

COMPLEX HYPERCONTRACTIVITY FOR SEVERAL CORRELATED GAUSSIAN FUNCTIONS BELOW THE DIAGONAL

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ABSTRACT. Let $\xi = (\xi_1, \dots, \xi_n)$ be a centred Gaussian vector whose blocks $\xi_j \in \mathbb{R}^{k_j}$ are standard, with covariance $\Sigma = \text{cov}(\xi)$, and let T_z be the Mehler transform ($T_z H_\beta = z^{|\beta|} H_\beta$). We study when

$$\left\| \prod_{j=1}^n |T_{z_j} f_j(\xi_j)|^{p_j} \right\|_\alpha \leq \left\| \prod_{j=1}^n |f_j(\xi_j)|^{p_j} \right\|_1,$$

with emphasis on the range below the diagonal, i.e. outer exponent $\alpha < 1$, where the second-order (local) conditions that characterise the regime $\alpha \geq 1$ are no longer sufficient. The paper has three parts, all self-contained. *(I) Imaginary parameters, every outer exponent.* For $z_j = is_j$ and every $\alpha > 0$ the inequality holds if and only if the covariance condition $\frac{1}{\alpha} \text{diag}(\frac{1+s_j^2}{s_j^{2p_j}}) \geq \Sigma \geq \text{diag}(\frac{1+s_j^2}{p_j})$ holds and $\max_j |s_j| \leq 1$; the second condition is automatic for $\alpha \geq 1$ and genuinely needed below the diagonal. We deduce a Gaussian-weighted Hausdorff–Young inequality for every outer exponent, and for equal parameters the sharp region $y^2 \leq \min\{1, p\lambda_{\min} - 1, (r\lambda_{\max} - 1)^{-1}\}$. *(II) Two ρ -correlated functions.* On the imaginary axis we determine the region exactly for all $p > 1$, $r > 0$; on the real axis it is $[-1, 1]$; for general z we give inner and outer bounds. *(III) General complex z and joint narrow bumps.* For two equally weighted ρ -correlated functions with $r \geq 2$ and $Q = r(1+\rho)$ we show the inequality *fails* whenever $\delta_Q(z) < 0$, or $\delta_Q(z) > 0$ and $\kappa_Q(z) > \frac{1-\rho}{4Q}$, using witnesses $f = g = 1 - 2b$ with a narrow far-out Gaussian bump b ; the excluded set contains points at which all second-order conditions and all one-function conditions hold, and two such points are certified in interval (ball) arithmetic. Together with a Brascamp–Lieb inner bound this sandwiches the admissible set between $\Omega_{Q,Q}$ and an explicit larger set $\mathcal{J}_{\rho,r} \cap \Omega_{r,r}$; the two agree on both axes and differ off them, and the region between is left open. The same mechanism gives bounds for n equicorrelated functions, with an admissible region of width $\asymp (n\rho)^{-1/2}$ around $(-1, 1)$ as $n \rightarrow \infty$. Proofs are by hand; several numerical facts are certified in ball arithmetic and the remaining ones are stated as numerical evidence.

Disclosure. This text is an AI-generated proof, produced by Grok Bots via Cursor agents and prompted by Paata Ivanisvili in October 2026. It was subsequently prompted several times for improvement and for publishing, and revised and polished by Grok Bots. It is essentially the raw output obtained after a series of prompts.

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1. INTRODUCTION

Let γ_k denote the standard Gaussian measure on $\mathbb{R}^k$, let $\mathcal{P}_k$ be the complex polynomials on $\mathbb{R}^k$, and let H_β be the (probabilists') Hermite polynomials, an orthogonal basis of $L^2(\gamma_k)$. The Mehler transform T_z , $z \in \mathbb{C}$, acts on $f = \sum_\beta c_\beta H_\beta$ by

$$(1) \quad T_z f = \sum_\beta z^{|\beta|} c_\beta H_\beta.$$

For real $z \in [-1, 1]$ this is the Ornstein–Uhlenbeck (noise) operator $T_z f(x) = \mathbb{E}f(zx + \sqrt{1-z^2}Y)$, $Y \sim \gamma_k$; for $z = i\sqrt{p-1}$ it is Beckner's operator, whose boundedness from L^p to $L^{p'}$ is the sharp Hausdorff–Young inequality in Gaussian coordinates [1]. The exact set of z for which T_z is a contraction of $L^q(\gamma_k)$ (the diagonal case $p = q$) is classical: it is

$$\Omega_q := \Omega_{q,q} = \{z : |q-2||1-z^2| \leq q(1-|z|^2)\}$$

(Weissler [11], Epperson [4], Lieb [7], Janson [6]).

Recently Ivanisvili and Kalantzopoulos [5] introduced a multifunction framework: for a Gaussian vector $\xi = (\xi_1, \dots, \xi_n)$ with standard blocks and covariance $\Sigma = \text{cov}(\xi)$, and exponents $p_j > 0$, $\alpha > 0$, one asks when

$$(2) \quad \left\| \prod_{j=1}^n |T_{z_j} f_j(\xi_j)|^{p_j} \right\|_\alpha \leq \left\| \prod_{j=1}^n |f_j(\xi_j)|^{p_j} \right\|_1 \quad (f_j \in \mathcal{P}_{k_j}),$$

where $\|W\|_\alpha = (\mathbb{E}|W|^\alpha)^{1/\alpha}$. They proved [5, Thm. 1] that for $\alpha \geq 1$ the inequality (2) holds if and only if the second-order (local) condition

$$(3) \quad \sum_{i,j} (\text{Re } w_i) \text{cov}(\xi_i, \xi_j) (\text{Re } w_j)^T - \alpha (\text{Re } z_i w_i) \text{cov}(\xi_i, \xi_j) (\text{Re } z_j w_j)^T \geq \text{Re} \sum_j \frac{(1-z_j^2) w_j w_j^T}{p_j}$$

holds for all $w_j \in \mathbb{C}^{k_j}$, and derived sharp multifunction Hausdorff–Young, hypercontractive and Brascamp–Lieb inequalities. The outer exponent $\alpha = q/p$ compares an L^q -type quantity on the left with an L^p -type quantity on the right; $\alpha \geq 1$ is “above or on the diagonal” ($q \geq p$), and $\alpha < 1$ is *below the diagonal* ($q < p$).

The question. Below the diagonal the local condition (3) is no longer sufficient. Already for one function ($n = 1$, real $p, q < p$) the exact contraction region of T_z from L^p to L^q is strictly smaller than the one predicted by the second-order expansion: it is governed by $\Omega_{s,s}$ with $s = \min\{p, \max\{q, 2\}\}$, not by the local inequality. This paper studies what replaces (3) below the diagonal for several correlated functions. We treat three cases in which a complete or nearly complete answer is available, and we are explicit, throughout, about what is proved, what is partial, and what is only numerical.

Contributions. The results are organised into three parts and summarised in Table 1; Figure 1 shows which part of the parameter space each is responsible for, and Figure 2 shows how the proofs depend on one another.

Part I (Section 6): imaginary parameters, every outer exponent. Theorem 6.3 shows that for $z_j = is_j$ and every $\alpha > 0$, (2) holds iff the matrix condition (10) holds and $\max_j |s_j| \leq 1$. For $\alpha \geq 1$ the extra condition is automatic, and the proof gives a new, short derivation of [5, Thm. 2]; for $\alpha < 1$ it is new. Corollary 6.4 is the corresponding Gaussian-weighted Hausdorff–Young inequality, and Corollary 6.5 the sharp equal-parameter region.

Part II (Section 7): two ρ -correlated functions. Theorem 7.1 determines the imaginary-axis region exactly for all $p > 1$, $r > 0$ and $\rho \in (-1, 1)$; Proposition 7.3 gives the real axis; Proposition 7.4 gives inner and outer bounds for general z . We give a second, independent proof of the two-function Gaussian Hölder inequalities through Nelson’s and Borell’s hypercontractivity (Lemma 4.2).

Part III (Sections 9–11): general complex z , joint bumps. Theorem 10.4 is the main obstruction specific to several functions: for $r \geq 2$ and $Q = r(1 + \rho)$, (2) fails for every p whenever $\delta_Q(z) < 0$, or $\delta_Q(z) > 0$ and $\kappa_Q(z) > \frac{1-\rho}{4Q}$. Theorem 10.7 sandwiches the admissible region between $\Omega_{Q,Q}$ and $\mathcal{J}_{\rho,r} \cap \Omega_r$. Corollary 10.8 exhibits two points at which all local and all one-function conditions hold yet the inequality fails; both are certified in ball arithmetic (Section 12). Section 11 extends the method to n equicorrelated functions (Theorem 11.4).

To make the paper self-contained we include, with full proofs, the facts about the Mehler integral (Section 3), the Gaussian Hölder inequalities (Section 4), the bump inequality and the key algebraic identity (Section 5), and the one-function problem below the diagonal (Section 9). Only published results are cited; the single external input that we quote rather than reprove is the classical diagonal theorem and, for the real axis of Part II, [5, Thm. 1] at $\alpha = 1$ (discussed where it is used).

2. NOTATION AND STANDING ASSUMPTIONS

Throughout, γ_k is the standard Gaussian measure on $\mathbb{R}^k$, and $\|W\|_s = \|W\|_{L^s} = (\mathbb{E}|W|^s)^{1/s}$ for $s > 0$ (so $\|\cdot\|_s$ is a genuine norm only for $s \geq 1$, but we use the notation for all $s > 0$). For a multi-index β we write H_β for the probabilists’ Hermite polynomials and $h_\beta = H_\beta/\sqrt{\beta!}$ for the orthonormal ones, so that $\mathbb{E}h_\beta h_{\beta'} = \delta_{\beta\beta'}$. The Mehler transform T_z is defined by (1); it is linear, $T_z 1 = 1$, and $T_z T_w = T_{zw}$ on $\mathcal{P}_k$. We write $\mathbb{D} = \{|z| < 1\}$ for the open unit disc and $\varsigma = 1 - z^2$.

For $s > 0$ put $|w|_s^2 = (s-1)(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2$ for $w \in \mathbb{C}$ (an indefinite form when $s < 1$), and for $a, b > 0$ let

$$\Omega_{a,b} = \{z \in \mathbb{C} : |zw|_b^2 \leq |w|_a^2 \text{ for all } w \in \mathbb{C}\}, \quad \Omega_q := \Omega_{q,q}.$$

For $q > 1$ one has (Lemma 5.1) $\Omega_q \cap \mathbb{D} = \{z \in \mathbb{D} : |q-2||1-z^2| \leq q(1-|z|^2)\}$, and Ω_q is the set where T_z contracts $L^q(\gamma_k)$ for all k . For $z \in \mathbb{D}$ set

$$(4) \quad A = \operatorname{Re} \frac{1}{\varsigma}, \quad C = \operatorname{Re} \frac{z}{\varsigma}, \quad \delta_q = 1 - q + qA, \quad \kappa_q = -\frac{A}{2} + \frac{1}{2q} + \frac{qC^2}{2\delta_q} \quad (\delta_q \neq 0),$$

and for $q > 1$

$$(5) \quad F_1^{(q)}(z) = q(1-|z|^4) - (q-2)|1-z^2|^2, \quad F_2^{(q)}(z) = (q-2)^2|1-z^2|^2 - q^2(1-|z|^2)^2.$$

We shall use the ρ -correlated setting repeatedly: (X, Y) denotes a centred Gaussian vector in $\mathbb{R}^k \times \mathbb{R}^k$ with $X, Y \sim \gamma_k$ and $\operatorname{cov}(X, Y) = \rho \operatorname{Id}_k$, $\rho \in (-1, 1)$. Negative ρ reduce to $|\rho|$ by replacing $g(y)$ with $g(-y)$, which commutes with T_z ; so we assume $\rho \in [0, 1)$ wherever convenient. More generally, for n equicorrelated functions $\xi_1, \dots, \xi_n \sim \gamma_k$ with $\operatorname{cov}(\xi_i, \xi_j) = \rho \operatorname{Id}_k$ ($i \neq j$) we write $\lambda_m = 1 + (m-1)\rho$, $Q_m = r\lambda_m$; the matrix with unit diagonal and off-diagonal ρ has eigenvalue λ_m on $\mathbf{1}$ and $1 - \rho$ on $\mathbf{1}^\perp$.

Result	Status	How established
Thm. 6.3 (imaginary, every α)	proved	by hand
Cor. 6.4 (Hausdorff–Young)	proved	by hand
Cor. 6.5 (equal parameters)	proved	by hand
Thm. 7.1 (two functions, $i\mathbb{R}$)	proved	by hand
Prop. 7.3 (two functions, $\mathbb{R}$)	proved	by hand; sufficiency via [5, Thm. 1] ($\alpha = 1$)
Prop. 7.4 (two functions, general z)	partial	inner/outer bounds; region between open off the axes
Thm. 10.4 (joint-bump exclusion)	proved	by hand
Thm. 10.7 (sandwich, $r \geq 2$)	partial	bounds proved; region between Ω_Q and $\mathcal{J}_{\rho,r}$ open
Cor. 10.8 (local conds. insufficient)	proved	computer-assisted (ball arithmetic)
Prop. 10.5 ($1 < r < 2$)	partial	by hand, only $z \in \Omega_r$
Cor. 11.2, Prop. 11.3 (n functions)	proved	by hand
Thm. 11.4 (many functions, $n \rightarrow \infty$)	partial	profiles differ off $i\mathbb{R}$
“No finite bump wins inside $\mathcal{J}_{\rho,r}$ ” (Rmk. 10.9)	numerical	scan only; stated as evidence

TABLE 1. Status of the main results. “By hand” means a complete mathematical proof; “computer-assisted” means a finite rigorous computation in interval (ball) arithmetic; “numerical” means floating-point evidence, stated as such.

3. THE MEHLER INTEGRAL

For $|z| < 1$ the Mehler transform has an integral representation that extends T_z beyond polynomials to all bounded measurable functions, and that we shall use to feed the operator non-polynomial test functions (Gaussian bumps and plane waves). Everything in this section is classical in substance; we include proofs because the constants and the domain of validity matter later.

For $|z| < 1$ put $\varsigma = 1 - z^2$, so $\operatorname{Re} \varsigma > 0$, and take principal branches of square roots. Mehler’s formula states that, for $f \in \mathcal{P}_1$,

$$(6) \quad T_z f(x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi\varsigma}} \exp\left(-\frac{(y - zx)^2}{2\varsigma}\right) dy = \int_{\mathbb{R}} f(y) K_z(x, y) d\gamma_1(y),$$

with the Mehler kernel $K_z(x, y) = \varsigma^{-1/2} \exp\left(-\frac{z^2(x^2 + y^2) - 2zxy}{2\varsigma}\right)$. For real $z \in (-1, 1)$ this is the Ornstein–Uhlenbeck formula $T_z f(x) = \mathbb{E}f(zx + \sqrt{1 - z^2}Y)$, $Y \sim \gamma_1$. For fixed x both sides of (6) are analytic in $z \in \mathbb{D}$ (the left side is a polynomial in z ; the integrand on the right is analytic in z and dominated, locally uniformly in z , by an integrable function of y), so (6) extends from $(-1, 1)$ to all of $\mathbb{D}$. In k variables the kernel is the product of one-dimensional kernels. For measurable g on $\mathbb{R}^k$ we define the *Mehler integral*

$$\mathbf{T}_z g(x) := \int_{\mathbb{R}^k} g(y) K_z(x, y) d\gamma_k(y)$$

whenever the integral converges absolutely.

Lemma 3.1 (Basic properties of the Mehler integral). *Let $|z| < 1$. With $A = \operatorname{Re} \frac{1}{\varsigma}$, $a = \operatorname{Re} \frac{z^2}{\varsigma} = A - 1$ and $C = \operatorname{Re} \frac{z}{\varsigma}$:*

Which section proves which part of the results

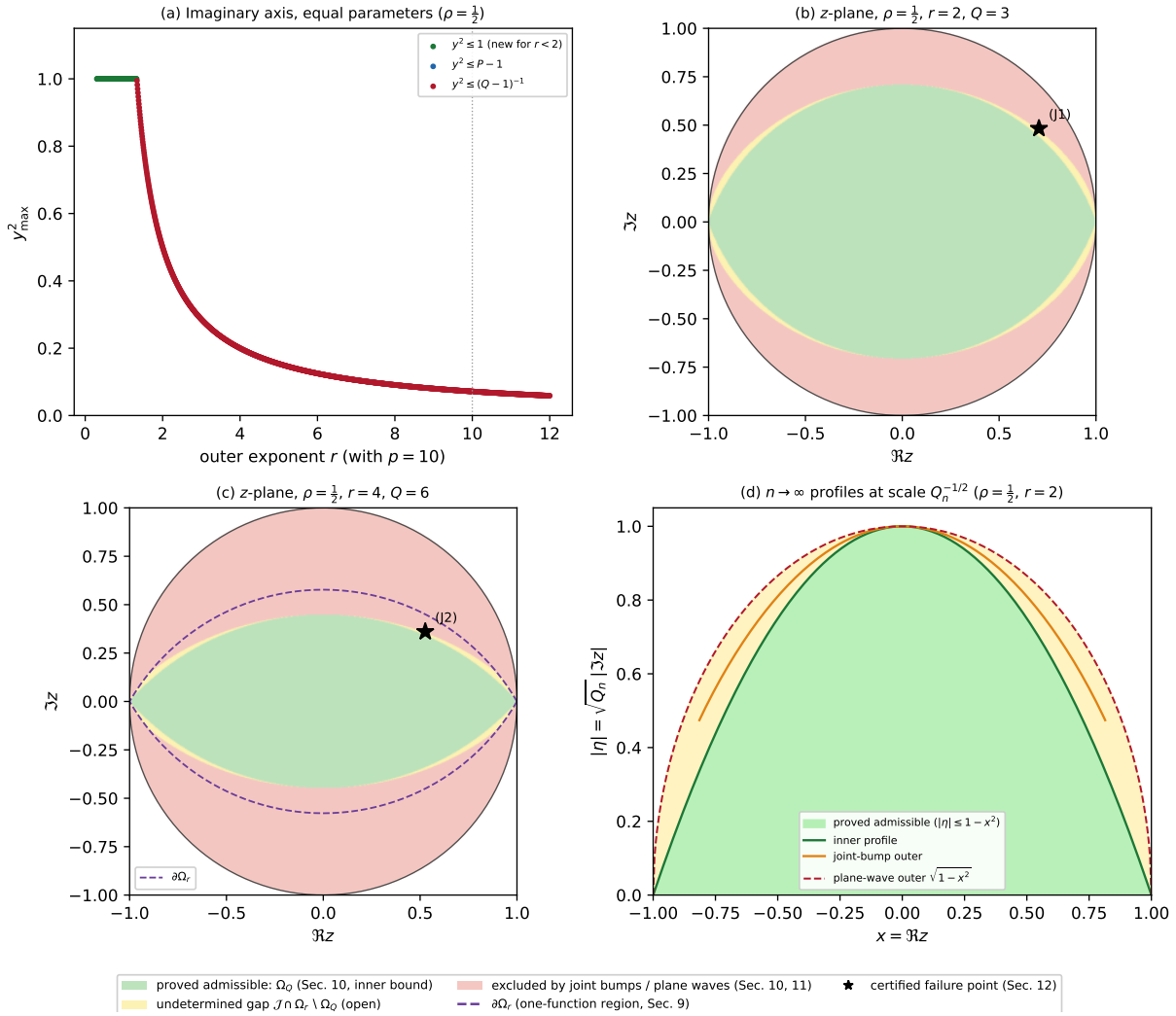


FIGURE 1. Which section proves which part of the results. (a) The equal-parameter imaginary-axis threshold $y_{\max}^2$ as a function of r (with $p = 10$, $\rho = \frac{1}{2}$), coloured by the binding constraint of Corollary 6.5; the constraint $y^2 \leq 1$ (green) is the piece that is new below the diagonal. (b), (c) The complex z -plane for $(\rho, r) = (\frac{1}{2}, 2)$ and $(\frac{1}{2}, 4)$: the green region $\Omega_Q \cap \mathbb{D}$ is proved admissible (Theorem 10.7, inner bound), the thin yellow band $\mathcal{J}_{\rho,r} \cap \Omega_r \setminus \Omega_Q$ is undetermined, and the red region is excluded by joint bumps (Theorem 10.4); the stars are the certified failure points (J1), (J2) of Corollary 10.8, and the dashed curve in (c) is $\partial\Omega_r$. (d) The $n \rightarrow \infty$ profiles of Theorem 11.4 at scale $Q_n^{-1/2}$.

- (a) $|K_z(x, y)| = |\varsigma|^{-1/2} \exp(-\frac{a}{2}(x^2 + y^2) + Cxy)$ (in one variable), and $A > |C|$;
- (b) $\int_{\mathbb{R}} |K_z(x, y)| d\gamma_1(y) = (|\varsigma|A)^{-1/2} e^{Ex^2/2}$ with $E := \frac{C^2}{A} - A + 1 < 1$;
- (c) for every bounded measurable g on $\mathbb{R}^k$, $\mathbf{T}_z g$ is defined, $\mathbf{T}_z 1 = 1$, $\int \mathbf{T}_z g d\gamma_k = \int g d\gamma_k$, and $\mathbf{T}_z = T_z$ on $\mathcal{P}_k$;
- (d) (semigroup) for $0 \leq \rho \leq 1$ and bounded g , $\mathbf{T}_{\rho z} g = T_\rho(\mathbf{T}_z g)$ a.e., where T_ρ is the Ornstein–Uhlenbeck operator.

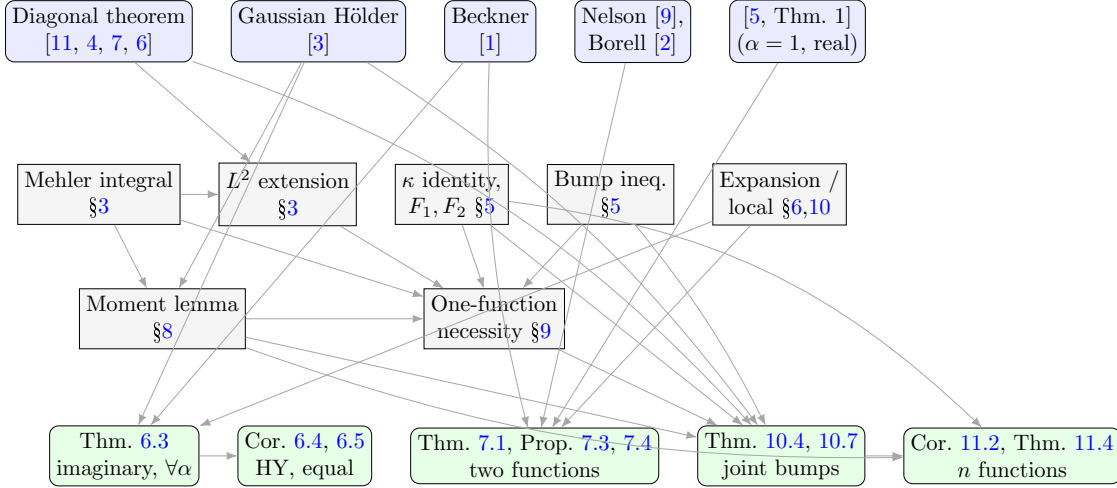


FIGURE 2. Schematic of the proof structure. Blue boxes are published results used as inputs; grey boxes are the self-contained lemmas proved here; green boxes are the main theorems. Every dependency on the unpublished companion material has been replaced by an internal lemma (grey) or a cited published result (blue).

Proof. (a) From (6), $|K_z(x, y)| = |\varsigma|^{-1/2} \exp(-\operatorname{Re} \frac{z^2(x^2+y^2)-2zxy}{2\varsigma})$, which is the stated expression since $\operatorname{Re} \frac{z^2}{\varsigma} = a$ and $\operatorname{Re} \frac{z}{\varsigma} = C$. The identities $\frac{1}{\varsigma} \pm \frac{z}{\varsigma} = \frac{1}{1 \mp z}$ give $A \pm C = \operatorname{Re} \frac{1}{1 \mp z} > \frac{1}{2}$ for $z \in \mathbb{D}$ (the Möbius map $w \mapsto \frac{1}{1-w}$ sends $\mathbb{D}$ into $\{\operatorname{Re} > \frac{1}{2}\}$); hence $A > |C|$.

(b) The one-dimensional Gaussian integral with weight $d\gamma_1(y) = e^{-y^2/2} dy / \sqrt{2\pi}$ and quadratic exponent $-\frac{a}{2}y^2 + Cxy - \frac{1}{2}y^2$ has y -coefficient $-\frac{a+1}{2} = -\frac{A}{2} < 0$, so it converges, and equals $|\varsigma|^{-1/2}(A)^{-1/2} \exp(\frac{C^2x^2}{2A} - \frac{a}{2}x^2) = (|\varsigma|A)^{-1/2} e^{Ex^2/2}$ with $E = \frac{C^2}{A} - a = \frac{C^2}{A} - A + 1$. Since $A > |C|$, $E < \frac{A^2}{A} - A + 1 = 1$.

(c) By (b), $\int |g(y)K_z(x, y)| d\gamma_1(y) \leq \|g\|_\infty (|\varsigma|A)^{-1/2} e^{Ex^2/2} < \infty$ (in k variables, take the product over coordinates), so $\mathbf{T}_z g$ is defined for bounded g . Taking $g = 1$ and using (6) for the polynomial 1 gives $\mathbf{T}_z 1 = T_z 1 = 1$. The kernel is symmetric, $K_z(x, y) = K_z(y, x)$, and $\int K_z(x, y) d\gamma_1(x) = 1$ (that is $\mathbf{T}_z 1 = 1$ with the roles of x, y exchanged); Fubini's theorem (justified by the finiteness just shown, together with $\int e^{Ex^2/2} d\gamma_k(x) < \infty$ as $E < 1$) gives $\int \mathbf{T}_z g d\gamma_k = \int g d\gamma_k$. Finally $\mathbf{T}_z = T_z$ on $\mathcal{P}_k$ is (6) in each variable.

(d) For $\rho = 1$ there is nothing to prove. For $\rho < 1$, $T_\rho h(x) = \int K_\rho(x, u) h(u) d\gamma_k(u)$ with $K_\rho > 0$. The Chapman–Kolmogorov identity $\int K_\rho(x, u) K_z(u, y) d\gamma_k(u) = K_{\rho z}(x, y)$ is a Gaussian integral, valid for real $z \in (-1, 1)$ and then by analyticity for all $z \in \mathbb{D}$. Fubini applies because, by (b), $\int K_\rho(x, u) \int |K_z(u, y)| |g(y)| d\gamma_k(y) d\gamma_k(u) \leq \|g\|_\infty (|\varsigma|A)^{-k/2} \mathbb{E} \exp(\frac{E}{2} |\rho x + \sqrt{1 - \rho^2} Y|^2) < \infty$, since $E(1 - \rho^2) < 1$. $\square$

The next closed form is the workhorse of Parts II and III: it evaluates $\mathbf{T}_z$ on Gaussians and gives the exact L^q -norm of the image.

Lemma 3.2 (Gaussian closed form). *Let $|z| < 1$, $B > 0$, $t \in \mathbb{R}$, and $g_{B,t}(x) = e^{-Bx^2/2+tx}$ on $\mathbb{R}$. Put $D = 1 + B\varsigma$ (so $\operatorname{Re} D > 0$) and $\beta' = -Bz^2/D$. Then $\mathbf{T}_z g_{B,t}$ converges absolutely and*

$$\mathbf{T}_z g_{B,t}(x) = D^{-1/2} \exp\left(\frac{\beta'}{2} x^2 + \frac{tz}{D} x + \frac{\varsigma t^2}{2D}\right).$$

For $r \geq 1$, $\log \|g_{B,t}\|_r = -\frac{1}{2r} \log(1 + rB) + \frac{rt^2}{2(1+rB)}$. If $1 - q \operatorname{Re} \beta' > 0$, then

$$\log \|\mathbf{T}_z g_{B,t}\|_q = -\frac{1}{2} \log |D| - \frac{1}{2q} \log(1 - q \operatorname{Re} \beta') + \frac{t^2}{2} \operatorname{Re} \frac{\zeta}{D} + \frac{qt^2(\operatorname{Re} \frac{\zeta}{D})^2}{2(1 - q \operatorname{Re} \beta')}.$$

Proof. Absolute convergence holds because $\operatorname{Re} \frac{1}{\zeta} + B = A + B > 0$. Completing the square in $-\frac{(y-zx)^2}{2\zeta} - \frac{B}{2}y^2 + ty$ gives the stated formula for real z by the real Gaussian integral, and it extends analytically to $z \in \mathbb{D}$. For real $\alpha < 1/r$ and real β , the Gaussian integral $\mathbb{E}_{\gamma_1} e^{r(\alpha x^2/2 + \beta x)} = (1 - r\alpha)^{-1/2} \exp(\frac{r^2\beta^2}{2(1-r\alpha)})$. Applying it to $|g_{B,t}| = g_{B,t}$ (with $\alpha = -B$, $\beta = t$) gives $\log \|g_{B,t}\|_r$. Applying it to $|\mathbf{T}_z g_{B,t}(x)| = |D|^{-1/2} \exp(\frac{\operatorname{Re} \beta'}{2}x^2 + (\operatorname{Re} \frac{\zeta}{D})tx + \frac{t^2}{2} \operatorname{Re} \frac{\zeta}{D})$ (with $\alpha = \operatorname{Re} \beta'$ and $\beta = t \operatorname{Re} \frac{\zeta}{D}$, which requires $1 - q \operatorname{Re} \beta' > 0$) and dividing $\log \mathbb{E} |\mathbf{T}_z g_{B,t}|^q$ by q gives $\log \|\mathbf{T}_z g_{B,t}\|_q$. $\square$

We also record that, on bounded functions, $\mathbf{T}_z$ coincides with the (unique) L^2 -contraction extension of the diagonal operator, and that T_z contracts $L^q(\gamma_k)$ on bounded functions when $z \in \Omega_q$, $q \geq 2$.

Lemma 3.3 (L^2 extension and diagonal contraction). *Let $|z| < 1$.*

- (a) T_z extends from $\mathcal{P}_k$ to a contraction $\tilde{T}_z$ of $L^2(\gamma_k)$, given on $h = \sum_{\beta} c_{\beta} H_{\beta}$ by $\tilde{T}_z h = \sum_{\beta} z^{|\beta|} c_{\beta} H_{\beta}$; and $\tilde{T}_z g = \mathbf{T}_z g$ a.e. for every bounded measurable g .
- (b) If $q \geq 2$ and $z \in \Omega_q$, then $\|\mathbf{T}_z g\|_{L^q(\gamma_k)} \leq \|g\|_{L^q(\gamma_k)}$ for every bounded measurable g , and $\|\mathbf{T}_z g\|_{L^q} \leq \|g\|_{L^q}$ for g with $\|g\|_q < \infty$.

Proof. (a) On $\mathcal{P}_k$, $\|\tilde{T}_z h\|_2^2 = \sum_{\beta} |z|^{2|\beta|} |c_{\beta}|^2 \beta! \leq \sum_{\beta} |c_{\beta}|^2 \beta! = \|h\|_2^2$ since $|z| < 1$, so $\tilde{T}_z$ is an L^2 -contraction on the dense subspace $\mathcal{P}_k$ and extends uniquely. To identify it with $\mathbf{T}_z$ on bounded g , we use a Schur–Hölder bound. For $\sigma \geq 1$ with $\sigma a + 1 > 0$ and $\frac{\sigma C^2}{\sigma a + 1} - a < 1$ (both hold at $\sigma = 1$, the second being $\frac{C^2}{A} - A + 1 = E < 1$ by Lemma 3.1), Hölder's inequality in y gives $|\mathbf{T}_z g(x)| \leq \|K_z(x, \cdot)\|_{L^{\sigma}(\gamma_k)} \|g\|_{L^{\sigma'}(\gamma_k)}$, and a Gaussian integral shows $\|K_z(x, \cdot)\|_{L^{\sigma}(\gamma_1)} = |\zeta|^{-1/2} (\sigma a + 1)^{-1/(2\sigma)} \exp(\frac{x^2}{2} [\frac{\sigma C^2}{\sigma a + 1} - a])$, whose $L^1(\gamma_1)$ norm in x is finite under the stated condition. Hence $\mathbf{T}_z$ is bounded from $L^{\sigma'}(\gamma_k)$ to $L^1(\gamma_k)$ for some finite $\sigma' \geq 2$. For bounded g take polynomials $P_n \rightarrow g$ in $L^{\sigma'}(\gamma_k)$; then $\tilde{T}_z g = \lim_n T_z P_n = \lim_n \mathbf{T}_z P_n = \mathbf{T}_z g$ in L^1 , using $T_z = \mathbf{T}_z$ on $\mathcal{P}_k$ and L^2 -continuity for the first limit, $L^{\sigma'} \rightarrow L^1$ continuity of $\mathbf{T}_z$ for the second.

(b) By the classical diagonal theorem [11, 4, 7, 6], for $z \in \Omega_q$ one has $\|T_z h\|_{L^q(\gamma_k)} \leq \|h\|_{L^q(\gamma_k)}$ for all $h \in \mathcal{P}_k$ (for $q = 2$ this is part (a)). Let g be bounded and let $P_n \rightarrow g$ in $L^{\max\{2, q\}}(\gamma_k)$ be polynomials. Then $T_z P_n \rightarrow \tilde{T}_z g = \mathbf{T}_z g$ in L^2 , hence a.e. along a subsequence; by Fatou's lemma $\|\mathbf{T}_z g\|_q \leq \liminf_n \|T_z P_n\|_q \leq \liminf_n \|P_n\|_q = \|g\|_q$. The same argument with g replaced by the truncations $g \mathbf{1}_{\{|g| \leq N\}}$ of a general $g \in L^q$ (and monotone convergence) gives the last claim. $\square$

Finally, plane waves are mapped to Gaussians; this gives the cleanest outer bounds.

Lemma 3.4 (Plane waves). *Let $|z| < 1$ and $\tau \in \mathbb{C}$. Then $\mathbf{T}_z e^{\tau y}(x) = e^{\tau z x + \varsigma \tau^2/2}$ a.e. In particular, for $\omega \in \mathbb{R}$ and $f_{\omega}(x) = e^{i\omega x}$, $|f_{\omega}| \equiv 1$ and*

$$\mathbf{T}_z f_{\omega}(x) = \exp\left(iz\omega x - \frac{1}{2}(1-z^2)\omega^2\right), \quad |\mathbf{T}_z f_{\omega}(x)| = \exp\left(-\omega(\operatorname{Im} z)x - \frac{\omega^2}{2}(1 - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2)\right).$$

Proof. The identity $\mathbf{T}_z e^{\tau y}(x) = e^{\tau z x + \varsigma \tau^2/2}$ is the case $B \rightarrow 0$ of Lemma 3.2 (directly: a Gaussian integral), valid for all $\tau \in \mathbb{C}$ by analyticity and absolute convergence (Lemma 3.1(b)). With $\tau = i\omega$, $\varsigma \tau^2/2 = -(1-z^2)\omega^2/2$ and $|\mathbf{T}_z f_{\omega}(x)| = \exp(\operatorname{Re}(iz\omega x) - \frac{\omega^2}{2} \operatorname{Re}(1-z^2))$; now $\operatorname{Re}(iz\omega x) = -\omega(\operatorname{Im} z)x$ and $\operatorname{Re}(1-z^2) = 1 - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$. $\square$

4. GAUSSIAN HÖLDER INEQUALITIES

All three parts use the Gaussian forward and reverse Hölder inequalities (equivalently, the forward and inverse Brascamp–Lieb inequalities) in the covariance form of Chen, Dafnis and Paouris [3]. We state them in the form we need and give, for two functions, a second self-contained proof through Nelson’s and Borell’s hypercontractivity.

Lemma 4.1 (Gaussian Hölder, covariance form [3]). *Let $\xi = (\xi_1, \dots, \xi_n)$ be a centred Gaussian vector with standard blocks $\xi_j \in \mathbb{R}^{k_j}$ and covariance Σ , and let $a_j > 0$. Then*

$$(7) \quad \begin{aligned} \Sigma \leq \text{diag}(a_j) &\Rightarrow \mathbb{E} \prod_j G_j(\xi_j) \leq \prod_j \|G_j\|_{L^{a_j}(\gamma_{k_j})}, \\ \Sigma \geq \text{diag}(a_j) &\Rightarrow \mathbb{E} \prod_j G_j(\xi_j) \geq \prod_j \|G_j\|_{L^{a_j}(\gamma_{k_j})} \end{aligned}$$

for all measurable $G_j \geq 0$, where $\text{diag}(a_j)$ is the block-diagonal matrix with blocks $a_j \text{Id}_{k_j}$.

Proof. This is [5, Thm. B], restating [3]: for $c_j \in \mathbb{R} \setminus \{0\}$, $\prod_j (\mathbb{E} F_j(\xi_j))^{c_j} \leq \mathbb{E} \prod_j F_j(\xi_j)^{c_j}$ for all measurable $F_j > 0$ iff $\text{diag}(1/c_j) \leq \Sigma$, and the inequality reverses iff $\text{diag}(1/c_j) \geq \Sigma$. Put $a_j = 1/c_j$ and $F_j = G_j^{1/c_j} = G_j^{a_j}$; apply the statement to $\min\{G_j, N\} + \delta$, let $\delta \downarrow 0$ (dominated convergence) and $N \uparrow \infty$ (monotone convergence). This yields (7). $\square$

For two ρ -correlated functions the case $a_1 = a_2 = 1 \pm \rho$ of (7) can be obtained directly, without [3], from one-function hypercontractivity.

Lemma 4.2 (Two-function Hölder via Nelson and Borell). *Let $\rho \in [0, 1)$ and let $F, G \geq 0$ be measurable on $\mathbb{R}^k$, with (X, Y) ρ -correlated. Then*

$$(a) \quad \mathbb{E} F(X)G(Y) \leq \|F\|_{L^{1+\rho}} \|G\|_{L^{1+\rho}}, \quad (b) \quad \mathbb{E} F(X)G(Y) \geq \|F\|_{L^{1-\rho}} \|G\|_{L^{1-\rho}}.$$

Proof. For $\rho = 0$ both are equalities. Let $\rho > 0$ and let P_ρ be the Ornstein–Uhlenbeck operator with parameter ρ , so $\mathbb{E} F(X)G(Y) = \mathbb{E} F(X)(P_\rho G)(X)$. (a) By Hölder with exponents $1 + \rho$ and $\frac{1+\rho}{\rho}$, $\mathbb{E} F(P_\rho G) \leq \|F\|_{1+\rho} \|P_\rho G\|_{(1+\rho)/\rho}$, and Nelson’s hypercontractivity [9], $\|P_\rho G\|_b \leq \|G\|_a$ whenever $(b-1)\rho^2 \leq a-1$, applies with $a = 1 + \rho$, $b = \frac{1+\rho}{\rho}$ (then $(b-1)\rho^2 = \rho = a-1$). (b) We may assume F, G bounded and $\geq \delta > 0$ (apply the argument to $\min\{F, N\} + \delta$ and $\min\{G, N\} + \delta$, then let $\delta \downarrow 0$ and $N \uparrow \infty$). By the reverse Hölder inequality $\mathbb{E} F H \geq \|F\|_a \|H\|_{a'}$ for $0 < a < 1$, $a' = \frac{a}{a-1} < 0$, with $a = 1 - \rho$, $a' = -\frac{1-\rho}{\rho}$, and Borell’s reverse hypercontractivity [2] (see also [8], [10, Ch. 10–11]), $\|P_\rho G\|_b \geq \|G\|_a$ for $b < a < 1$ with $\rho^2 \leq \frac{1-a}{1-b}$, applied with $b = a'$: here $\frac{1-a}{1-a'} = \frac{\rho}{1/\rho} = \rho^2$. Hence $\mathbb{E} F(X)G(Y) = \mathbb{E} F(P_\rho G) \geq \|F\|_{1-\rho} \|P_\rho G\|_{a'} \geq \|F\|_{1-\rho} \|G\|_{1-\rho}$. $\square$

5. GEOMETRY OF Ω , THE IDENTITY, AND THE BUMP INEQUALITY

We collect the elementary facts about the regions $\Omega_{a,b}$, the algebraic identity relating κ_q to F_2 , the convexity (“bump”) inequality, and Beckner’s inequality. The symbolic identities below are also verified in `check_identities.py` (Appendix A).

Lemma 5.1 (Geometry of Ω). *Let $z = \xi + i\eta \in \mathbb{C}$ and write $y = \eta = \text{Im } z$.*

- (a) *For $a, b > 0$, $iy \in \Omega_{a,b}$ iff $(b-1)y^2 \leq 1$ and $y^2 \leq a-1$.*
- (b) *For $a \geq 1$, a real $x \in \Omega_{a,a}$ iff $x^2 \leq 1$.*
- (c) *For $q > 1$, $\Omega_q \cap \mathbb{D} = \{z \in \mathbb{D} : F_2^{(q)}(z) \leq 0\} = \{z \in \mathbb{D} : |q-2||1-z^2| \leq q(1-|z|^2)\}$, and for $q \geq 2$ this equals $\{z \in \mathbb{D} : (q-2)|y| \leq \sqrt{q-1}(1-|z|^2)\}$.*
- (d) *For $2 \leq q \leq q'$, $\Omega_{q'} \subseteq \Omega_q$.*

Proof. Write $w = \alpha + i\beta$. (a) For $z = iy$, $zw = iy\alpha - y\beta$, so $\operatorname{Re}(zw) = -y\beta$, $\operatorname{Im}(zw) = y\alpha$, and $|zw|_b^2 = (b-1)y^2\beta^2 + y^2\alpha^2$, while $|w|_a^2 = (a-1)\alpha^2 + \beta^2$. The inequality $|zw|_b^2 \leq |w|_a^2$ for all α, β holds iff the coefficient of α^2 obeys $y^2 \leq a-1$ and that of β^2 obeys $(b-1)y^2 \leq 1$. (b) For real x and $a = b$, $|xw|_a^2 = x^2|w|_a^2$ (the form $|\cdot|_a^2$ is homogeneous of degree two and x is real), so $x \in \Omega_{a,a}$ iff $x^2|w|_a^2 \leq |w|_a^2$ for all w ; since $a \geq 1$ makes $|\cdot|_a^2 \geq 0$ and positive somewhere, this is $x^2 \leq 1$. (c) Multiplication by z has real matrix $R = \begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix}$, and $z \in \Omega_{q,q}$ iff $R^T G R \leq G$ with $G = \operatorname{diag}(q-1, 1)$, i.e. iff $N := G^{1/2} R G^{-1/2}$ has operator norm ≤ 1 . For the 2×2 symmetric matrix $I - N^T N$ this means $\det(I - N^T N) \geq 0$ and $\operatorname{tr}(I - N^T N) \geq 0$. A direct computation gives $\det(I - N^T N) = 1 - \|N\|_F^2 + |z|^4 = -F_2^{(q)}/(4(q-1))$ with $\|N\|_F^2 = 2\xi^2 + (q-1)\eta^2 + \eta^2/(q-1)$; so the determinant condition is $F_2^{(q)} \leq 0$. On $\{F_2^{(q)} \leq 0\} \cap \mathbb{D}$ the trace condition $\|N\|_F^2 \leq 2$ holds as well, so $\Omega_q \cap \mathbb{D} = \{F_2^{(q)} \leq 0\}$. Since $1 - |z|^2 > 0$ on $\mathbb{D}$, $F_2^{(q)} \leq 0$ is $|q-2||1-z^2| \leq q(1-|z|^2)$; substituting $|1-z^2|^2 = (1-|z|^2)^2 + 4y^2$ turns $(q-2)^2|1-z^2|^2 \leq q^2(1-|z|^2)^2$ into $(q-2)^2y^2 \leq (q-1)(1-|z|^2)^2$, which for $q \geq 2$ is the last form. (d) For $q \geq 2$, $\frac{q-2}{q}$ increases in q , so $\frac{q-2}{q}|1-z^2| \leq \frac{q'-2}{q'}|1-z^2| \leq 1 - |z|^2$ shows $\Omega_{q'} \subseteq \Omega_q$. $\square$

Lemma 5.2 (The identity). *For $z \in \mathbb{D}$ and $q > 1$ with $\delta_q \neq 0$,*

$$(8) \quad F_1^{(q)} = 2\delta_q |1-z^2|^2, \quad \kappa_q = \frac{F_2^{(q)}}{4q F_1^{(q)}}.$$

Consequently, for $\delta_q > 0$ one has $\kappa_q \leq 0 \iff z \in \Omega_q$.

Proof. Write $|\varsigma|^2 = |1-z^2|^2$. Since $A|\varsigma|^2 = \operatorname{Re}(1/\varsigma)|\varsigma|^2 = \operatorname{Re}\bar{\varsigma} = \operatorname{Re}\varsigma = 1 - \operatorname{Re}z^2$ and $|\varsigma|^2 = 1 - 2\operatorname{Re}z^2 + |z|^4$, we get $\operatorname{Re}\varsigma = \frac{1}{2}(1 - |z|^4 + |\varsigma|^2)$, whence

$$\delta_q |\varsigma|^2 = (1-q)|\varsigma|^2 + q\operatorname{Re}\varsigma = (1-q)|\varsigma|^2 + \frac{q}{2}(1 - |z|^4) + \frac{q}{2}|\varsigma|^2 = \frac{q}{2}(1 - |z|^4) + (1 - \frac{q}{2})|\varsigma|^2 = \frac{1}{2}F_1^{(q)},$$

which is the first identity. For the second, from (4), $2q\delta_q\kappa_q = (1-qA)\delta_q + q^2C^2 = q^2C^2 - (qA-1)(qA-q+1)$ (as $\delta_q = qA-q+1$). Putting $u = \operatorname{Re}z^2$, $\tau = |z|^2$, $S = |\varsigma|^2 = 1 - 2u + \tau^2$, one has $A = (1-u)/S$ and $C = (1-\tau)\operatorname{Re}z/S$ with $(\operatorname{Re}z)^2 = \frac{\tau+u}{2}$; substituting and clearing denominators gives

$$q^2C^2 - (qA-1)(qA-q+1) = \frac{q^2(1-|z|^2)^2 - (q-2)^2|\varsigma|^2}{4|\varsigma|^2} = \frac{F_2^{(q)}}{4|\varsigma|^2}$$

(a polynomial identity in u, τ , verified symbolically in `check_identities.py`). Hence $\kappa_q = \frac{F_2^{(q)}}{8q\delta_q|\varsigma|^2} = \frac{F_2^{(q)}}{4qF_1^{(q)}}$ by the first identity. Finally, for $\delta_q > 0$ we have $F_1^{(q)} = 2\delta_q|\varsigma|^2 > 0$, so $\kappa_q \leq 0 \iff F_2^{(q)} \leq 0 \iff z \in \Omega_q$. $\square$

Lemma 5.3 (Bump inequality). *For $q \geq 2$ and $w \in \mathbb{C}$, $|1+w|^q \geq 1 + q\operatorname{Re}w + 2^{1-q}|w|^q$.*

Proof. For $x, y \in \mathbb{C}$, the parallelogram law, the superadditivity of $t \mapsto t^{q/2}$ on $[0, \infty)$ and the convexity of the same function give Clarkson's inequality

$$\left|\frac{x+y}{2}\right|^q + \left|\frac{x-y}{2}\right|^q \leq \left(\frac{|x|^2+|y|^2}{2}\right)^{q/2} \leq \frac{|x|^q+|y|^q}{2}.$$

With $x = 1+w$, $y = 1$ this reads $|1+w|^q \geq 2|1+\frac{w}{2}|^q + 2^{1-q}|w|^q - 1$. Moreover $|1+\frac{w}{2}|^q \geq 1 + \frac{q}{2}\operatorname{Re}w$: if $1 + \frac{1}{2}\operatorname{Re}w \geq 0$ this is Bernoulli's inequality applied to $|1+\frac{w}{2}| \geq 1 + \frac{1}{2}\operatorname{Re}w$, and otherwise the right side is negative. Combining gives the claim. $\square$

We shall also use Beckner's inequality in the following form.

Lemma 5.4 (Beckner [1]). *For $0 < |s| \leq 1$, all k and all $h \in \mathcal{P}_k$,*

$$(9) \quad \|T_{is}h\|_{L^{1+s^{-2}}(\gamma_k)} \leq \|h\|_{L^{1+s^2}(\gamma_k)}.$$

Proof. This is the sharp Gaussian Hausdorff–Young inequality of Beckner [1]: $T_{i\sqrt{p-1}}$ maps L^p to $L^{p'}$ contractively for $1 \leq p \leq 2$, which is (9) with $p = 1 + s^2 \in (1, 2]$, $p' = 1 + s^{-2}$. For $s < 0$ use $T_{-is}h(x) = T_{is}h(-x)$ (Lemma 3.1 and $H_\beta(-x) = (-1)^{|\beta|}H_\beta(x)$). $\square$

6. PART I: IMAGINARY PARAMETERS, FOR EVERY OUTER EXPONENT

Throughout this part $z_j = is_j$ with $s_j \in \mathbb{R}$, and we characterise (2) for *every* $\alpha > 0$. We first record the second-order expansion that produces the necessary conditions, then prove Theorem 6.3 and its corollaries. Recall (2) and the matrix condition

$$(10) \quad \frac{1}{\alpha} \operatorname{diag} \left(\frac{1 + s_j^2}{s_j^2 p_j} \right) \geq \Sigma \geq \operatorname{diag} \left(\frac{1 + s_j^2}{p_j} \right).$$

Lemma 6.1 (Second-order expansion). *Let $q_j > 0$, $c_j \in \mathbb{C}$, $w_j \in \mathbb{C}^{k_j}$ and $\varphi_j(x) = w_j \cdot x$. With $Q = \operatorname{diag}(q_j)$ and $(cw)_j = c_j w_j$, as $\varepsilon \rightarrow 0$,*

$$\mathbb{E} \prod_{j=1}^n |1 + \varepsilon c_j \varphi_j(\xi_j)|^{q_j} = 1 + \frac{\varepsilon^2}{2} \left[(Q \operatorname{Re}(cw))^T \Sigma (Q \operatorname{Re}(cw)) - \sum_j q_j \operatorname{Re}((c_j w_j) \cdot (c_j w_j)) \right] + O(\varepsilon^3).$$

Proof. Write $|1 + \varepsilon c_j \varphi_j|^{q_j} = (1 + u_j)^{q_j/2}$ with $u_j = 2\varepsilon \operatorname{Re}(c_j \varphi_j) + \varepsilon^2 |c_j \varphi_j|^2 \geq -1$. On $[-1, \infty)$, $(1 + u)^{q/2} = 1 + \frac{q}{2}u + \frac{q(q-2)}{8}u^2 + R_q(u)$ with $|R_q(u)| \leq C_q(|u|^3 + |u|^{M_q})$, $M_q = \max\{3, q/2\}$. All u_j are polynomials in a Gaussian vector, so all moments are finite and every product containing a factor R_{q_j} or three factors u contributes $O(\varepsilon^3)$; thus

$$\mathbb{E} \prod_j (1 + u_j)^{q_j/2} = 1 + \sum_j \left(\frac{q_j}{2} \mathbb{E} u_j + \frac{q_j(q_j-2)}{8} \mathbb{E} u_j^2 \right) + \sum_{i < j} \frac{q_i q_j}{4} \mathbb{E} u_i u_j + O(\varepsilon^3).$$

Since $\operatorname{Re}(c_j \varphi_j(\xi_j)) = \operatorname{Re}(c_j w_j) \cdot \xi_j$ is a centred Gaussian, $\mathbb{E} u_j = \varepsilon^2 |c_j w_j|^2$, $\mathbb{E} u_j^2 = 4\varepsilon^2 |\operatorname{Re}(c_j w_j)|^2 + O(\varepsilon^3)$, and $\mathbb{E} u_i u_j = 4\varepsilon^2 \operatorname{Re}(c_i w_i)^T \Sigma_{ij} \operatorname{Re}(c_j w_j) + O(\varepsilon^3)$. Collecting terms and using $|v|^2 - 2|\operatorname{Re} v|^2 = -\operatorname{Re}(v \cdot v)$ gives the claim. $\square$

Lemma 6.2 (Necessity). *Let $\alpha > 0$, $p_j > 0$, $z_j \in \mathbb{C}$, and suppose (2) holds. Then:*

- (a) *for all w , $(\operatorname{Re} w)^T \Sigma (\operatorname{Re} w) - \alpha (\operatorname{Re} z w)^T \Sigma (\operatorname{Re} z w) \geq \sum_j \frac{1}{p_j} \operatorname{Re}((1 - z_j^2) w_j \cdot w_j)$, where $(zw)_j = z_j w_j$;*
- (b) *$|z_j| \leq 1$ for every j .*

Proof. (a) Take $f_j = 1 + \varepsilon \varphi_j$ with $\varphi_j(x) = w_j \cdot x / p_j$; then $T_{z_j} f_j = 1 + \varepsilon z_j \varphi_j$ (as $T_{z_j}(w_j \cdot x) = z_j w_j \cdot x$). Apply Lemma 6.1 with $q_j = \alpha p_j$, $c_j = z_j$ to the left side (raised to $1/\alpha$) and with $q_j = p_j$, $c_j = 1$ to the right side; comparing the coefficients of ε^2 and using $p_j \operatorname{Re}(c_j w_j / p_j) = \operatorname{Re}(c_j w_j)$ gives (a). (b) Take $f_i = 1$ for $i \neq j$: (2) says T_{z_j} contracts $L^{p_j} \rightarrow L^{\alpha p_j}$ on polynomials. With $h_m = H_m / \sqrt{m!}$ in one variable and $f_j = 1 + \varepsilon w h_m$, the one-function case of Lemma 6.1 gives $|z_j^m w|_{\alpha p_j}^2 \leq |w|_{p_j}^2$ for all $w \in \mathbb{C}$, $m \geq 1$. Choosing $|w| = 1$ with $z_j^m w \in i\mathbb{R}$ gives $|z_j|^{2m} \leq \max\{p_j - 1, 1\}$ for all m , so $|z_j| \leq 1$. $\square$

Theorem 6.3 (Imaginary parameters, every α). *Let $\alpha > 0$, $p_j > 0$ and $s_j \in \mathbb{R} \setminus \{0\}$. Then (2) holds with $z_j = is_j$ if and only if (10) holds and $\max_j |s_j| \leq 1$. If $\alpha \geq 1$, then (10) alone implies $\max_j |s_j| \leq 1$, so (2) holds iff (10) holds (this is [5, Thm. 2]).*

Proof. Necessity. With $z_j = is_j$, $\operatorname{Re}(z_j w_j) = -s_j \operatorname{Im} w_j$ and $\operatorname{Re}((1-z_j^2)w_j \cdot w_j) = (1+s_j^2)(|\operatorname{Re} w_j|^2 - |\operatorname{Im} w_j|^2)$. Put $D = \operatorname{diag}((1+s_j^2)/p_j)$ and $S = \operatorname{diag}(s_j)$. Lemma 6.2(a) reads $(\operatorname{Re} w)^T(\Sigma - D)(\operatorname{Re} w) + (\operatorname{Im} w)^T(D - \alpha S \Sigma S)(\operatorname{Im} w) \geq 0$ for all w ; as $\operatorname{Re} w, \operatorname{Im} w$ are arbitrary, $\Sigma \geq D$ and $\alpha S \Sigma S \leq D$, i.e. $\Sigma \leq \frac{1}{\alpha} S^{-1} D S^{-1}$, which is (10). Lemma 6.2(b) gives $|s_j| \leq 1$.

Sufficiency. Let $a_j = \frac{1+s_j^2}{\alpha s_j^2 p_j}$ and $b_j = \frac{1+s_j^2}{p_j}$, so (10) is $\operatorname{diag}(b_j) \leq \Sigma \leq \operatorname{diag}(a_j)$. By (7) with $G_j = |T_{is_j} f_j|^{\alpha p_j}$ and exponents a_j (note $\alpha p_j a_j = 1 + s_j^{-2}$), $\mathbb{E} \prod_j |T_{is_j} f_j|^{\alpha p_j} \leq \prod_j \|T_{is_j} f_j\|_{1+s_j^{-2}}^{\alpha p_j}$. By (7) with $G_j = |f_j|^{p_j}$ and exponents b_j ($p_j b_j = 1 + s_j^2$), $\mathbb{E} \prod_j |f_j|^{p_j} \geq \prod_j \|f_j\|_{1+s_j^2}^{p_j}$. By Beckner's inequality (9) (as $|s_j| \leq 1$), $\|T_{is_j} f_j\|_{1+s_j^{-2}} \leq \|f_j\|_{1+s_j^2}$. Combining,

$$\left(\mathbb{E} \prod_j |T_{is_j} f_j|^{\alpha p_j} \right)^{1/\alpha} \leq \prod_j \|f_j\|_{1+s_j^2}^{p_j} \leq \mathbb{E} \prod_j |f_j|^{p_j},$$

which is (2).

The case $\alpha \geq 1$. The diagonal blocks of Σ are identities, so (10) forces $\frac{1+s_j^2}{\alpha s_j^2 p_j} \geq 1 \geq \frac{1+s_j^2}{p_j}$; hence $\alpha s_j^2(1+s_j^2) \leq \alpha s_j^2 p_j \leq 1+s_j^2$, i.e. $s_j^2 \leq 1/\alpha \leq 1$. $\square$

For $\alpha < 1$ the condition $\max_j |s_j| \leq 1$ is not implied by (10): for $n = 1$, $\Sigma = \operatorname{Id}$, $p_1 = p$, $\alpha p = r$, (10) reads $s^2 \leq \min\{p-1, (r-1)^{-1}\}$, which allows $|s| > 1$ when $r < 2 < p$, whereas (2) is then the contractivity of T_{is} from L^p to L^r , which forces $|s| \leq 1$. Thus Theorem 6.3 for $\alpha < 1$ genuinely extends [5, Thm. 2].

Corollary 6.4 (*n*-function Gaussian-weighted Hausdorff–Young). *Let $\alpha > 0$, $p_j > 0$, $t_j > 1$ and $\eta_j = \frac{\sqrt{t_j-1}}{t_j} \xi_j$, with $e_t(x) = e^{-|x|^2/(2t)}$ and the unitary Fourier transform $\widehat{\cdot}$. The inequality*

$$\left\| \prod_{j=1}^n |(\widehat{g_j}/\widehat{e_{t_j}})(\eta_j)|^{p_j} \right\|_{\alpha} \leq \left\| \prod_{j=1}^n |(g_j/e_{t_j})(\xi_j)|^{p_j} \right\|_1 \quad (g_j \in \mathcal{P}_{k_j} e_{t_j})$$

holds if and only if $\frac{1}{\alpha} \operatorname{diag} \left(\frac{t_j}{(t_j-1)p_j} \right) \geq \Sigma \geq \operatorname{diag} \left(\frac{t_j}{p_j} \right)$ and $\max_j t_j \leq 2$.

Proof. By the identity $T_{i\sqrt{t-1}} f(x) = \frac{\widehat{f e_t}}{\widehat{e_t}} \left(-\frac{\sqrt{t-1}}{t} x \right)$ [5, (2.6)] and the symmetry $\xi \mapsto -\xi$, the stated inequality is (2) with $s_j = \sqrt{t_j-1}$ and $g_j = f_j e_{t_j}$. Then $\frac{1+s_j^2}{s_j^2 p_j} = \frac{t_j}{(t_j-1)p_j}$, $\frac{1+s_j^2}{p_j} = \frac{t_j}{p_j}$, and $|s_j| \leq 1 \iff t_j \leq 2$; apply Theorem 6.3. $\square$

Corollary 6.5 (Equal parameters). *Let $p > 0$, $r > 0$, $y \in \mathbb{R} \setminus \{0\}$, and let $\lambda_{\min}, \lambda_{\max}$ be the extreme eigenvalues of Σ . Then $\| \prod_j T_{iy} f_j(\xi_j) \|_{L^r} \leq \| \prod_j f_j(\xi_j) \|_{L^p}$ for all $f_j \in \mathcal{P}_{k_j}$ holds if and only if*

$$y^2 \leq \min \left\{ 1, p\lambda_{\min} - 1, (r\lambda_{\max} - 1)^{-1} \right\},$$

with $(r\lambda_{\max} - 1)^{-1} := \infty$ if $r\lambda_{\max} \leq 1$. The same holds at $y = 0$ iff $p\lambda_{\min} \geq 1$; if $p\lambda_{\min} < 1$ the inequality fails for every $y \in \mathbb{R}$.

Proof. This is (2) with $p_j = p$, $\alpha = r/p$, $s_j = y$. Both sides of (10) are multiples of Id , so (10) reads $\lambda_{\max} \leq \frac{1+y^2}{ry^2}$ and $\lambda_{\min} \geq \frac{1+y^2}{p}$, i.e. $(r\lambda_{\max} - 1)y^2 \leq 1$ and $y^2 \leq p\lambda_{\min} - 1$; combine with $y^2 \leq 1$ (Theorem 6.3). At $y = 0$ the left side is $|\prod_j \mathbb{E} f_j|$ and the inequality is the reverse part of (7) with all exponents $1/p$, which holds iff $\Sigma \geq \frac{1}{p} \operatorname{Id}$, i.e. $p\lambda_{\min} \geq 1$ (necessity by Lemma 6.2(a) with $z = 0$, real w). If $p\lambda_{\min} < 1$, Lemma 6.2(a) with real w in the bottom eigenspace of Σ fails for every $z = iy$. $\square$

For $n = 2$ and $\Sigma = \begin{pmatrix} \text{Id} & \rho \text{Id} \\ \rho \text{Id} & \text{Id} \end{pmatrix}$ the eigenvalues are $1 \pm |\rho|$, so Corollary 6.5 gives the two-function imaginary-axis region $y^2 \leq \min\{1, p(1 - |\rho|) - 1, (r(1 + |\rho|) - 1)^{-1}\}$ used in Part II. For $r \geq p$, $y^2 = p\lambda_{\min} - 1$ and $\frac{1}{p\lambda_{\min}} + \frac{1}{r\lambda_{\max}} = 1$ it contains the multiversion of Beckner's inequality [5, Thm. 3] and the ρ -correlated Beckner inequality [5, Thm. 4].

7. PART II: TWO ρ -CORRELATED FUNCTIONS

Let (X, Y) be ρ -correlated standard Gaussian vectors in $\mathbb{R}^k$, $\rho \in (-1, 1)$, and for $p > 1$, $r > 0$, $z \in \mathbb{C}$ consider

$$(11) \quad \|T_z f(X) T_z g(Y)\|_{L^r} \leq \|f(X) g(Y)\|_{L^p} \quad \text{for all } f, g \in \mathcal{P}_k \text{ and all } k.$$

Replacing $g(y)$ by $g(-y)$ (which commutes with T_z) turns ρ into $-\rho$, so we assume $\rho \in [0, 1]$; then Σ has eigenvalues $1 \pm \rho$. Put $P = p(1 - \rho)$ and $Q = r(1 + \rho)$.

Theorem 7.1 (Imaginary axis). *Let $p > 1$, $r > 0$, $\rho \in (-1, 1)$ and $y \in \mathbb{R}$. Then (11) holds at $z = iy$ if and only if $P \geq 1$ and $y^2 \leq \min\{1, P - 1, (Q - 1)^{-1}\}$ (with $(Q - 1)^{-1} := \infty$ if $Q \leq 1$).*

Proof. This is Corollary 6.5 for $n = 2$, $\lambda_{\min} = 1 - \rho$, $\lambda_{\max} = 1 + \rho$: the three constraints are $y^2 \leq 1$, $y^2 \leq p(1 - \rho) - 1 = P - 1$ (which also encodes $P \geq 1$), and $(r(1 + \rho) - 1)y^2 = (Q - 1)y^2 \leq 1$; the degenerate case $y = 0$ needs $P \geq 1$. $\square$

The constraint $y^2 \leq 1$ binds only below the diagonal: if $r \geq p$ then $Q \geq P$ and it follows from the other two. Theorem 7.1 is, for $r \geq p$, the imaginary-axis case of [5, Thm. 1]; for $r < p$ it is new. Two Gaussian Hölder steps reduce (11) to one function, and for $n = 2$ these steps may be taken either from (7) or, independently of [3], from Lemma 4.2.

Lemma 7.2 (Reduction to one function). *Let $\rho \in [0, 1]$. If $\|T_z h\|_{L^Q(\gamma_k)} \leq \|h\|_{L^P(\gamma_k)}$ for all $h \in \mathcal{P}_k$, then (11) holds. More generally the same implication holds for all bounded measurable f, g if $\mathbf{T}_z$ is an $L^P \rightarrow L^Q$ contraction on bounded functions.*

Proof. By Lemma 4.2(a) with $F = |T_z f|^r$, $G = |T_z g|^r$ (using $a_1 = a_2 = 1 + \rho$) and (b) with $F = |f|^p$, $G = |g|^p$ (using $a_1 = a_2 = 1 - \rho$),

$$\|T_z f(X) T_z g(Y)\|_{L^r} \leq \|T_z f\|_{L^Q} \|T_z g\|_{L^Q} \leq \|f\|_{L^P} \|g\|_{L^P} \leq \|f(X) g(Y)\|_{L^p}. \quad \square$$

Proposition 7.3 (Real axis). *Let $p > 1$, $r \in (0, p)$, $\rho \in (-1, 1)$ with $P \geq 1$, and $x \in \mathbb{R}$. Then (11) holds at $z = x$ iff $|x| \leq 1$.*

Proof. Necessity is $|z| \leq 1$ (Lemma 6.2(b)). For sufficiency let $|x| \leq 1$ and pick $s \in [\max\{r, (1 - \rho)^{-1}\}, p]$, which exists since $P \geq 1$. By [5, Thm. 1] at $\alpha = 1$, $n = 2$, $p_1 = p_2 = s$, the inequality $\mathbb{E}|T_x f(X) T_x g(Y)|^s \leq \mathbb{E}|f(X) g(Y)|^s$ holds iff Lemma 6.2(a) holds with $\alpha = 1$, $p_1 = p_2 = s$, $z_1 = z_2 = x$. In coordinates $u = (w_1 + w_2)/\sqrt{2}$, $v = (w_1 - w_2)/\sqrt{2}$ the form splits (Σ acts as $1 + \rho$ on u , $1 - \rho$ on v , and $w_1 \cdot w_1 + w_2 \cdot w_2 = u \cdot u + v \cdot v$), giving $x \in \Omega_{s(1+\rho)} \cap \Omega_{s(1-\rho)}$; as $s(1 \pm \rho) \geq 1$, by Lemma 5.1(b) this is $x^2 \leq 1$. Hence $\|T_x f(X) T_x g(Y)\|_{L^r} \leq \|T_x f(X) T_x g(Y)\|_{L^s} \leq \|f(X) g(Y)\|_{L^s} \leq \|f(X) g(Y)\|_{L^p}$ by Lyapunov. $\square$

For general z only bounds are available. Let $\mathcal{C}_{a,b}$ be the set of z for which T_z contracts $L^a(\gamma_k) \rightarrow L^b(\gamma_k)$ for all k .

Proposition 7.4 (General z : inner and outer bounds). *Let $p > 1$, $r \in (0, p)$, $\rho \in [0, 1]$, $P \geq 1$. Then $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \{z : (11) \text{ holds}\} \subseteq \mathcal{N} \cap \mathcal{C}_{p,r}$, where*

$$\mathcal{S}_1 = \bigcup_{s \in [\max\{r, 1\}, p]} (\Omega_{s(1+\rho)} \cap \Omega_{s(1-\rho)}), \quad \mathcal{S}_2 = \mathcal{C}_{P,Q},$$

$$\mathcal{N} = \{z : z^m \in \Omega_{p(1 \pm \rho^m), r(1 \pm \rho^m)} \text{ for all } m \geq 1 \text{ and both signs}\}.$$

On the real axis $\mathcal{S}_1 \supseteq (-1, 1) \supseteq \mathcal{N} \cap \mathbb{R}$, and on the imaginary axis $\mathcal{S}_2 \cap i\mathbb{R}$ and $\mathcal{N} \cap i\mathbb{R}$ both equal the region of Theorem 7.1; off the axes the inner and outer bounds differ in general (Section 12).

Proof. $\mathcal{S}_1$: as in Proposition 7.3, [5, Thm. 1] at $\alpha = 1$ with exponent s , then Lyapunov. $\mathcal{S}_2$: Lemma 7.2. $\mathcal{N}$: Lemma 6.2(a) applied with $f = 1 + \varepsilon w h_m(x_1)$, $g = 1 \pm \varepsilon w h_m(y_1)$ and $\mathbb{E} h_m(X_1) h_m(Y_1) = \rho^m$ gives $|z^m w|_{r(1 \pm \rho^m)}^2 \leq |w|_{p(1 \pm \rho^m)}^2$. $\mathcal{C}_{p,r}$: take $g = 1$. On $i\mathbb{R}$: $\mathcal{S}_2 \cap i\mathbb{R}$ contains the region of Theorem 7.1 (by Lemma 7.2 and the one-function imaginary region, i.e. Corollary 6.5 at $n = 1$), and $\mathcal{N} \cap i\mathbb{R}$ is contained in it (the $m = 1$ conditions with $|z| \leq 1$). On $\mathbb{R}$, Proposition 7.3. $\square$

The outer bound is not sharp: for $r \geq 2$, Part III shows with joint bumps that (11) fails at points of $\mathcal{N} \cap \mathcal{C}_{p,r}$. We turn to that mechanism, after isolating the one-function input it requires.

8. EXACT MOMENTS OF A NARROW BUMP

The obstruction below the diagonal is driven by a narrow, far-out Gaussian bump. Fix $B > 0$, $t > 0$ and

$$b(x) = b_{B,t}(x) = e^{-Bx^2/2 + tx - t^2/(2B)} = e^{-B(x-t/B)^2/2} \in (0, 1] \quad (x \in \mathbb{R}).$$

With $\varsigma = 1 - z^2$, $D = 1 + \varsigma B$ (so $\operatorname{Re} D > 0$), put

$$\mu_B = rB \operatorname{Re} \frac{z^2}{D}, \quad \nu_B = r \operatorname{Re} \frac{z}{D}, \quad \Phi_m = \frac{m}{\lambda_m} \left(Q_m \kappa_{Q_m} - \frac{1}{2} \right) + \frac{1}{2} \quad (\delta_{Q_m} \neq 0),$$

where $\lambda_m = 1 + (m-1)\rho$, $Q_m = r\lambda_m$. Recall the m equicorrelated vectors $\xi_1, \dots, \xi_m$ of Section 2 (covariance ρ off the diagonal); for $m = 1$ this is a single γ_k , for $m = 2$ the ρ -correlated pair.

Lemma 8.1 (Bump moments). *Let $|z| < 1$ and $r \geq 1$.*

- (a) $\mathbf{T}_z b(x) = c \exp\left(-\frac{Bz^2}{2D}x^2 + \frac{tz}{D}x\right)$ with $c = D^{-1/2} \exp\left(-\frac{t^2}{2B} + \frac{\varsigma t^2}{2D}\right)$, and $|\mathbf{T}_z b(x)|^r = |c|^r \exp\left(-\frac{\mu_B}{2}x^2 + t\nu_B x\right)$.
- (b) $\mathbb{E} b(X) = (1+B)^{-1/2} \exp\left(-\frac{t^2}{2B(1+B)}\right)$.
- (c) Let $1 \leq m \leq n$ and $\lambda = \lambda_m$. If $1 + \mu_B \lambda > 0$, then $J_m := \mathbb{E} \prod_{j=1}^m |\mathbf{T}_z b(\xi_j)|^r$ is finite and

$$\log \frac{J_m}{\mathbb{E} b} = t^2 R_m(B) + c_{m,B}, \quad R_m(B) = m \left(-\frac{r}{2B} + \frac{r}{2} \operatorname{Re} \frac{\varsigma}{D} \right) + \frac{m\lambda\nu_B^2}{2(1+\mu_B\lambda)} + \frac{1}{2B(1+B)},$$

with $c_{m,B}$ independent of t . If $1 + \mu_B \lambda < 0$, then $J_m = \infty$.

- (d) As $B \rightarrow \infty$: $\mu_B \rightarrow r(A-1)$, $1 + \mu_B \lambda_m \rightarrow \delta_{Q_m}$, and, if $\delta_{Q_m} \neq 0$, $B^2 R_m(B) \rightarrow \Phi_m$. In particular $\Phi_1 = r\kappa_r$ and $\Phi_2 = 2r\kappa_Q - \frac{1-\rho}{2(1+\rho)}$ with $Q = Q_2 = r(1+\rho)$.

Proof. (a) Since $b = g_{B,t} e^{-t^2/(2B)}$ with $g_{B,t}$ as in Lemma 3.2, the closed form and c follow from that lemma (with $\beta' = -Bz^2/D$); taking moduli, $r \operatorname{Re}(-\frac{Bz^2}{2D}) = -\frac{\mu_B}{2}$ and $r \operatorname{Re} \frac{tz}{D} = t\nu_B$. (b) is the Gaussian integral $\mathbb{E} e^{-BX^2/2 + tX} = (1+B)^{-1/2} e^{t^2/(2(1+B))}$ times $e^{-t^2/(2B)}$. (c) The covariance Σ_m of $(\xi_1, \dots, \xi_m)$ (one dimension each) has eigenvalue λ_m on $\mathbf{1}$ and $1 - \rho$ on $\mathbf{1}^\perp$. For real μ, h with $1 + \mu\lambda_m > 0$ (which gives $\Sigma_m^{-1} + \mu \operatorname{Id} > 0$),

$$\mathbb{E} \exp \left(-\frac{\mu}{2} \sum_{j \leq m} \xi_j^2 + h \sum_{j \leq m} \xi_j \right) = \frac{\exp \left(\frac{m\lambda_m h^2}{2(1+\mu\lambda_m)} \right)}{\sqrt{(1+\mu\lambda_m)(1+\mu(1-\rho))^{m-1}}},$$

since $\mathbf{1}^T (\Sigma_m^{-1} + \mu \operatorname{Id})^{-1} \mathbf{1} = \frac{m\lambda_m}{1+\mu\lambda_m}$. Apply this with $\mu = \mu_B$, $h = t\nu_B$, and use (a), (b) and $\log |c| = -\frac{1}{2} \log |D| - \frac{t^2}{2B} + \frac{t^2}{2} \operatorname{Re} \frac{\varsigma}{D}$. If $1 + \mu_B \lambda_m < 0$, the integrand grows along $\mathbf{1}$ and $J_m = \infty$. (d) From $\frac{\varsigma}{D} = \frac{1}{B} - \frac{1}{\varsigma B^2} + O(B^{-3})$, $\frac{z}{D} = \frac{z}{\varsigma B} + O(B^{-2})$ and $B \frac{z^2}{D} = \frac{z^2}{\varsigma} + O(B^{-1}) = \frac{1}{\varsigma} - 1 + O(B^{-1})$, we get $-\frac{r}{B} + r \operatorname{Re} \frac{\varsigma}{D} = -\frac{rA}{B^2} + O(B^{-3})$, $\nu_B = \frac{rC}{B} + O(B^{-2})$, $\mu_B \rightarrow r(A-1)$, $1 + \mu_B \lambda_m \rightarrow 1 + Q_m(A-1) =$

δ_{Q_m} and $\frac{1}{2B(1+B)} = \frac{1}{2B^2} + O(B^{-3})$. Hence $B^2 R_m(B) \rightarrow -\frac{mrA}{2} + \frac{m\lambda_m r^2 C^2}{2\delta_{Q_m}} + \frac{1}{2}$, which equals Φ_m because $Q_m \kappa_{Q_m} - \frac{1}{2} = -\frac{Q_m A}{2} + \frac{Q_m^2 C^2}{2\delta_{Q_m}}$ and $Q_m = r\lambda_m$. For $m = 1$, $\Phi_1 = Q_1 \kappa_{Q_1} = r\kappa_r$; for $m = 2$ a direct computation from $\lambda_2 = 1 + \rho$, $Q_2 = Q$ gives $\Phi_2 = 2r\kappa_Q - \frac{1-\rho}{2(1+\rho)}$. $\square$

9. THE ONE-FUNCTION PROBLEM BELOW THE DIAGONAL

The only input from the one-function theory that Part III needs is the failure of the $L^\infty \rightarrow L^r$ contraction outside Ω_r . We prove it here from scratch, by the narrow bump; this is the single place where the below-diagonal phenomenon is produced, and it replaces the unpublished companion material.

Proposition 9.1 (One function, $L^\infty \rightarrow L^r$). *Let $r \geq 2$ and $z \in \mathbb{D}$. Then $\|\mathbf{T}_z f\|_{L^r(\gamma_k)} \leq \|f\|_\infty$ for all bounded measurable f and all k if and only if $z \in \Omega_r$. If $z \notin \Omega_r$, there is already in dimension one a bounded f with $\|f\|_\infty \leq 1$ and $\|\mathbf{T}_z f\|_{L^r(\gamma_1)} > 1$; one may take $f = 1 - 2b_{B,t}$ for suitable $B, t > 0$.*

Proof. Sufficiency. If $z \in \Omega_r$ and $r \geq 2$, then by Lemma 3.3(b), $\|\mathbf{T}_z f\|_{L^r} \leq \|f\|_{L^r} \leq \|f\|_\infty$.

Necessity. Let $z \notin \Omega_r$; we work in $k = 1$. Put $f = 1 - 2b$ with $b = b_{B,t}$, so $|f| \leq 1$ (as $b \in (0, 1]$). Then $\mathbf{T}_z f = 1 - 2\mathbf{T}_z b$, and $w := \mathbf{T}_z f - 1 = -2\mathbf{T}_z b$ satisfies, by the bump inequality (Lemma 5.3, $r \geq 2$), $|\mathbf{T}_z f|^r = |1 + w|^r \geq 1 + r \operatorname{Re} w + 2^{1-r}|w|^r$ pointwise. Taking expectations and using $\mathbb{E}\mathbf{T}_z b = \mathbb{E}b$ (real; Lemma 3.1(c)) and $2^{1-r}|w|^r = 2|\mathbf{T}_z b|^r$,

$$(12) \quad \mathbb{E}|\mathbf{T}_z f|^r \geq 1 - 2r \mathbb{E}b + 2J_1, \quad J_1 = \mathbb{E}|\mathbf{T}_z b|^r.$$

Since $r \geq 2$ and $z \notin \Omega_r$, either $\delta_r < 0$ or $\delta_r > 0$ with $\kappa_r > 0$. Indeed for $\delta_r > 0$, $\kappa_r \leq 0 \iff z \in \Omega_r$ (Lemma 5.2); and $\delta_r < 0$ means $F_1 = 2\delta_r|\varsigma|^2 < 0$, which forces $F_2 > 0$ (for $r \geq 2$, $F_2 \leq 0$ gives $(r-2)|1-z^2|^2 \leq r(1-|z|^2)|1-z^2| \leq r(1-|z|^4)$, i.e. $F_1 \geq 0$), hence $z \notin \Omega_r$.

If $\delta_r < 0$: by Lemma 8.1(d), $1 + \mu_B \lambda_1 = 1 + \mu_B \rightarrow \delta_r < 0$, so for B large $1 + \mu_B < 0$ and $J_1 = \infty$ by Lemma 8.1(c); then (12) is $+\infty > 1$.

If $\delta_r > 0$ and $\kappa_r > 0$: by Lemma 8.1(d), $B^2 R_1(B) \rightarrow \Phi_1 = r\kappa_r > 0$ and $1 + \mu_B \rightarrow \delta_r > 0$, so fix B with $R_1(B) > 0$ and $1 + \mu_B > 0$. By Lemma 8.1(c), $J_1/\mathbb{E}b = \exp(t^2 R_1(B) + c_{1,B}) \rightarrow \infty$ as $t \rightarrow \infty$, so (12) gives $\mathbb{E}|\mathbf{T}_z f|^r \geq 1 + 2\mathbb{E}b(J_1/\mathbb{E}b - r) > 1$ for t large. In both cases $\|\mathbf{T}_z f\|_{L^r} > 1 = \|f\|_\infty$. $\square$

For $r \geq 2$, Proposition 9.1 says $\mathcal{C}_{\infty,r} \cap \mathbb{D} = \Omega_r$ (bounded functions), the fact used below. The companion unpublished note determines $\mathcal{C}_{p,q}$ for all p, q ; we do not need the full statement.

10. PART III: TWO CORRELATED FUNCTIONS AND JOINT NARROW BUMPS

Fix $\rho \in (0, 1)$ and $1 \leq r < p \leq \infty$, and let $\mathcal{R}_p = \mathcal{R}_p(\rho, r)$ be the set of $z \in \mathbb{D}$ for which (11) holds, read with the Mehler integral $\mathbf{T}_z$, for all k and all bounded measurable f, g . Since (X, Y) has a strictly positive density, $\|f(X)g(Y)\|_{L^\infty} = \|f\|_\infty \|g\|_\infty \geq \|f(X)g(Y)\|_{L^p}$, so $\mathcal{R}_p \subseteq \mathcal{R}_\infty$. For $p < \infty$ the class of test functions does not matter.

Lemma 10.1 (Function classes). *Let $p < \infty$, $z \in \mathbb{D}$, $k \geq 1$. Then (11) holds for all bounded measurable f, g on $\mathbb{R}^k$ iff it holds for all polynomials f, g .*

Proof. Two facts: (a) $\|F(X)G(Y)\|_{L^s} \leq \|F\|_{L^{s(1+\rho)}} \|G\|_{L^{s(1+\rho)}}$ for $s > 0$, by the forward inequality (7) (or Lemma 4.2), so $(F, G) \mapsto F(X)G(Y)$ is continuous from $L^{s(1+\rho)} \times L^{s(1+\rho)}$ to L^s ; (b) $\mathbf{T}_z = \tilde{T}_z$ on bounded functions and $\tilde{T}_z$ is an L^2 -contraction (Lemma 3.3(a)). Let f, g violate (11). Approximate in $L^{\max\{2, p(1+\rho)\}}(\gamma_k)$: if f, g are bounded take polynomials f_n, g_n ; if f, g are polynomials take truncations $f\mathbf{1}_{\{|f| \leq n\}}$. Then $\tilde{T}_z f_n \rightarrow \tilde{T}_z f$ in L^2 , hence a.e. along a subsequence, also with respect to the (absolutely continuous) law of (X, Y) ; by Fatou $\liminf_n \|\tilde{T}_z f_n(X)\tilde{T}_z g_n(Y)\|_{L^r} \geq \|\tilde{T}_z f(X)\tilde{T}_z g(Y)\|_{L^r}$, and by (a), $\|f_n(X)g_n(Y)\|_{L^p} \rightarrow \|f(X)g(Y)\|_{L^p}$. So f_n, g_n violate (11) for n large. $\square$

Lemma 10.2 (Local conditions). *Let $1 < r < p < \infty$ and $z \in \mathbb{C}$. The second-order conditions of (11) at $f = g = 1$ (i.e. (11) with $f = 1 + \varepsilon\varphi$, $g = 1 + \varepsilon\psi$ holds to $o(\varepsilon^2)$ for all polynomials φ, ψ hold iff $z \in \mathcal{N}(p)$, where $\mathcal{N}(p) = \{z : z^m \in \Omega_{p(1 \pm \rho^m), r(1 \pm \rho^m)} \forall m \geq 1, \text{ both signs}\}$.*

Proof. Homogeneity lets us assume $\mathbb{E}\varphi = \mathbb{E}\psi = 0$ (the mean enters either side only through the vanishing first-order term). Writing $a = T_z\varphi(X) + T_z\psi(Y)$, the expansion $|1 + w|^s = 1 + s \operatorname{Re} w + \frac{s}{2}|w|^2 + \frac{s(s-2)}{2}(\operatorname{Re} w)^2 + O(|w|^3)$ with $w = \varepsilon a + \varepsilon^2 T_z\varphi(X)T_z\psi(Y)$ gives $\log(\text{LHS}) = \varepsilon^2 L + O(\varepsilon^3)$ with $L = \operatorname{Re} \mathbb{E}[T_z\varphi(X)T_z\psi(Y)] + \frac{1}{2}\mathbb{E}|a|^2 + \frac{r-2}{2}\mathbb{E}(\operatorname{Re} a)^2$, and $\log(\text{RHS}) = \varepsilon^2 R + O(\varepsilon^3)$ with R the same at $z = 1$, $r = p$. Decomposing $\varphi = \sum_\beta c_\beta H_\beta$, $\psi = \sum_\beta d_\beta H_\beta$ and using $\mathbb{E}H_\beta(X)H_{\beta'}(Y) = \rho^{|\beta|}\beta!\delta_{\beta\beta'}$, $L - R$ is a sum over β of $\beta!$ times a quadratic form in (c_β, d_β) depending only on $|\beta| = m$; with $c_\pm = (c \pm d)/\sqrt{2}$ and $\operatorname{Re}(\zeta^2) = 2(\operatorname{Re} \zeta)^2 - |\zeta|^2$,

$$2L = |z^m c_+|_{r(1+\rho^m)}^2 + |z^m c_-|_{r(1-\rho^m)}^2, \quad 2R = |c_+|_{p(1+\rho^m)}^2 + |c_-|_{p(1-\rho^m)}^2.$$

So $L \leq R$ for all φ, ψ iff $z^m \in \Omega_{p(1+\rho^m), r(1+\rho^m)} \cap \Omega_{p(1-\rho^m), r(1-\rho^m)}$ for every m , i.e. $z \in \mathcal{N}(p)$. $\square$

Lemma 10.3 (Joint lower bound). *Let $r \geq 2$, $|z| < 1$, $b : \mathbb{R} \rightarrow [0, 1]$ measurable, $f = 1 - 2b$, $S = \mathbb{E}|\mathbf{T}_z b(X)|^r < \infty$, and $J = \mathbb{E}|\mathbf{T}_z b(X)\mathbf{T}_z b(Y)|^r \in [0, \infty]$. Then $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^r \geq 1 - 8r\mathbb{E}b + 4J - 2^{r+1}S$.*

Proof. Put $u = \mathbf{T}_z b(X)$, $v = \mathbf{T}_z b(Y)$, so $\mathbf{T}_z f(X)\mathbf{T}_z f(Y) = (1 - 2u)(1 - 2v) = 1 + w$, $w = -2u - 2v + 4uv$. By Lemma 5.3, $\mathbb{E}|1 + w|^r \geq 1 + r \operatorname{Re} \mathbb{E}w + 2^{1-r}\mathbb{E}|w|^r$. Now $\mathbb{E}u = \mathbb{E}v = \mathbb{E}b$. Conditioning on X and using the semigroup $T_\rho \mathbf{T}_z = \mathbf{T}_{\rho z}$ (Lemma 3.1(d)), $\mathbb{E}[uv] = \mathbb{E}[\mathbf{T}_z b \cdot \mathbf{T}_{\rho z} b] = \sum_{m \geq 0} \rho^m z^{2m} \hat{b}_m^2$ in the orthonormal Hermite basis, so $|\mathbb{E}[uv]| \leq \sum_m \hat{b}_m^2 = \mathbb{E}b^2 \leq \mathbb{E}b$; hence $\operatorname{Re} \mathbb{E}w \geq -4\mathbb{E}b - 4\mathbb{E}b = -8\mathbb{E}b$. Finally $4uv = w + 2u + 2v$ gives $|4uv|^r \leq 2^{r-1}(|w|^r + |2u + 2v|^r)$ and $|2u + 2v|^r \leq 2^{2r-1}(|u|^r + |v|^r)$, whence $2^{1-r}|w|^r \geq 4|uv|^r - 2^r(|u|^r + |v|^r)$; take expectations. $\square$

Theorem 10.4 (Joint narrow bumps). *Let $\rho \in (0, 1)$, $r \geq 2$, $Q = r(1 + \rho)$ and $z \in \mathbb{D}$. Suppose*

$$(13) \quad F_1(z) < 0, \quad \text{or} \quad F_1(z) > 0 \text{ and } F_2(z) > (1 - \rho)F_1(z),$$

equivalently $\delta_Q(z) < 0$, or $\delta_Q(z) > 0$ and $\kappa_Q(z) > \frac{1-\rho}{4Q}$ (here $F_i = F_i^{(Q)}$). Then there are measurable $f, g : \mathbb{R} \rightarrow \mathbb{C}$ with $|f|, |g| \leq 1$ and $\|\mathbf{T}_z f(X)\mathbf{T}_z g(Y)\|_{L^r} > 1$; hence (11) fails for every $p \in [1, \infty]$, i.e. $z \notin \mathcal{R}_p$. If $z \in \Omega_r$ one may take $f = g = 1 - 2b_{B,t}$; for $p < \infty$, (11) then also fails for a polynomial $f = g$.

Proof. If $z \notin \Omega_r$: by Proposition 9.1 there is bounded f , $\|f\|_\infty \leq 1$, $\|\mathbf{T}_z f\|_{L^r} > 1$; take $g = 1$.

If $z \in \Omega_r$: then $\mathbf{T}_z$ contracts $L^r(\gamma_1)$ on bounded functions (Lemma 3.3(b)), so $S = \mathbb{E}|\mathbf{T}_z b|^r \leq \mathbb{E}b^r \leq \mathbb{E}b$ for $0 \leq b \leq 1$. By Lemma 10.3,

$$(14) \quad \mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^r \geq 1 + 4J - (8r + 2^{r+1})\mathbb{E}b \quad (f = 1 - 2b).$$

If $F_1 < 0$, then $\delta_Q < 0$ (as $F_1 = 2\delta_Q|\zeta|^2$); by Lemma 8.1(d) choose B with $1 + \mu_B\lambda_2 = 1 + \mu_B(1 + \rho) < 0$, so $J = J_2 = \infty$ and (14) is $+\infty$. If $F_1 > 0$ and $F_2 > (1 - \rho)F_1$, then $\delta_Q > 0$ and $\kappa_Q = \frac{F_2}{4QF_1} > \frac{1-\rho}{4Q}$, so $\Phi_2 = 2r\kappa_Q - \frac{1-\rho}{2(1+\rho)} > \frac{2r(1-\rho)}{4Q} - \frac{1-\rho}{2(1+\rho)} = 0$. By Lemma 8.1(d), as $B \rightarrow \infty$, $1 + \mu_B\lambda_2 \rightarrow \delta_Q > 0$ and $1 + \mu_B(1 - \rho) \rightarrow 1 + r(A - 1)(1 - \rho)$ (which is ≥ 1 if $A \geq 1$ and $> \delta_Q > 0$ if $A < 1$), so we may fix B with $R_2(B) > 0$ and $1 + \mu_B(1 \pm \rho) > 0$. Then $J = J_2$, $J/\mathbb{E}b = \exp(t^2 R_2(B) + c_{2,B}) \rightarrow \infty$ as $t \rightarrow \infty$, and (14) gives $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^r > 1$ for t large. For $p < \infty$ the polynomial statement follows from Lemma 10.1 (same approximant for $f = g$). $\square$

Written out, the second alternative of (13) is $F_2 > (1 - \rho)F_1 > 0$, i.e. joint bumps exclude points outside Ω_Q ($F_2 > 0$) only beyond a margin $(1 - \rho)F_1$ that vanishes as $\rho \rightarrow 1$.

Proposition 10.5 (Outer exponents $1 < r < 2$). *Let $\rho \in (0, 1)$, $1 < r < 2$, $Q = r(1 + \rho)$, and $z \in \Omega_r \cap \mathbb{D}$ satisfy (13). Then $f = g = 1 - 2b_{B,t}$ gives $\|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)\|_{L^r} > 1$ for suitable B, t ; hence (11) fails for every $p \in [1, \infty]$, and for polynomials when $p < \infty$.*

Proof. For $1 < r < 2$ replace Lemma 5.3 by two facts on $\phi_r(w) = |1 + w|^r - 1 - r \operatorname{Re} w$: (i) $\phi_r \geq 0$ (convexity of $v \mapsto |v|^r$ on $\mathbb{R}^2$, gradient $(r, 0)$ at $v = 1$); (ii) $\phi_r(w) \geq c_r |w|^r$ for $|w| \geq K_r$, with $c_r = \frac{1}{2}(\frac{3}{4})^r$ and $K_r = \max\{4, (2(1+r)(\frac{4}{3})^r)^{1/(r-1)}\}$ (for $|w| \geq 4$: $|1 + w| \geq \frac{3}{4}|w|$ and $1 + r \operatorname{Re} w \leq (1+r)|w|$). With u, v, w as in Lemma 10.3, $\Lambda = \max\{4, K_r^{1/2}\}$ and $G = \{|u| \geq \Lambda, |v| \geq \Lambda\}$: on G , $|w| \geq 4|uv| - 2|u| - 2|v| \geq 3|uv| \geq 3\Lambda^2 \geq K_r$, so $\phi_r(w) \geq c_r 3^r |uv|^r$. Using $\mathbf{1}_G \geq 1 - \mathbf{1}_{\{|u| < \Lambda\}} - \mathbf{1}_{\{|v| < \Lambda\}}$ and $S \leq \mathbb{E}b$ (as $z \in \Omega_r$),

$$\mathbb{E}|1 + w|^r \geq 1 + r \operatorname{Re} \mathbb{E}w + c_r 3^r (J - 2\Lambda^r S) \geq 1 - 8r \mathbb{E}b + c_r 3^r (J - 2\Lambda^r \mathbb{E}b).$$

Lemma 8.1 does not use $r \geq 2$, so the choice of B and the limit $t \rightarrow \infty$ are as before. $\square$

Proposition 10.5 has content only for $\frac{2+\rho}{1+\rho} < r < 2$: $F_2 \leq 0$ forces $F_1 \geq 0$ (for $Q > 2$, $(Q-2)|1 - z^2|^2 \leq Q(1 - |z|^2)|1 - z^2| \leq Q(1 - |z|^4)$), so $\Omega_Q \cap \mathbb{D} \subseteq \mathcal{J}_{\rho,r}$; and for $1 < r < 2$, $\Omega_r = \Omega_{r'} \subseteq \Omega_Q$ exactly when $r \leq \frac{2+\rho}{1+\rho}$.

Remark 10.6 (General cross-covariance). If $X \in \mathbb{R}^k$, $Y \in \mathbb{R}^l$ are jointly Gaussian with standard marginals and cross-covariance $C = \mathbb{E}XY^T$ of operator norm $\rho \in (0, 1)$, the covariance of (X, Y) has spectrum in $[1 - \rho, 1 + \rho]$, so the Brascamp–Lieb steps apply verbatim; and for unit u, v with $u^T C v = \rho$, $(u \cdot X, v \cdot Y)$ is a ρ -correlated standard pair on which $\mathbf{T}_z$ acts as in one dimension. Hence Theorems 10.4, 10.7 and Proposition 10.5 hold for (X, Y) with this ρ .

Theorem 10.7 (Sandwich). *Let $\rho \in (0, 1)$, $r \geq 2$, $Q = r(1 + \rho)$ and $p \in [\frac{Q}{1-\rho}, \infty]$. Then*

$$\Omega_Q \cap \mathbb{D} = \{z \in \mathbb{D} : F_2(z) \leq 0\} \subseteq \mathcal{R}_p(\rho, r) \subseteq \mathcal{J}_{\rho,r} \cap \Omega_r,$$

$$\mathcal{J}_{\rho,r} = \{z \in \mathbb{D} : F_1 \geq 0, \text{ and } F_2 \leq (1 - \rho)F_1 \text{ if } F_1 > 0\}.$$

For $p < \infty$ also $\mathcal{R}_p \subseteq \mathcal{N}(p)$. On the axes the bounds agree: $\mathcal{R}_p \cap \mathbb{R} = (-1, 1)$ and $\mathcal{R}_p \cap i\mathbb{R} = \{iy : y^2 \leq \frac{1}{Q-1}\}$. Off the axes $\Omega_Q \cap \mathbb{D}$ is a proper subset of $\mathcal{J}_{\rho,r}$: every $z_0 \in \partial\Omega_Q \cap \mathbb{D}$ with $z_0 \notin i\mathbb{R}$ is interior to $\mathcal{J}_{\rho,r}$.

Proof. Inner bound. Let $z \in \Omega_Q \cap \mathbb{D}$, $P = p(1 - \rho) \geq Q$ (or $p = \infty$), f, g bounded. Forward (7) ($\Sigma \leq (1 + \rho)\operatorname{Id}$) gives $\|\mathbf{T}_z f(X) \mathbf{T}_z g(Y)\|_{L^r} \leq \|\mathbf{T}_z f\|_{L^Q} \|\mathbf{T}_z g\|_{L^Q}$; since $Q \geq 2$, $z \in \Omega_Q$, Lemma 3.3(b) gives $\|\mathbf{T}_z f\|_{L^Q} \leq \|f\|_{L^Q} \leq \|f\|_{L^P}$ (and $\leq \|f\|_\infty$ if $p = \infty$). For $p < \infty$, reverse (7) ($\Sigma \geq (1 - \rho)\operatorname{Id}$, exponents $1 - \rho$) gives $\mathbb{E}|f(X)|^p |g(Y)|^p \geq \|f\|_{L^P}^p \|g\|_{L^P}^p$. Combine.

Outer bound. $\mathcal{R}_p \subseteq \mathcal{R}_\infty$, and Theorem 10.4 excludes every z satisfying (13), so $\mathcal{R}_\infty \subseteq \mathcal{J}_{\rho,r}$. With $g = 1$, $z \in \mathcal{R}_\infty$ says $\mathbf{T}_z$ is an $L^\infty \rightarrow L^r$ contraction on bounded functions, so $z \in \Omega_r$ by Proposition 9.1. Thus $\mathcal{R}_p \subseteq \mathcal{R}_\infty \subseteq \mathcal{J}_{\rho,r} \cap \Omega_r$. For $p < \infty$, $\mathcal{R}_p \subseteq \mathcal{N}(p)$ by Lemmas 10.1, 10.2.

Axes. For $x \in (-1, 1)$, $\Omega_Q \cap \mathbb{R} \supseteq (-1, 1)$. For $z = iy$: $A = \frac{1}{1+y^2}$, $C = 0$, $\delta_Q(1 + y^2) = 1 - (Q - 1)y^2$, $\kappa_Q = \frac{1}{2}(\frac{1}{Q} - \frac{1}{1+y^2})$. So $F_1 \geq 0 \iff y^2 \leq \frac{1}{Q-1}$, and then $1 + y^2 \leq \frac{Q}{Q-1} < Q$, so $\kappa_Q < 0$, i.e. $F_2 < 0$; thus $\mathcal{J}_{\rho,r} \cap i\mathbb{R} = \{y^2 \leq \frac{1}{Q-1}\} = \Omega_Q \cap i\mathbb{R}$.

Strictness. $Q > 2$. For $z_0 \in \partial\Omega_Q \cap \mathbb{D}$, $F_2(z_0) = 0$, i.e. $(Q - 2)|1 - z_0^2| = Q(1 - |z_0|^2)$, so $F_1(z_0) = Q(1 - |z_0|^2)[(1 + |z_0|^2) - |1 - z_0^2|]$. As $|1 - z_0^2| \leq 1 + |z_0|^2$ with equality iff $z_0 \in i\mathbb{R}$, we get $F_1(z_0) > 0 = F_2(z_0)$ for $z_0 \notin i\mathbb{R}$, so a neighbourhood of z_0 lies in $\mathcal{J}_{\rho,r}$. $\square$

Corollary 10.8 (Local and one-function conditions are not sufficient). *Let $\rho = \frac{1}{2}$. The points*

$$(J1) \quad z = \frac{171}{200}e^{3i/5}, \quad r = 2; \quad (J2) \quad z = \frac{319}{500}e^{3i/5}, \quad r = 4$$

satisfy, for every $p \in [10, \infty)$, all second-order conditions ($z \in \mathcal{N}(p)$) and the one-function condition $z \in \Omega_r$ (so T_z contracts $L^p(\gamma_k) \rightarrow L^r(\gamma_k)$ for all k). Yet (11) fails at them for every $p \in [1, \infty]$, with $f = g = 1 - 2b$ ($b(x) = e^{-Bx^2/2+tx-t^2/(2B)}$), and for a polynomial $f = g$ when $p < \infty$:

- (J1), $B = 60$, $t = 1000$: $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^2 - 1 = 6.9151 \dots \times 10^{-57} > 0$ (*exact closed form*);
- (J2), $B = 45$, $t = 1500$: $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^4 - 1 \geq 6.7204 \dots \times 10^{-233} > 0$.

All quantities are certified in ball arithmetic (Section 12). At $r = 2$, $\Omega_r = \mathbb{D}$, so the failure at (J1) is caused by the correlation alone.

Proof. That $z \in \mathcal{N}(p)$ for all $p \geq 10$ and $z \in \Omega_r$, and that (13) holds, are the certified facts of Section 12; the failure then follows from Theorem 10.4 (with $r = 2 \geq 2$ and $r = 4 \geq 2$). The displayed excesses are the certified values of $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^r$ at the stated B, t (for (J2), the lower bound of Lemma 10.3). $\square$

Remark 10.9 (Mechanism). The forward Brascamp–Lieb inequality gives $J \leq \|\mathbf{T}_z b\|_{L^Q}^{2r}$, and the one-function large deviation at exponent Q (Lemma 8.1 with $m = 1$, exponent Q) yields $\log \|\mathbf{T}_z b\|_{L^Q}^{2r} = t^2 B^{-2}(2r\kappa_Q - \frac{1}{1+\rho} + o(1))$, while $\log J = \log \mathbb{E}b + t^2 R_2(B) + c_{2,B} = t^2 B^{-2}(-\frac{1}{2} + \Phi_2 + o(1))$ with $-\frac{1}{2} + \Phi_2 = 2r\kappa_Q - \frac{1}{1+\rho}$. So the joint bump attains the Brascamp–Lieb bound in the leading large-deviation exponent, and the threshold $\frac{1-\rho}{4Q}$ is the point where the gain $(\mathbb{E}b)^{2/(1+\rho)}$ balances the first-order cost $\sim \mathbb{E}b$. Inside $\mathcal{J}_{\rho,r}$ (with $\delta_Q > 0$), $B^2 R_2(B) \rightarrow \Phi_2 < 0$, so equal bumps with large B do not win; that *no* finite B wins is only numerical (Section 12), so the region between Ω_Q and $\mathcal{J}_{\rho,r}$ is left open.

11. MORE THAN TWO FUNCTIONS

Let $n \geq 2$ and let $\xi_1, \dots, \xi_n \sim \gamma_k$ be equicorrelated, $\text{cov}(\xi_i, \xi_j) = \rho \text{Id}_k$ ($i \neq j$). For $1 \leq r < p \leq \infty$ consider

$$(15) \quad \left\| \prod_{j=1}^n \mathbf{T}_z f_j(\xi_j) \right\|_{L^r} \leq \left\| \prod_{j=1}^n f_j(\xi_j) \right\|_{L^p},$$

with admissible set $\mathcal{R}_p^{(n)} \subseteq \mathbb{D}$ (bounded f_j , all k). Thus $\mathcal{R}_p^{(2)} = \mathcal{R}_p$ and $\mathcal{R}_p^{(n)} \subseteq \mathcal{R}_p^{(m)}$ for $m \leq n$. Recall $\lambda_m = 1 + (m-1)\rho$, $Q_m = r\lambda_m$, $\Phi_m = \frac{m}{\lambda_m}(Q_m\kappa_{Q_m} - \frac{1}{2}) + \frac{1}{2}$; by (8), for $\delta_{Q_m} > 0$, $\Phi_m > 0 \iff F_2^{(Q_m)} > \frac{2(m-1)(1-\rho)}{m} F_1^{(Q_m)}$.

Proposition 11.1 (Joint bumps on m of n functions). *Let $r \geq 2$, $2 \leq m \leq n$, $z \in \Omega_r \cap \mathbb{D}$, with $z \in \Omega_{\lambda_m}$ if $\lambda_m > 2$. Suppose $\delta_{Q_{m'}}(z) > 0$ for $2 \leq m' < m$, and either (i) $\delta_{Q_m}(z) < 0$, or (ii) $\delta_{Q_m}(z) > 0$, $\Phi_m(z) > 0$ and $\Phi_{m'}(z) < \Phi_m(z)$ for $2 \leq m' < m$. Then $f_1 = \dots = f_m = 1 - 2b_{B,t}$, $f_{m+1} = \dots = f_n = 1$ give $\|\prod_j \mathbf{T}_z f_j(\xi_j)\|_{L^r} > 1$ for suitable B, t ; hence $z \notin \mathcal{R}_p^{(n)}$ for every $p \in [1, \infty]$.*

Proof. Let $b = b_{B,t}$, $u_j = \mathbf{T}_z b(\xi_j)$, $u_S = \prod_{j \in S} u_j$; by exchangeability $\mathbb{E}|u_S|^r = J_{|S|}$. Then $\prod_{j \leq m} (1 - 2u_j) = 1 + w$, $w = \sum_{\emptyset \neq S \subseteq [m]} (-2)^{|S|} u_S$.

Mixed means. $\mathbb{E}u_j = \mathbb{E}b$. For $|S| \geq 2$, $\lambda = \lambda_{|S|}$: the covariance of $(\xi_j)_{j \in S}$ is $\leq \lambda \text{Id}$, so forward (7) gives $\mathbb{E}|u_S| \leq \|\mathbf{T}_z b\|_{L^\lambda}^{|S|}$; if $\lambda \leq 2$ then $\|\mathbf{T}_z b\|_{L^\lambda} \leq \|\mathbf{T}_z b\|_{L^2} \leq \|b\|_{L^2} \leq (\mathbb{E}b)^{1/2}$ (Lemma 3.3), so $\mathbb{E}|u_S| \leq \mathbb{E}b$; if $\lambda > 2$ then $z \in \Omega_{\lambda_m} \subseteq \Omega_\lambda$ (as Ω_q decreases in $q \geq 2$, Lemma 5.1(d)), so $\mathbb{E}|u_S| \leq \|b\|_{L^\lambda}^{|S|} \leq (\mathbb{E}b)^{|S|/\lambda} \leq \mathbb{E}b$ (since $|S| \geq \lambda_{|S|}$). As $\sum_S 2^{|S|} = 3^m$, $\text{Re } \mathbb{E}w \geq -3^m \mathbb{E}b$.

Lower bound. By Lemma 5.3, $\mathbb{E}|1 + w|^r \geq 1 + r \text{Re } \mathbb{E}w + 2^{1-r} \mathbb{E}|w|^r$. Since $(-2)^m u_{[m]} = w - w'$ with $w' = \sum_{\emptyset \neq S \subsetneq [m]} (-2)^{|S|} u_S$ ($M = 2^m - 2$ terms), the inequalities $|a - b|^r \leq 2^{r-1}(|a|^r + |b|^r)$ and $|\sum_{i \leq M} a_i|^r \leq M^{r-1} \sum |a_i|^r$ give, on taking expectations and using $J_1 \leq \mathbb{E}b$,

$$(16) \quad \mathbb{E}|1 + w|^r \geq 1 + c_\# J_m - C_\# \left(\mathbb{E}b + \sum_{m'=2}^{m-1} J_{m'} \right), \quad c_\# = 2^{(m-2)r+2},$$

with $C_\# < \infty$ depending only on m, r . The moments $J_{m'}$ and their exponents $R_{m'}$, $\Phi_{m'}$ are given by Lemma 8.1(c),(d).

Choice of B, t . For $2 \leq m' < m$, $\delta_{Q_{m'}} > 0$, so $J_{m'}$ is finite for large B . In case (i) choose B with $1 + \mu_B \lambda_m < 0$, so $J_m = \infty$ and (16) is $+\infty$. In case (ii) fix B large with $R_m(B) > 0$ and $R_m(B) > R_{m'}(B)$ for $m' < m$ (possible since $B^2 R_{m'}(B) \rightarrow \Phi_{m'}$ and $\Phi_m > \max_{m' < m} \Phi_{m'}$, $\Phi_m > 0$); as $t \rightarrow \infty$, $J_m/\mathbb{E}b \rightarrow \infty$ and $J_{m'}/J_m \rightarrow 0$, so (16) exceeds 1. Finally $\|\prod_j f_j(\xi_j)\|_{L^p} \leq 1$. $\square$

Corollary 11.2. *Let $r \geq 2$, $n \geq 2$, $z \in \Omega_r \cap \mathbb{D}$, with $z \in \Omega_{\lambda_n}$ if $\lambda_n > 2$, and $\delta_{Q_m}(z) \neq 0$ for $2 \leq m \leq n$. If $\delta_{Q_n}(z) < 0$, or $\delta_{Q_n}(z) > 0$ and $\max_{2 \leq m \leq n} \Phi_m(z) > 0$, then $z \notin \mathcal{R}_p^{(n)}$ for every $p \in [1, \infty]$.*

Proof. Since $\delta_q = 1 - q(1 - A)$ and Q_m increases in m : if $A \geq 1$ all $\delta_{Q_m} > 0$; if $A < 1$, δ_{Q_m} decreases in m . If $\delta_{Q_n} < 0$, apply Proposition 11.1(i) with the least m having $\delta_{Q_m} < 0$. If $\delta_{Q_n} > 0$, then $\delta_{Q_m} > 0$ for all $m \leq n$; apply Proposition 11.1(ii) with the least m attaining $\max_{2 \leq m' \leq n} \Phi_{m'}$. $\square$

Proposition 11.3 (Inner bound and plane waves). *Let $r \geq 2$, $n \geq 2$.*

- (a) *If $p = \infty$ or $p \geq Q_n/(1 - \rho)$, then $\Omega_{Q_n} \cap \mathbb{D} \subseteq \mathcal{R}_p^{(n)}$.*
- (b) *For $\omega \in \mathbb{R}$, $f_1 = \dots = f_n = e^{i\omega x}$: $\mathbb{E}|\prod_j \mathbf{T}_z f_j(\xi_j)|^r = \exp(\frac{nr\omega^2}{2}[(Q_n - 1)(\text{Im } z)^2 + (\text{Re } z)^2 - 1])$ and $\|\prod_j f_j(\xi_j)\|_{L^p} = 1$. Hence $\mathcal{R}_p^{(n)} \subseteq \{z : (Q_n - 1)(\text{Im } z)^2 + (\text{Re } z)^2 \leq 1\}$ for every p .*
- (c) *If $p = \infty$ or $p \geq Q_n/(1 - \rho)$, then $\mathcal{R}_p^{(n)} \cap \mathbb{R} = (-1, 1)$ and $\mathcal{R}_p^{(n)} \cap i\mathbb{R} = \{iy : y^2 \leq \frac{1}{Q_n - 1}\}$.*

Proof. (a) The inner bound of Theorem 10.7 with $\lambda_n \text{Id}$ as the upper bound for the covariance: forward (7) gives $\|\prod_j \mathbf{T}_z f_j(\xi_j)\|_{L^r} \leq \prod_j \|\mathbf{T}_z f_j\|_{L^{Q_n}} \leq \prod_j \|f_j\|_{L^{Q_n}} \leq \prod_j \|f_j\|_{L^{p(1-\rho)}}$ ($z \in \Omega_{Q_n}$, $Q_n \geq 2$), and reverse (7) closes it (or $\|\cdot\|_\infty$ for $p = \infty$). (b) By Lemma 3.4, $|\mathbf{T}_z f_j(x)|^r = \exp(-r\omega x \text{Im } z - \frac{r}{2}(1 - (\text{Re } z)^2 + (\text{Im } z)^2)\omega^2)$; $\sum_j \xi_j$ is centred Gaussian with variance $n\lambda_n$, so $\mathbb{E} \exp(-r\omega \text{Im } z \sum_j \xi_j) = \exp(\frac{1}{2}r^2\omega^2(\text{Im } z)^2 n\lambda_n)$, and $r\lambda_n = Q_n$ gives the formula. (c) $(-1, 1) \subseteq \Omega_{Q_n}$ and $\Omega_{Q_n} \cap i\mathbb{R} = \{y^2 \leq \frac{1}{Q_n - 1}\}$ as in Theorem 10.7; by (b) the same set bounds $i\mathbb{R}$ from outside. $\square$

Theorem 11.4 (Many functions). *Let $\rho \in (0, 1)$, $r \geq 2$, and write $z = x + i\eta Q_n^{-1/2}$ with $x, \eta \in \mathbb{R}$.*

- (a) *For every $\varepsilon \in (0, 1)$ there is n_0 such that for $n \geq n_0$ and $p \in \{\infty\} \cup [Q_n/(1 - \rho), \infty)$, every z with $|x| < 1$ and $|\eta| \leq (1 - \varepsilon)(1 - x^2)$ lies in $\mathcal{R}_p^{(n)}$.*
- (b) *For all $n \geq 2$, $p \in [1, \infty]$, every $z \in \mathcal{R}_p^{(n)}$ satisfies $\eta^2 \leq \frac{Q_n}{Q_n - 1}(1 - x^2)$.*
- (c) *Let $0 < x^2 < \frac{r-1}{r-\rho}$ and $\eta^2 > (1 - x^2)(1 - \rho x^2)$. Then there is n_0 such that $z \notin \mathcal{R}_p^{(n)}$ for all $n \geq n_0$, $p \in [1, \infty]$.*

Thus at scale $Q_n^{-1/2} \asymp (n\rho)^{-1/2}$ the admissible region lies between $|\eta| = 1 - x^2$ (inner) and $|\eta| = \sqrt{(1 - x^2)(1 - \rho x^2)}$ (outer, for $x^2 < \frac{r-1}{r-\rho}$), inside $|\eta| \leq \sqrt{1 - x^2}$ for all x ; the two profiles agree only at $x = 0$.

Proof. Put $a = 1 - x^2$, $Q = Q_n$, $y = \eta/\sqrt{Q}$, and use $|1 - z^2|^2 = (1 - |z|^2)^2 + 4y^2$, $F_2^{(q)} = 4q^2 y^2 - 4(q - 1)|1 - z^2|^2$, so $\Omega_q \cap \mathbb{D} = \{(q - 2)|y| \leq \sqrt{q - 1}(1 - |z|^2)\}$ for $q \geq 2$.

(a) If $|\eta| \leq (1 - \varepsilon)a$ then $|z|^2 \leq x^2 + a^2/Q < 1$ and $1 - |z|^2 \geq a(1 - \frac{1}{Q})$, so $(Q - 2)|y| \leq \sqrt{Q}(1 - \varepsilon)a$ while $\sqrt{Q - 1}(1 - |z|^2) \geq \sqrt{Q}(1 - \frac{1}{Q})^{3/2}a$; hence $z \in \Omega_Q$ once $1 - \varepsilon \leq (1 - \frac{1}{Q})^{3/2}$, i.e. for large n . Apply Proposition 11.3(a). (b) is Proposition 11.3(b).

(c) *Level m .* For $\tilde{\eta}^2 \in (a(1 - \rho x^2), ra^2)$ and $z_m = x + i\eta_m Q_m^{-1/2}$ with $\eta_m \rightarrow \tilde{\eta}$: as $m \rightarrow \infty$, $z_m \rightarrow x$, $1 - |z_m|^2 \rightarrow a$, $|1 - z_m^2|^2 \rightarrow a^2$. Then $z_m \in \Omega_r$ for large m ($(r - 2)|y_m| \rightarrow 0 < \sqrt{r - 1}a$); if $\lambda_m > 2$, $z_m \in \Omega_{\lambda_m}$ reduces (divide by $\sqrt{\lambda_m}$, $Q_m = r\lambda_m$) to $|\tilde{\eta}|/\sqrt{r} \leq a$, true as $|\tilde{\eta}| < \sqrt{ra}$. Also $F_1^{(Q_m)}(z_m)/Q_m \rightarrow 2x^2 a > 0$ and $F_2^{(Q_m)}(z_m)/Q_m \rightarrow 4(\tilde{\eta}^2 - a^2)$, so $\delta_{Q_m}(z_m) > 0$ for large m , and

$Q_m \kappa_{Q_m} = F_2^{(Q_m)} / (4F_1^{(Q_m)})$ with $m/\lambda_m \rightarrow 1/\rho$ gives $\Phi_m(z_m) \rightarrow \frac{1}{\rho}(\frac{\tilde{\eta}^2 - a^2}{2x^2a} - \frac{1}{2}) + \frac{1}{2}$, which is > 0 iff $\tilde{\eta}^2 > a(1 - \rho x^2)$. So all hypotheses of Corollary 11.2 hold at level m for large m .

Conclusion. Since $x^2 < \frac{r-1}{r-\rho}$, $a(1 - \rho x^2) < ra^2$; pick $\theta \in (0, 1]$ with $a(1 - \rho x^2) < \theta \eta^2 < ra^2$ and set $m = \lceil \theta n \rceil$. Then $Q_m/Q_n \rightarrow \theta$ and $z = x + i\eta_m Q_m^{-1/2}$ with $\eta_m \rightarrow \sqrt{\theta} \eta =: \tilde{\eta}$, $\tilde{\eta}^2 \in (a(1 - \rho x^2), ra^2)$. By the level- m statement $z \notin \mathcal{R}_p^{(m)} \supseteq \mathcal{R}_p^{(n)}$ for large n . $\square$

12. CERTIFICATES AND NUMERICS

All scripts are in the accompanying `code/` bundle; those that we rely on for a *proof* use exact or interval (ball) arithmetic and are flagged as such. The symbolic identities of Sections 5 and 11 are verified in `check_identities.py` (sympy, exact) and `kappa_identity.py`.

Certified failure points (Corollary 10.8). `certify_joint_bump.py` (python-flint/Arb, 600 and 900 bits, exact rational inputs $z = \frac{171}{200}e^{3i/5}$ and $\frac{319}{500}e^{3i/5}$, $\rho = \frac{1}{2}$) certifies, with rigorous enclosures:

- (i) *Second-order conditions for all $p \geq 10$.* For $m = 1, 2$ and both signs the 2×2 matrix of $|w|_a^2 - |z^m w|_b^2$ ($a = p(1 \pm \rho^m)$, $p = 10$; $b = r(1 \pm \rho^m)$) is certified positive definite; for $m \geq 3$, $|z^m w|_b^2 \leq (Q - 1)|z|^6|w|^2 \leq |w|^2 \leq |w|_a^2$ (here $(Q - 1)|z|^6 = 0.781\dots$ for (J1), $0.337\dots$ for (J2)). Since $|w|_a$ increases in a , these hold for every $p \geq 10$.
- (ii) *One-function condition.* (J1): $|z| < 1$ (so $z \in \Omega_2 = \mathbb{D}$). (J2): $4(1 - |z|^2) - 2|1 - z^2| = 0.5056\dots > 0$, so $z \in \Omega_4$.
- (iii) *Hypotheses of Theorem 10.4.* (J1): $\delta_Q = 0.1951\dots$, $\Phi_2 = 0.1337\dots > 0$; (J2): $\delta_Q = 0.8746\dots$, $\Phi_2 = 0.1120\dots > 0$; the explicit form (13) is checked independently.
- (iv) *Witnesses.* At $r = 2$, the identity $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^2 = 1 - 8\mathbb{E}b + 8S + 8\text{Re}(P + P_c) - 32\text{Re}R + 16J$ ($P = \mathbb{E}[uv]$, $P_c = \mathbb{E}[u\bar{v}]$, $R = \mathbb{E}[u|v|^2]$) is evaluated exactly, giving $\mathbb{E}|\mathbf{T}_z f(X)\mathbf{T}_z f(Y)|^2 - 1 = 6.9151\dots \times 10^{-57}$. At $r = 4$ the lower bound of Lemma 10.3 is evaluated, giving $\geq 6.7204\dots \times 10^{-233}$. Every closed form ($\mathbb{E}b$, S , J , P , P_c , R , the $r = 2$ identity, the Mehler formula) is cross-checked against direct `mpmath` quadrature (relative errors $\leq 1.3 \times 10^{-16}$).

Runtime is about eight minutes, nearly all in the two-dimensional quadrature cross-check.

Region radii (floating point). `joint_bump_regions.py` computes, along rays, the radii of Ω_Q , $\mathcal{J}_{\rho,r}$ and $\mathcal{N}(p)$. For $(\rho, r) = (\frac{1}{2}, 2)$, $p = \infty$, $\arg z = 0.6$: Ω_3 (inner) 0.8201, $\mathcal{J}$ (outer) 0.8438, $\mathcal{N}(10^9)$ 0.8708; for $(\frac{1}{2}, 4)$, $\arg z = 0.6$: Ω_6 0.6153, $\mathcal{J}$ 0.6294, $\mathcal{N}(10)$ 0.6466, Ω_4 0.7258. Across $\arg z \in \{0.3, 0.6, 0.9, 1.2\}$, $\rho \in \{0.3, 0.5, 0.7\}$, $r \in \{2, 3, 4\}$, $\mathcal{J}$ lies strictly inside $\mathcal{N}(p) \cap \Omega_r$ off the real axis and the three agree at $\arg z = \pi/2$; near the real axis ($r \geq 3$) the binding outer constraint is $\Omega_r \cap \mathcal{N}(p)$.

Converse scans (floating point). `joint_bump_lib.py` with `joint_bump_scan2.py` optimises the large-deviation exponent over pairs of bumps of independent widths ($B \leq e^{14}$) and centres; no win is found inside $\mathcal{J}_{\rho,r}$, and in 19 of 20 rays the first win is within 5×10^{-4} of $\partial\mathcal{J}$. `joint_bump_converse_scan.py` finds, at 4000 random points of the gap between Ω_Q and $\mathcal{J}_{\rho,r}$, that $R_2(B) \leq 0$ for all B in a grid of 2000 values in $[10^{-4}, 10^7]$ (evidence for Remark 10.9, not a proof).

Parts I–II and n functions. `two_function_imaginary.py` checks, for 154 triples (ρ, p, r) , that the second-order threshold ($m \leq 60$), the one-function $L^P \rightarrow L^Q$ threshold, and the formula of Theorem 7.1 agree to 10^{-12} . `two_function_gaussian_test.py` finds no violation at $y^2 = 0.98Y_*$ and violations at $y^2 = 1.05Y_*$, Y_* the threshold of Theorem 7.1. `two_function_regions.py` gives the radii of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{N}$ (Proposition 7.4). `check_n_function.py` checks the closed forms, the limits $B^2 R_m(B) \rightarrow \Phi_m$, the plane-wave formula and the identities of Section 11 (working precision 30 digits; two limit checks to 10^{-6}). `n_function_regions.py` computes, at $n = 1000$, $(\rho, r) = (\frac{1}{2}, 2)$,

$x = \frac{1}{2}$, the inner, joint-bump and plane-wave profiles 0.7506, 0.8106, 0.8665, against the limits 0.75, 0.8101, 0.8660 of Theorem 11.4.

APPENDIX A. THE ACCOMPANYING CODE

The code/ bundle is self-contained (relative paths, a README, no external data). Proof-grade scripts use exact or ball arithmetic; the rest are floating-point explorations, labelled as such.

- `certify_joint_bump.py` — (*ball/exact*) certificates for Corollary 10.8.
- `check_identities.py`, `kappa_identity.py` — (*exact, sympy*) the identities of Lemmas 5.1, 5.2 and Section 11.
- `joint_bump_regions.py`, `joint_bump_lib.py`, `joint_bump_scan2.py`, `joint_bump_converse_scan.py` — (*float*) regions and converse scans (Section 12, Remark 10.9).
- `check_n_function.py`, `n_function_regions.py` — (*float*) Section 11.
- `two_function_imaginary.py`, `two_function_regions.py`, `two_function_gaussian_test.py` — (*float*) Parts I–II.
- `make_figure.py` — regenerates Figure 1.

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