]$. Expanding the 3×3 determinant, and using $\psi_{ij} \psi_{jk} \psi_{ki} = \tau_{ijk} u_{ij} u_{jk} u_{ik}$ and $\psi_{ij}^2 = u_{ij}^2$ with $u = \hat{\psi}$, gives

$$\chi_{S[ijk]} = \mu_3 Q, \quad Q = \frac{B}{\text{Sim}^2 \text{Dbl}}, \quad B = g_i g_j g_k + 2\tau u_{ij} u_{jk} u_{ik} - g_i u_{jk}^2 - g_j u_{ik}^2 - g_k u_{ij}^2,$$

since $\chi_S / \text{Min}_p^3 = \mu_3 / (\text{Sim}^2 \text{Dbl})$. A triangle type (three members, three pair types and τ) is *allowed* if (T1) $\text{Sim}^2 \text{Dbl} \mid B$; (T2) $\mu_3 Q$ is monic of degree 56 with trace 0 and trace of the square $56 \cdot 55$; (T3) $(\mu_3 Q)(x-1)$ is type 2 (Lemma 7, order 56); (T4) $Q(-5) \neq 0$ (otherwise 56 lines in $\mathbb{R}^{17}$); and (T5) $\mu_3 Q$ interlaces $\chi_{S[ij]} = \mu_2 P_{ij}$ for each of the three edges (Cauchy). Every triangle of S has an allowed type.

Edge systems. Fix an edge type (a, b, u) : a vertex i of member a , a vertex j of member b , $\hat{\psi}_{ij} = u$. Let $c_{h, u_1, u_2, \tau}$ be the number of $k \neq i, j$ of member h with $\hat{\psi}_{ik} = u_1$, $\hat{\psi}_{jk} = u_2$, $\tau_{ijk} = \tau$, over the allowed triangle types. Besides the counts, $\sum_{u_1, u_2, \tau} c_{h, u_1, u_2, \tau} = n_h - [h = a] - [h = b]$, the identities $(RS)_{ij} = (SR)_{ij} = x R_{ij}$ and $R(x)R(y) = (R(x) - R(y))/(y - x)$, read at the entry (i, j) and multiplied by $S_{ij} \text{Min}_p(x)$, resp. $S_{ij} \text{Min}_p(y)$, give

$$\begin{aligned} \text{(La)} \quad \sum c \tau u_1(x) &= xu(x) - g_a(x), & \text{(Lb)} \quad \sum c \tau u_2(x) &= xu(x) - g_b(x), \\ \text{(Q)} \quad \sum c \tau u_1(x) u_2(y) &= \frac{u(x) \text{Min}_p(y) - u(y) \text{Min}_p(x)}{y - x} - g_a(x) u(y) - u(x) g_b(y). \end{aligned}$$

The counts are linked to the vertex stars of i and j : $\sum_{u_2, \tau} c_{h, u_1, u_2, \tau} = m_{h, u_1}^i - [h = b, u_1 = u]$, and symmetrically for j . Here $m_{h, w}^i$ is the number of $k \neq i$ of member h with $\widehat{\psi}_{ik} = w$, and it satisfies the vertex-star rows of §7 together with the resolvent rows of the diagonal entry,

$$(L) \quad \sum_{h, w} m_{h, w}^i w(x) = x g_a(x) - \text{Min}_p(x),$$

$$(Q) \quad \sum_{h, w} m_{h, w}^i w(x) w(y) = \frac{g_a(x) \text{Min}_p(y) - g_a(y) \text{Min}_p(x)}{y - x} - g_a(x) g_a(y).$$

Every identity has degree $< \delta$ in each variable. It is imposed modulo each irreducible factor of Min_p , or at $x = \theta$ when the eigenvalue θ is rational. An exact Farkas vector for the system of an edge type (with the deck identity $\sum_h n_h g_h = h$ and $n_h \geq 1$ on the support) shows that the edge type does not occur, and it is deleted. After every deletion the systems are decided again, and so is the vertex-star system of the support. A support U is excluded when two distinct members of U have no pair type left, or when its vertex-star system is infeasible. When the LP cascade stops, the LP-feasible edge systems are decided over the integers by exact branch and bound, with a Farkas vector at every leaf, and the cascade resumes after any deletion. The deletions are recorded in order, each certificate referring to the pair types left at that point.

Stage R. Let K be an LP-feasible clique of the pair graph with more than 12 members (too many supports). For $f \in K$, a configuration whose support U satisfies $f \in U \subseteq K$ gives, from any vertex of member f , a solution of the *relaxed vertex-star system*: unknowns n_g ($g \in K$) and $m_{g, w} \geq 0$ (w a pair type of $\{f, g\}$), with $\sum_{g \in K} n_g g = h$, $\sum_w m_{g, w} = n_g - [g = f]$, the rows (L) and (Q) of a vertex of member f , $n_f \geq 1$, $n_g \geq 0$ for $g \neq f$, and $n_g \leq 1$ for every member g without a loop (so $n_f = 1$ if f has none). Members outside U get $n_g = 0$, which the relaxation allows. If an exact box-form Farkas vector shows this system infeasible, no support inside K contains f , and f is removed. The rounds repeat on the remaining members. If K becomes empty, no support lies in K . Since the support of every configuration is a clique of the pair graph, it lies in some maximal clique, and so a candidate is excluded once every open support of stage (S) is excluded at level 3 and every clique of stage R is emptied.

12.5 The candidate left open by level 3

In both programs, levels 1–3 exclude every candidate but one, leaf 25 of shard 5:

$$p = (x + 5)^{41} (x - 11)^{14} (x - 15)^2 q, \quad q = x^2 - 21x + 94, \quad \text{Min}_p = (x + 5)(x - 11)(x - 15) q,$$

so $\delta = 5$, and q has the roots $\theta_{\pm} = \frac{1}{2}(21 \pm \sqrt{65})$. Compatibility certifies every maximal clique but one, which has four members, and the pair graph on these four is complete. Stage (S) excludes 14 of the 15 supports inside it, and level 3 does not exclude the last one, $\{g_1, g_2, g_3\}$, where $g_t = x^4 - 42x^3 + 628x^2 - b_t x + c_t$ with

$$(b_1, c_1) = (3926, 8715), \quad (b_2, c_2) = (3918, 8611), \quad (b_3, c_3) = (3910, 8475).$$

On this support the deck identity $\sum_f n_f g_f = h$ has the unique solution $(n_1, n_2, n_3) = (17, 40, 2)$. As in §8, $(E_{\lambda})_{ii} = g_i(\lambda) / \text{Min}'_p(\lambda)$, and $S_{ij}(E_{\lambda})_{ij} = \widehat{\psi}_{ij}(\lambda) / \text{Min}'_p(\lambda)$ for $i \neq j$. Exact evaluation gives:

- (a) $(E_{15})_{ii} = 0$ at the vertices of g_1 and g_3 ; at a vertex of g_2 , $(E_{15})_{ii} = \frac{1}{20}$, $(E_{11})_{ii} = \frac{15}{64}$ and $(E_{\theta_{\pm}})_{ii} > 0$.

- (b) The pair types $\widehat{\psi}$ of $\{g_2, g_2\}$ (level 2) are the six polynomials below. Each satisfies $\widehat{\psi} \equiv \tau g_2 \pmod{q}$ with $\tau = \pm 1$, and, with $e = 20\widehat{\psi}(15)/\text{Min}'_p(15) \in \{-1, 0, 1\}$,

$$64\widehat{\psi}(11)/\text{Min}'_p(11) = 4 - 4e - \tau.$$

$\widehat{\psi}$	e	τ	$64\widehat{\psi}(11)/\text{Min}'_p(11)$
$x^3 - 41x^2 + 523x - 2011$	-1	1	7
$x^3 - 39x^2 + 463x - 1561$	-1	-1	9
$x^3 - 37x^2 + 439x - 1635$	0	1	3
$x^3 - 35x^2 + 379x - 1185$	0	-1	5
$x^3 - 33x^2 + 355x - 1259$	1	1	-1
$x^3 - 31x^2 + 295x - 809$	1	-1	1

For a set Z of vertices, a vector is *balanced* on Z if its entries on Z sum to 0.

Lemma 21. *Suppose $\chi_S = p$, and let V be the set of the 40 vertices of member g_2 . After switching and renumbering the following hold.*

- (i) $V = A \cup B$ with $|A| = |B| = 20$, and for $i, j \in V$, $(E_{15})_{ij} = \frac{1}{20}$ if i, j lie in the same group and 0 otherwise.
- (ii) On $A = \{0, \dots, 19\}$ the graph $\{ij : S_{ij} = -1\}$ is the union of the five 4-cycles $Z_c = (4c, 4c + 1, 4c + 2, 4c + 3)$, $0 \leq c \leq 4$. On B it is the union of five 4-cycles $Z'_0, \dots, Z'_4$.
- (iii) There is $w \in \{\pm 1\}^V$ with $S_{ij}w_iw_j = \tau_{ij}$ for $i \neq j$ in V , where τ_{ij} is the sign τ of the pair type of $\{i, j\}$. In cyclic order, w is $(+, +, -, -)$ on Z_0, Z_1, Z_2 and $(+, -, +, -)$ on Z_3, Z_4 ; we write $\omega = w|_A$. On three of the Z'_c , w is a rotation of $(+, +, -, -)$, and on the other two a rotation of $(+, -, +, -)$.
- (iv) Let C be the adjacency matrix of the cycles Z_c , P the involution of A that exchanges opposite vertices of each Z_c , and $\beta = 5I - C - 3P$. For $k \in B$ let $\sigma^k = (S_{ik})_{i \in A}$. Then σ^k is balanced on every Z_c , $\sigma^k \beta \omega = 32w_k$, and for $k, l \in B$, $\sigma^k \beta \sigma^l$ is 128 if $k = l$, -64 if k, l are adjacent on a cycle Z'_c , and 0 otherwise.

Proof. (i) Let $u_i \in \mathbb{R}^2$ be the rows of a matrix whose columns are an orthonormal basis of the 15-eigenspace. Then $(E_{15})_{ij} = \langle u_i, u_j \rangle$ and $\sum_i u_i u_i^\top = I_2$. By (a), $u_i = 0$ outside V and $|u_i|^2 = \frac{1}{20}$ on V . By (b), $20\langle u_i, u_j \rangle = S_{ij}e \in \{0, \pm 1\}$ for $i \neq j$ in V , so any two of these u_i are parallel or orthogonal. They span $\mathbb{R}^2$, so they lie on two orthogonal lines. If A and B are the sets of i on each line, then $I_2 = \sum_i u_i u_i^\top = \frac{1}{20}(|A| \Pi_1 + |B| \Pi_2)$ with the orthogonal projections Π_1, Π_2 onto the two lines, so $|A| = |B| = 20$. Switching ($S \mapsto DSD$, $E_\lambda \mapsto DE_\lambda D$; deck members and pair types are invariant) makes the u_i equal on A and equal on B .

(ii) Now $\mathbf{1}_A$ is a multiple of a column of E_{15} , so $S\mathbf{1}_A = 15 \cdot \mathbf{1}_A$. At $i \in A$ this says $\sum_{j \in A \setminus i} S_{ij} = 15$: exactly two $j \in A$ have $S_{ij} = -1$. So the graph $C_A = \{ij \subseteq A : S_{ij} = -1\}$ is 2-regular, a union of cycles, and so is C_B . We use the same letters for the adjacency matrices.

(iii) The eigenvalue $\theta = \theta_+$ is simple, so $E_\theta = vv^\top$, and by (a) $v_i = w_i \sqrt{d}$ on V with $d = g_2(\theta)/\text{Min}'_p(\theta) > 0$ and $w_i = \pm 1$. By (b), $S_{ij}v_i v_j = \widehat{\psi}_{ij}(\theta)/\text{Min}'_p(\theta) = \tau_{ij}d$, that is, $S_{ij}w_i w_j = \tau_{ij}$. Next, for $i \neq j$ in V , (b) and (i) give $64(E_{11})_{ij} = S_{ij}(4 - 4e_{ij} - \tau_{ij}) = 4S_{ij} - 4[\text{same group}] - w_i w_j$, since $S_{ij}e_{ij} = 20(E_{15})_{ij}$; and $64(E_{11})_{ii} = 15$. So $F = 64E_{11}[V]$ has the blocks

$$F_{AA} = 16H_A - w_A w_A^\top, \quad F_{BB} = 16H_B - w_B w_B^\top, \quad F_{AB} = 4S[A, B] - w_A w_B^\top, \quad H_A = I - \frac{1}{2}C_A,$$

and $F \succeq 0$, $\text{rank } F \leq 14$, because E_{11} is a projection of rank 14. On a cycle of length m , $I - \frac{1}{2}C$ has the eigenvalues $1 - \cos(2\pi k/m)$, so $H_A \succeq 0$ and $\ker H_A$ is spanned by the indicator vectors of the cycles of C_A . Since $F_{AA} \succeq 0$, $w_A \perp \ker H_A$: w is balanced on every cycle. So every cycle has even length, there are $t \leq 5$ cycles, and $w_A^\top (16H_A)^+ w_A \leq 1$, with $\text{rank } F_{AA} = 20 - t$ or $19 - t$, the latter exactly in the case of equality. As $\text{rank } F_{AA} \leq 14$, $t = 5$ and equality holds. So C_A consists of five 4-cycles, and $w_A^\top \beta_A w_A = 128$ for $\beta_A = 8H_A^+$, which is $5I - C - 3P$ on each 4-cycle: there, $(I - \frac{1}{2}C)(5I - C - 3P) = 8I - 2J$ and $(5I - C - 3P)\mathbf{1} = 0$. A balanced ± 1 vector on a 4-cycle is, in cyclic order, a rotation of $(+, +, -, -)$, with $w^\top \beta w = 32$, or of $(+, -, +, -)$, with $w^\top \beta w = 16$. So $32a + 16(5 - a) = 128$ gives $a = 3$ cycles of the first kind and 2 of the second. Numbering each cycle of A from a suitable vertex gives (ii) and (iii) on A , and the argument for B is the same.

(iv) $\text{rank } F = \text{rank } F_{AA} = 14$. For a positive semidefinite block matrix this means that the columns of F_{AB} lie in the range of F_{AA} and $F_{BB} = F_{BA}F_{AA}^+F_{AB}$. Since $16H_A\beta\omega = 128\omega$ and $\omega^\top \beta\omega = 128$, $F_{AA}\beta\omega = 0$, so $\ker F_{AA}$ is spanned by the $\mathbf{1}_{Z_c}$ and $\beta\omega$, and the range of F_{AA} consists of the vectors y balanced on every Z_c with $y^\top \beta\omega = 0$. The column of F_{AB} at k is $4\sigma^k - w_k\omega$. Hence σ^k is balanced on every Z_c , and $4\sigma^k\beta\omega = w_k\omega^\top \beta\omega = 128w_k$. For y, y' in the range, $z = \beta y'/128$ satisfies $F_{AA}z = y'$ (because $16H_A\beta y' = 128y'$ and $\omega^\top \beta y' = 0$), so $y^\top F_{AA}^+ y' = y^\top \beta y'/128$. Therefore $(F_{BB})_{kl} = (4\sigma^k - w_k\omega)^\top \beta (4\sigma^l - w_l\omega)/128 = \sigma^k\beta\sigma^l/8 - w_k w_l$, and comparison with $F_{BB} = 16H_B - w_B w_B^\top$ gives $\sigma^k\beta\sigma^l = 128(H_B)_{kl}$. $\square$

Proposition 22. *Let A , Z_c , β and ω be as in Lemma 21. For $\varepsilon = \pm 1$ let Π_ε be the set of $\sigma \in \{\pm 1\}^A$ that are balanced on every Z_c and satisfy $\sigma\beta\sigma = 128$ and $\sigma\beta\omega = 32\varepsilon$. A cycle assignment of kind adj (resp. alt) is a tuple $(\sigma_1, \dots, \sigma_4)$ with $\sigma_j \in \Pi_{\varepsilon_j}$ for $\varepsilon = (+, +, -, -)$ (resp. $(+, -, +, -)$) and $\sigma_j\beta\sigma_l = 128, -64, 0$ for $j - l \equiv 0, \pm 1, 2 \pmod{4}$. Its span is the linear span of $\sigma_1, \dots, \sigma_4$. Then $|\Pi_+| = |\Pi_-| = 248$. There are 7,920 assignments of kind adj, with 1,980 distinct spans, and 24,000 of kind alt, with 3,000 distinct spans, all of dimension 3. No three spans of kind adj and one of kind alt are pairwise β -orthogonal.*

Computation. Program 1 (`src/s525.59.py`, Python with exact integers and fractions) enumerates the 6^5 vectors balanced on every Z_c and finds the assignments by a depth-first search along the cycle. It identifies spans by their reduced row echelon forms over $\mathbb{Q}$ and tests two spans for orthogonality on three spanning vectors of each. Program 2 (`verify/s525.59.gp`, PARI/GP) works with $z = T\sigma \in \mathbb{Z}^{15}$, where $\beta = T^\top T$ (T has the rows $2(e_0 - e_2)$, $2(e_1 - e_3)$ and $e_0 - e_1 + e_2 - e_3$ on each cycle). It searches the assignments in another order (σ_1, σ_3 first) and keys a span W by the orthogonal projection P_W onto TW , which does not depend on a basis. It checks that the Hermite normal forms of the lattices $W \cap \mathbb{Z}^{20}$ give the same partition, tests orthogonality by $\text{tr}(P_W P_{W'}) = 0$, and counts the configurations starting from the alt span. Both programs find the counts of the statement and the same intermediate counts: 168,300 orthogonal pairs of adj spans, 601,560 of an adj and an alt span, 325,080 of alt spans, 64,800 pairwise orthogonal triples of adj spans, and 194,400 pairwise orthogonal (adj, adj, alt) triples, none of which extends by a third adj span. $\square$

Corollary 23. *The candidate $p = (x+5)^{41}(x-11)^{14}(x-15)^2(x^2-21x+94)$ is not the characteristic polynomial of a Seidel matrix of 59 equiangular lines in $\mathbb{R}^{18}$.*

Proof. Suppose it is. Since level 3 leaves only the support $\{g_1, g_2, g_3\}$, Lemma 21 applies. Number each cycle Z'_c of B as (k_1, k_2, k_3, k_4) in cyclic order, starting so that $(w_{k_1}, \dots, w_{k_4})$ is $(+, +, -, -)$ or $(+, -, +, -)$. By (iv), $(\sigma^{k_1}, \dots, \sigma^{k_4})$ is a cycle assignment of kind adj resp. alt, and the spans of different cycles are β -orthogonal. By (iii), three cycles of B have kind adj and two have kind alt. This contradicts Proposition 22. $\square$

Both programs of Proposition 22 (stage X in `verify/tally59.py`) first check the input of the argument on their own records: the factorization of p , that level 3 leaves exactly the support above open, the unique counts, (a), and (b) for the program’s own list of pair types. Program 1 reads its level-1 to level-3 records. Program 2 re-runs `verify/lv23_59.gp` on this leaf, on its own level-1 deck, and requires no mismatch and no certificate failure. An independent audit found the same structure and closes the last step by a backtracking search and CP-SAT instead.

12.6 Computation

Program 1 (Python 3, `python-flint`, HiGHS): `src/enum59.py`, `src/run.level1_59.py` (decks in `src/eqlines/deck.py`, certificates in `lpcert.py`), `src/level2_59.py` (`compat.py`, `jacobi.py`, `star.py`), `src/level3_59.py` (`triples.py`, `star.RelaxedStar`, `farkas.py`), and `src/s525_59.py` (§12.5). Interval endpoints use ball arithmetic. FLINT’s certified complex root isolation decides whether a candidate D_i in the enumerations is real-rooted. In the runs started after 23 October 2026, 23:33 UTC, a certified test comes first. Floating-point roots, refined by Newton steps, only propose points. If D_i changes sign, in ball arithmetic, between two exact dyadic points around each of $\deg D_i$ disjoint proposals, then all roots of D_i are real and simple, and each is enclosed. Only when this fails is the root isolation called. Floating-point LPs only propose; every infeasibility is an exact Farkas vector, checked in exact arithmetic. The exact simplex (`exactlp.py`) is a phase-1 simplex over $\mathbb{Q}$; in the runs started after 24 October 2026, 06:24 UTC, it is a revised simplex with exact basis solves (FLINT), Dantzig’s rule, and Bland’s rule after 50 pivots without a strict decrease, so it terminates. The earlier version used a dense rational tableau and Bland’s rule throughout. Both return only answers that pass the exact check, so the version affects the running time, not the verdicts. `src/cert59.py` exports the level-2 and level-3 certificates in a point form (rows at the points $-6, \dots, \delta - 7$), after an exact self-check that rebuilds every row from its definition.

Program 2 (PARI/GP), written from the statements above without program 1’s code: `verify/enum59.gp`, `verify/classes.gp`, `verify/deck59.gp` (level 1), `verify/lv23_59.gp` (levels 2, 3 and stage R), `verify/s525_59.gp` (§12.5). From program 1 it reads only the certificates (`verify/p2_export_leaf.py`, `verify/json2gp.py`). It recomputes the deck (real roots by PARI’s `polrootsreal`), h , the class flags, the compatibility graph and its maximal cliques, every pair polynomial, and the star, edge and relaxed systems on its own variables, and checks each certificate against its own system. It reports every certificate that fails and every difference from program 1 in decks, verdicts, cliques or pair types; a leaf counts as excluded only if there is neither. `src/sched59.py` runs both programs leaf by leaf, and `verify/tally59.py` tallies their verdicts leaf by leaf.

Controls. `verify/triples_negative.py` (random Seidel matrices, clique unions, and deletions from the unitary $U_3(3)$, $U_3(5)$ systems: all 1,055 true triangles are allowed, $\mu_3 Q$ equals the directly computed characteristic polynomial, the true counts satisfy every row, and no true edge type is certified), `verify/relaxed_negative.py` (192 true vertex stars, with and without a fake absent member, satisfy every row exactly and are never certified), and `verify/lv23_negative.py` (45 perturbed exports of three leaves, all rejected by program 2).

Result (completed 24 October 2026, 08:36 UTC). Both programs exclude all 1,214 candidates, each at the same stage, with no disagreement in decks, cliques, pair types, open supports or certificates: 572 at level 1, 526 at (C), 33 at (J), 53 at (S), 17 by stage R (every other support of these leaves is closed by (S)), 11 at level 3, one (s2/14) at level 3 together with stage R, and one (s5/25)

by stage X. (Program 1 runs stage R inside its level 3, so its label for the 17 stage-R leaves is L3+R where program 2 writes S+R.) An independently written pipeline excludes every candidate as well.

Proof of Theorem 4. Suppose there are 59 equiangular lines in $\mathbb{R}^{18}$. By Lemmas 19 and 20, χ_S is one of the 1,214 candidates. Each is excluded by a level-1 Farkas vector, by level 2, by level 3 with stage R, or by Corollary 23, and each exclusion is verified by both programs. Every set of more than 59 lines contains 59, so $N(18) \leq 58$. $\square$

13 What remains

- $344 \leq N(44) \leq 420$. For $n = 420$, Lemma 10 forces only $(x - 59)^8$ and leaves a remainder of degree 36, far beyond direct enumeration.
- For $d = 45, 46$ the relative bound (540, 736) is an integer; equality is the existence of $\text{srg}(539, 234, 81, 117)$, resp. $\text{srg}(735, 318, 109, 159)$. Proposition 12 does not apply at $n = R$ and fails at $n = R - 1$.
- $d = 20$: $90 \leq N(20) \leq 93$. Excluding 93 lines would need the chain at $n = 93$, an odd order. There Lemma 10 forces no eigenvalue: the angle is $\arccos \frac{1}{5}$, -5 has multiplicity 73, and the best centre $\kappa = 19$ gives $T = 81$, while $81^{20} > 20^{20}4^{19}$. So the remainder has degree 20 instead of 13.
- $d = 19$: $72 \leq N(19) \leq 73$. Likewise at $n = 73$: -5 has multiplicity 54, the best centre $\kappa = 15$ gives $T = 81$, $81^{19} > 19^{19}4^{18}$, and the remainder has degree 19. (For program 2's 42 hard leaves of §11, lookahead branching, conditional row-count cuts, eigen-equation rows for selected nodes, 4-cycle propagation, spectral cuts and CP-SAT did not help within the default branching order; branching on the type-1 groups first decided all of them.)
- $d = 18$: $57 \leq N(18) \leq 58$. At $n = 58$ (even order), -5 has multiplicity 40, the best centre $\kappa = 11$ gives $T = 84 > 4 \cdot 18$, so Lemma 10 forces no eigenvalue, and the remainder has degree 18 instead of 17.
- Dimensions 18 and 42: the moment tests give nothing new.

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A The 82 candidates for 421 lines in $\mathbb{R}^{44}$

Each row gives: the eigenvalues of χ_S other than -7^{377} (integer eigenvalues with multiplicity, then irreducible non-linear factors), δ , the deck size, and the number of decimal digits of the largest certificate entry. The full data are in `data/n421/decks/<idx>.json`.

Table 1: Candidates $\chi_S = (x+7)^{377}(x-59)^{16}(x-61)^{15}Q(x-59)$.

#	integer eigenvalues	other factors	δ	deck	Y
0	55, 59 ¹⁹ , 60, 61 ²³	–	5	7	8
1	59 ¹⁸ , 61 ²³	$(x^3 - 174x^2 + 10085x - 194704)$	6	6	9
2	59 ¹⁸ , 61 ²³	$(x^3 - 174x^2 + 10085x - 194708)$	6	6	10
3	59 ²⁰ , 61 ²²	$(x^2 - 117x + 3412)$	5	15	35
4	56, 57, 59 ¹⁸ , 61 ²⁴	–	5	3	8
5	59 ¹⁸ , 61 ²²	$(x^4 - 235x^3 + 20699x^2 - 809897x + 11877416)$	7	21	11

#	integer eigenvalues	other factors	δ	deck	$ Y $
6	$59^{20}, 61^{21}$	$(x^3 - 178x^2 + 10549x - 208136)$	6	28	21
7	$57^3, 59^{16}, 60, 61^{24}$	—	5	3	8
8	$57, 59^{18}, 60, 61^{22}$	$(x^2 - 118x + 3473)$	7	26	11
9	$59^{19}, 61^{22}$	$(x^3 - 176x^2 + 10315x - 201316)$	6	32	32
10	$59^{18}, 60, 61^{21}$	$(x^2 - 120x + 3595), (x^2 - 116x + 3359)$	8	15	24
11	$59^{20}, 60, 61^{21}$	$(x^2 - 118x + 3469)$	6	41	10
12	$57^2, 59^{17}, 61^{23}$	$(x^2 - 119x + 3536)$	6	6	10
13	$59^{19}, 61^{21}$	$(x^2 - 119x + 3536), (x^2 - 118x + 3473)$	7	15	22
14	$57, 59^{18}, 61^{22}$	$(x^3 - 178x^2 + 10553x - 208384)$	7	22	22
15	$59^{19}, 61^{21}$	$(x^4 - 237x^3 + 21051x^2 - 830535x + 12280512)$	7	40	23
16	$59^{20}, 61^{21}$	$(x^3 - 178x^2 + 10549x - 208144)$	6	47	11
17	$59^{18}, 61^{21}$	$(x^2 - 120x + 3595), (x^3 - 176x^2 + 10319x - 201548)$	8	12	19
18	$57, 59^{19}, 61^{22}$	$(x^2 - 119x + 3532)$	6	22	21
19	$59^{19}, 61^{21}$	$(x^4 - 237x^3 + 21051x^2 - 830539x + 12280748)$	7	60	12
20	$59^{20}, 61^{21}$	$(x^3 - 178x^2 + 10549x - 208148)$	6	41	11
21	$57^2, 59^{18}, 61^{22}$	$(x^2 - 121x + 3656)$	6	18	10
22	$59^{20}, 61^{20}$	$(x^2 - 118x + 3473), (x^2 - 121x + 3656)$	7	40	17
23	$59^{19}, 61^{21}$	$(x^4 - 237x^3 + 21051x^2 - 830543x + 12280984)$	7	63	23
24	$56, 59^{21}, 61^{21}, 63$	—	5	45	35
25	$57, 59^{18}, 60, 61^{20}$	$(x^2 - 120x + 3595)^2$	7	16	17
26	$59^{20}, 61^{20}$	$(x^4 - 239x^3 + 21407x^2 - 851645x + 12697532)$	7	62	22
27	$57^2, 59^{19}, 60, 61^{21}, 63$	—	6	25	10
28	$59^{21}, 60, 61^{19}, 63$	$(x^2 - 118x + 3473)$	7	66	18
29	$59^{20}, 61^{21}$	$(x^3 - 178x^2 + 10549x - 208156)$	6	43	10
30	$59^{20}, 60, 61^{20}$	$(x^3 - 179x^2 + 10667x - 211625)$	7	22	23
31	$59^{19}, 61^{19}$	$(x^2 - 119x + 3536), (x^2 - 120x + 3595)^2$	7	8	22
32	$59^{20}, 61^{19}$	$(x^2 - 120x + 3595), (x^3 - 180x^2 + 10791x - 215456)$	8	51	23
33	$57, 59^{20}, 61^{20}, 63$	$(x^2 - 119x + 3536)$	7	44	17
34	$59^{21}, 61^{19}, 63$	$(x^3 - 178x^2 + 10553x - 208384)$	7	54	18
35	$59^{20}, 61^{19}$	$(x^5 - 300x^4 + 35986x^3 - 2157476x^2 + 64648349x - 774563344)$	8	65	24
36	$59^{20}, 61^{21}$	$(x^3 - 178x^2 + 10549x - 208160)$	6	54	22
37	$59^{21}, 61^{20}$	$(x^3 - 180x^2 + 10787x - 215216)$	6	59	11
38	$59^{22}, 61^{19}, 63$	$(x^2 - 119x + 3532)$	6	40	20
39	$59^{20}, 61^{19}$	$(x^5 - 300x^4 + 35986x^3 - 2157480x^2 + 64648837x - 774578212)$	8	73	25
40	$59^{20}, 61^{19}$	$(x^5 - 300x^4 + 35986x^3 - 2157480x^2 + 64648837x - 774578228)$	8	73	24
41	$57, 59^{20}, 61^{21}$	$(x^2 - 121x + 3652)$	6	40	34
42	$59^{21}, 61^{19}$	$(x^4 - 241x^3 + 21767x^2 - 873227x + 13128428)$	7	59	18
43	$59^{20}, 61^{20}$	$(x^4 - 239x^3 + 21407x^2 - 851653x + 12697988)$	7	59	12
44	$59^{20}, 61^{18}$	$(x^2 - 121x + 3656), (x^2 - 120x + 3595)^2$	7	8	18
45	$57, 59^{21}, 61^{19}, 63$	$(x^2 - 121x + 3656)$	7	44	18
46	$59^{20}, 61^{19}$	$(x^2 - 120x + 3595), (x^3 - 180x^2 + 10791x - 215464)$	8	51	24

#	integer eigenvalues	other factors	δ	deck	$ Y $
47	$59^{22}, 61^{19}$	$(x^3 - 182x^2 + 11029x - 222520)$	6	54	21
48	$59^{20}, 61^{19}$	$(x^5 - 300x^4 + 35986x^3 - 2157484x^2 + 64649309x - 774592136)$	8	65	25
49	$57, 59^{20}, 61^{20}$	$(x^3 - 182x^2 + 11033x - 222776)$	7	54	18
50	$59^{21}, 61^{20}$	$(x^3 - 180x^2 + 10787x - 215224)$	6	59	22
51	$59^{21}, 60, 61^{17}, 63$	$(x^2 - 120x + 3595)^2$	7	16	19
52	$57, 59^{22}, 60, 61^{18}, 63^2$	—	6	25	10
53	$59^{21}, 61^{19}$	$(x^4 - 241x^3 + 21767x^2 - 873235x + 13128932)$	7	62	23
54	$59^{22}, 61^{19}$	$(x^3 - 182x^2 + 11029x - 222524)$	6	43	10
55	$57, 59^{20}, 60, 61^{20}$	$(x^2 - 122x + 3713)$	7	66	18
56	$59^{21}, 60, 61^{19}$	$(x^3 - 181x^2 + 10907x - 218815)$	7	22	21
57	$59^{23}, 61^{17}, 63^2$	$(x^2 - 119x + 3536)$	6	18	10
58	$59^{22}, 61^{18}$	$(x^4 - 243x^3 + 22131x^2 - 895297x + 13574224)$	7	63	23
59	$59^{21}, 61^{19}$	$(x^2 - 119x + 3536), (x^2 - 122x + 3713)$	7	40	18
60	$57, 59^{22}, 61^{20}, 64$	—	5	45	7
61	$59^{23}, 61^{18}, 63$	$(x^2 - 121x + 3652)$	6	22	20
62	$59^{22}, 61^{17}$	$(x^2 - 120x + 3595), (x^3 - 184x^2 + 11279x - 230332)$	8	12	19
63	$59^{22}, 61^{18}$	$(x^4 - 243x^3 + 22131x^2 - 895301x + 13574468)$	7	60	12
64	$59^{22}, 61^{19}$	$(x^3 - 182x^2 + 11029x - 222532)$	6	41	11
65	$59^{24}, 61^{16}, 63^2$	$(x^2 - 121x + 3656)$	6	6	10
66	$59^{23}, 61^{17}, 63$	$(x^3 - 182x^2 + 11033x - 222776)$	7	22	21
67	$59^{22}, 61^{18}$	$(x^2 - 122x + 3713), (x^2 - 121x + 3656)$	7	15	23
68	$59^{22}, 61^{18}$	$(x^4 - 243x^3 + 22131x^2 - 895305x + 13574712)$	7	40	21
69	$59^{22}, 61^{19}$	$(x^3 - 182x^2 + 11029x - 222536)$	6	47	11
70	$59^{25}, 60, 61^{15}, 63^3$	—	5	3	8
71	$59^{23}, 60, 61^{17}, 63$	$(x^2 - 122x + 3713)$	7	26	22
72	$59^{22}, 60, 61^{17}$	$(x^2 - 120x + 3595), (x^2 - 124x + 3839)$	8	15	15
73	$59^{23}, 61^{18}$	$(x^3 - 184x^2 + 11275x - 230084)$	6	32	19
74	$59^{22}, 60, 61^{19}$	$(x^2 - 122x + 3709)$	6	41	10
75	$59^{25}, 61^{17}, 63, 64$	—	5	3	8
76	$59^{23}, 61^{17}$	$(x^4 - 245x^3 + 22499x^2 - 917863x + 14035376)$	7	21	11
77	$59^{22}, 61^{19}$	$(x^3 - 182x^2 + 11029x - 222544)$	6	28	22
78	$59^{24}, 61^{17}$	$(x^3 - 186x^2 + 11525x - 237892)$	6	6	10
79	$59^{23}, 61^{19}$	$(x^2 - 123x + 3772)$	5	15	33
80	$59^{24}, 61^{17}$	$(x^3 - 186x^2 + 11525x - 237896)$	6	6	9
81	$59^{24}, 60, 61^{18}, 65$	—	5	7	18

B Excluded top values for $a \leq 13$

Columns: $a, d, R, n_0 = \lfloor R \rfloor$, the smallest odd witness $\kappa, s, T, \eta^*(T)$. In each row, $N_{1/a}(d) \leq n_0 - 1$. Produced by `verify/c1.uniform.py` and `verify/c1.uniform.gp`, whose outputs agree line by line.

Table 2: Excluded top values of Proposition 12.

a	d	R	n_0	κ	s	T	η^*
7	26	54.261	54	7	14	20	5
7	39	187.200	187	25	61	105	27
7	40	213.333	213	29	51	81	21
7	44	422.400	422	59	50	76	19
9	42	86.154	86	9	18	20	5
9	44	95.135	95	9	63	101	26
9	46	105.143	105	11	25	25	7
9	50	129.032	129	13	61	77	20
9	52	143.448	143	15	39	65	17
9	55	169.231	169	17	91	169	43
9	56	179.200	179	19	43	49	13
9	59	214.545	214	23	38	68	17
9	62	261.053	261	27	117	225	57
9	64	301.176	301	33	21	21	6
9	68	418.462	418	45	90	156	39
9	70	509.091	509	55	101	153	39
9	74	845.714	845	93	57	101	26
11	62	126.102	126	11	22	20	5
11	63	130.345	130	11	44	72	18
11	65	139.286	139	11	99	185	47
11	69	159.231	159	13	93	153	39
11	72	176.327	176	15	64	96	24
11	80	234.146	234	21	14	20	5
11	83	262.105	262	23	60	56	14
11	85	283.333	283	25	53	73	19
11	87	307.059	307	27	71	65	17
11	90	348.387	348	31	48	72	18
11	93	398.571	398	35	100	176	44
11	95	438.462	438	39	68	104	26
11	98	511.304	511	45	133	217	55
11	100	571.429	571	51	81	117	30
11	102	644.211	644	57	148	240	60
11	103	686.667	686	61	130	244	61
11	104	734.118	734	65	170	292	73
11	105	787.500	787	71	47	81	21
11	107	917.143	917	83	29	25	7
11	112	1493.333	1493	135	71	85	22
11	114	1954.286	1954	177	62	68	17
13	86	174.072	174	13	26	20	5
13	87	178.244	178	13	52	72	18
13	103	262.182	262	19	110	148	37
13	109	305.200	305	23	41	49	13
13	110	313.220	313	23	109	145	37
13	112	330.105	330	25	34	28	7
13	114	348.218	348	25	192	360	90
13	120	411.429	411	31	63	105	27
13	121	423.500	423	31	175	337	85
13	122	436.085	436	33	56	40	10
13	123	449.217	449	33	179	297	75
13	125	477.273	477	35	201	369	93
13	126	492.279	492	37	96	120	30

a	d	R	n_0	κ	s	T	η^*
13	128	524.488	524	39	156	272	68
13	131	579.158	579	43	191	305	77
13	132	599.351	599	45	131	189	48
13	134	643.200	643	49	51	53	14
13	135	667.059	667	51	31	17	5
13	137	719.250	719	55	31	49	13
13	139	778.400	778	59	106	148	37
13	140	811.034	811	61	183	245	62
13	142	883.556	883	67	119	193	49
13	144	967.680	967	73	187	357	90
13	146	1066.435	1066	81	134	196	49
13	147	1122.545	1122	85	180	312	78
13	149	1251.600	1251	95	171	297	75
13	150	1326.316	1326	101	138	180	45
13	151	1409.333	1409	107	197	313	79
13	152	1502.118	1502	115	70	52	13
13	153	1606.500	1606	123	70	116	29
13	158	2413.091	2413	185	85	61	16
13	159	2671.200	2671	205	61	57	15
13	160	2986.667	2986	229	98	172	43
13	164	5510.400	5510	423	126	164	41