A SHORT PROOF THAT THE BOOLEAN BOHNENBLUST–HILLE CONSTANT IS AT MOST $d^2(\log d)^{3/2+o(1)}$

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Abstract. For $f : \{-1,1\}^n \to \mathbb{C}$ of degree at most d, with $\lambda(d) = 1 + \log(1 + \log d)$ and with $\mathrm{BH}(d)$ the Boolean Bohnenblust–Hille constant defined below, we prove

$$\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^2\, \lambda(d)^{1/4}$$

and

$$\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^{3/2} \lambda(d)^{3/2} (1 + \log \lambda(d))^{1/4}.$$

In particular $\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^{5/2}$. The level-two bound $\Phi(M, 2) \le CM\sqrt{\log(M+1)}$ is proved below; the block-constant estimate uses Bernstein–Walsh on complex lines. From any such seed, one weighted bootstrap gives both bounds. Blei's inequality and the entropy form of AM–GM, applied at the ℓ^{p_r} level and grouped by a single block degree, produce a counting factor $r^{1/r}$. The critical contraction, with $L \approx \log M$ blocks, therefore starts at level $20L^2 \log L$. The weight $r^{1+1/L}$ meets that contraction and contributes only a constant at degree d.

1. Statement

Let $\{-1,1\}^n$ carry the uniform probability measure; $\mathbb{E}$ is expectation and $\|\cdot\|_q$ the L^q norm (the ℓ^q norm for arrays). Every $f : \{-1,1\}^n \to \mathbb{C}$ is $f = \sum_{S \subseteq [n]} \widehat{f}(S) x^S$; $\deg f = \max\{|S| : \widehat{f}(S) \neq 0\}$. For $r \ge 1$ put

$$p_r = \frac{2r}{r+1}, \quad A_r(f) = \Big(\sum_{|S|=r} |\widehat{f}(S)|^{p_r} \Big)^{1/p_r}, \quad \Phi(M, r) = \sup\{A_r(f) : \deg f \le M,\ \|f\|_\infty \le 1\},$$

the supremum over all n and all complex f, and let $\mathrm{BH}(d)$ be the least constant with $(\sum_S |\widehat{f}(S)|^{p_d})^{1/p_d} \le \mathrm{BH}(d) \|f\|_\infty$ whenever $\deg f \le d$. For $s \ge 2$ let $\sigma_s = ((s-1)/(s+1))^{1/2} = (p_s - 1)^{1/2}$, so $\log(1/\sigma_s) = \operatorname{artanh}(1/s)$. Logarithms are natural.

Theorem 1.1. *There is an absolute constant C such that for every $d \ge 1$, with $\lambda(d) = 1 + \log(1 + \log d)$,*

$$\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^2\, \lambda(d)^{1/4}, \tag{1}$$

$$\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^{3/2} \lambda(d)^{3/2} (1 + \log \lambda(d))^{1/4}. \tag{2}$$

In particular $\mathrm{BH}(d) \le C\, d^2 (1 + \log d)^{5/2}$.

Theorem 1.2. *Let $\beta \ge 1$, $K_0 \ge 0$, $C_0 > 0$, and suppose $\Phi(M, 2) \le C_0 M^\beta (1 + \log M)^{K_0}$ for all $M \ge 1$. Then, with C absolute and $C_0' = \max(C_0, \pi/2)$,*

$$\mathrm{BH}(d) \le 1 + C\, C_0'\, d^{1+\beta} (1 + \log d)^{K_0 + 3/2} \big(1 + \log(1 + \log d)\big)^{1/4} \qquad (d \ge 1),$$

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and, with $\lambda(d) = 1 + \log(1 + \log d)$,

$$\mathrm{BH}(d) \le 1 + C\, C_0'\, d^{1+\beta}(1+\log d)^{K_0+1}\lambda(d)^{3/2}(1+\log\lambda(d))^{1/4} \qquad (d \ge 1).$$

Theorem 1.3 (the seed)**.** $\Phi(M,2) \le C\, M\sqrt{\log(M+1)}$ *for all* $M \ge 1$.

The Boolean Bohnenblust–Hille inequality is due to Defant, Mastyło and Pérez [6]. Ivanisvili [8, Theorem 1.1] proved $\mathrm{BH}(d) \le Cd^{27}$. Slote, Tseng and Volberg [11] give a proof scheme for the bound $\mathrm{BH}(d) \le C_\varepsilon d^{3/2+1/\log 2+\varepsilon}$ for every $\varepsilon > 0$. Barbosa, Núñez-Alarcón, Pellegrino and Raposo [1] prove $\mathrm{BH}(d) \le Cd^{5/2}(1+\log d)^3$. Theorem 1.1 is Theorem 1.2 with $\beta = 1$, $K_0 = 1/2$ and Theorem 1.3. Sections 2–5 prove Theorem 1.2 from an arbitrary level-two seed, and Section 6 proves Theorem 1.3.

Outline. Lemma 3.1 aggregates at the ℓ^{p_r} level and passes to ℓ^2 only over the pairs consisting of a block and a block degree. The resulting counting factor is $|T|^{1/r} \le r^{1/r}$, which is what permits the critical stage to begin at level $\lceil 20L^2\log L\rceil$ rather than at a higher power of L. The weight on that stage is $r^{1+1/L}$ with $L = \lfloor 1+\log M\rfloor$: it produces the contraction $1-\kappa^2 \asymp L^{-2}$ and, at degree d, only the factor $d^{1/L} \le e$. Proposition 3.2 is applied twice for (1) and three times for (2). The powers of $\log d$ are then K_0 from the seed, $\frac12$ from the factor $R_*^{1/4}$ with $R_* = \lceil 20L^2\log L\rceil$ (the weight is $r^{5/4}$ below R_*), and 1 from the contraction $1-\kappa^2 \asymp L^{-2}$.

2. Tools

Lemma 2.1 (Bonami–Beckner)**.** *For* $1 \le p \le 2$, $0 \le \rho \le (p-1)^{1/2}$ *and* $g : \{-1,1\}^n \to \mathbb{C}$, $\|T_\rho g\|_2 \le \|g\|_p$, *where* $T_\rho x^S = \rho^{|S|}x^S$. *Hence for* $s \ge 2$, $t \ge 0$ *and every* G,

$$\sigma_s^t\Big(\sum_{|\xi|=t}|\widehat{G}(\xi)|^2\Big)^{1/2} \le \|G\|_{p_s}. \tag{3}$$

Also $\|h\|_q \le (q-1)^{k/2}\|h\|_2$ *for* $q \ge 2$ *and* $\deg h \le k$, *and* $\|h\|_2 \le 3^k\|h\|_1$ *for* $\deg h \le k$.

Proof. The real case of the first claim is the Bonami–Beckner inequality [3, 2], [9, Ch. 9]. For complex g, $T_\rho g(x) = \mathbb{E}_\xi g(x\xi)$ with independent ξ_i, $\mathbb{E}\xi_i = \rho$, so $|T_\rho g| \le T_\rho|g|$ and $\|T_\rho g\|_2 \le \||g|\|_p$. For (3) use $\|T_{\sigma_s}G\|_2 \le \|G\|_{p_s}$ and Parseval. The L^q bound is the dual statement $\|T_{(q-1)^{-1/2}}g\|_q \le \|g\|_2$ applied to $g = T_{(q-1)^{1/2}}h$. For the last claim, $\|h\|_2 \le \|h\|_1^{1/3}\|h\|_4^{2/3} \le \|h\|_1^{1/3}3^{k/3}\|h\|_2^{2/3}$. □

Lemma 2.2 (level one)**.** *If* $\deg F \le M$, *then* $A_1(F) \le \frac{\pi}{2}M\|F\|_\infty$; *if* F *is real,* $\sum_i|\widehat{F}(\{i\})| \le M\|F\|_\infty$.

Proof. Fix x. The function $t \mapsto T_tF(x)$ is a polynomial of degree $\le M$ bounded by $\|F\|_\infty$ on $[-1,1]$ (an average for $t \ge 0$, and $T_{-t}F(x) = T_tF(-x)$). Its derivative at 0 is $F_1(x) = \sum_j \widehat{F}(\{j\})x_j$. The function $\theta \mapsto T_{\cos\theta}F(x)$ is a trigonometric polynomial of order at most $n = \lfloor M\rfloor$. Bernstein's inequality [5, p. 98] gives that the absolute value of its derivative is at most $n\|F\|_\infty$, so at $\theta = \pi/2$ one has $|F_1(x)| \le n\|F\|_\infty \le M\|F\|_\infty$. For real F choose $x_j = \operatorname{sign}\widehat{F}(\{j\})$. For complex F take $x_j = \operatorname{sign}\mathrm{Re}(e^{-i\theta}\widehat{F}(\{j\}))$ and average over θ, using $\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}(e^{-i\theta}a)|\,d\theta = \frac{2}{\pi}|a|$. □

Lemma 2.3 (Blei's inequality for L blocks)**.** *Let* $L \ge 2$, *let* a *be an array on* $E_1 \times \cdots \times E_L$, $\mathbf{d} \in \mathbb{N}_0^L$, $r = |\mathbf{d}|$, $s_j = r - d_j \ge 1$, $\theta_j = s_j/((L-1)r)$, *and*

$$X_j(a) = \Big(\sum_{(i_k)_{k\ne j}}\Big(\sum_{i_j\in E_j}|a_{i_1\dots i_L}|^2\Big)^{p_{s_j}/2}\Big)^{1/p_{s_j}}.$$

Then $\sum_j\theta_j = 1$ *and* $\|a\|_{p_r} \le \prod_j X_j(a)^{\theta_j}$.

Proof. For $P \in [1,\infty)^L$ let $\|a\|_P$ be the iterated norm (ℓ^{P_L} in i_L first, ℓ^{P_1} in i_1 last). If $\theta_j \ge 0$, $\sum_j \theta_j = 1$ and $1/P_k = \sum_j \theta_j / P_k^{(j)}$ for every k, then $\|a\|_P \le \prod_j \|a\|_{P^{(j)}}^{\theta_j}$. For one index this is Hölder's inequality $\|\prod_j b_j^{\theta_j}\|_{\ell^P} \le \prod_j \|b_j\|_{\ell^{P^{(j)}}}^{\theta_j}$ for arbitrary $b_j \ge 0$ (exponents $P^{(j)}/(P\theta_j)$). For the induction step fix i_1, apply the claim to $a(i_1,\cdot)$, and then apply the one-index inequality in i_1 to $b_j(i_1) = \|a(i_1,\cdot)\|_{(P_2^{(j)},\dots,P_L^{(j)})}$. Let $P^{(j)}$ have 2 in position j and p_{s_j} elsewhere. Since $\theta_j/(2s_j) = 1/(2(L-1)r)$, $\frac{\theta_k}{2} + \sum_{j\ne k}\theta_j(\frac12 + \frac{1}{2s_j}) = \frac12 + \frac{1}{2r} = \frac{1}{p_r}$, so $P = (p_r,\dots,p_r)$. Finally, Minkowski's inequality $\|\|F\|_{\ell^q_v}\|_{\ell^2_u} \le \|\|F\|_{\ell^2_u}\|_{\ell^q_v}$ ($q \le 2$) moves the ℓ^2 norm innermost, so $\|a\|_{P^{(j)}} \le X_j(a)$. □

Random partitions. Fix $L \ge 2$ and let $\omega : [n] \to [L]$ be uniform, $I_j = \omega^{-1}(j)$. For $\mathbf{d} \in \mathbb{N}_0^L$ let $a_{\mathbf{d}}(\omega) = (\widehat{f}(S_1 \cup \cdots \cup S_L))_{S_j \subseteq I_j, |S_j| = d_j}$, an array on $\prod_j \{S \subseteq I_j : |S| = d_j\}$, and $X_j(\mathbf{d},\omega) = X_j(a_{\mathbf{d}}(\omega))$.

Lemma 2.4 (capture identity)**.** *For every set $\mathcal{C}$ of multidegrees with $|\mathbf{d}| = r$,*

$$\mathbb{E}_\omega \sum_{\mathbf{d}\in\mathcal{C}} \|a_{\mathbf{d}}(\omega)\|_{p_r}^{p_r} = \eta(\mathcal{C}) A_r(f)^{p_r}, \qquad \eta(\mathcal{C}) = \Pr\big[\mathrm{Mult}(r; \tfrac1L,\dots,\tfrac1L) \in \mathcal{C}\big].$$

Proof. Each S lies in exactly one array, at the multidegree $(|S \cap I_j|)_j$, which for $|S| = r$ has the multinomial law. □

Partial square functions. A *weight* is a function $w : \mathbb{N} \to (0,\infty)$, where $\mathbb{N} = \{1,2,\dots\}$; only its values at $s \le M$ enter G_w. For $N \ge 1$ and $1 \le R \le M$ let

$$G_w(R) = \sup\Big\{ \Big(\sum_{s=1}^{R} \frac{A_s(g)^2}{w(s)^2} \Big)^{1/2} : \ g : \{-1,1\}^n \to \mathbb{C},\ n \le N,\ \deg g \le M,\ \|g\|_\infty \le 1 \Big\}.$$

(M and N are fixed and suppressed.) $G_w(R) < \infty$, since $A_s(g) \le \binom{N}{s}^{1/(2s)} \|g\|_2$ by Hölder.

Lemma 2.5 (section estimate)**.** *Let Q be a function on $\{-1,1\}^I \times \{-1,1\}^J$ with $|I| + |J| \le N$ and $\deg Q \le M$. Put $b_{\alpha,\xi} = \widehat{Q}(\alpha \cup \xi)$ for $\alpha \subseteq I$, $\xi \subseteq J$, and $Y_{s,t} = \big(\sum_{|\alpha|=s} (\sum_{|\xi|=t} |b_{\alpha,\xi}|^2)^{p_s/2}\big)^{1/p_s}$. Then for every weight w and $2 \le R \le M$,*

$$\sum_{s=2}^{R} \sum_{t \ge 0} \frac{\sigma_s^{2t} Y_{s,t}^2}{w(s)^2} \le G_w(R)^2 \|Q\|_\infty^2.$$

Proof. Fix $s \ge 2$, $p = p_s$, and let $g_\alpha(y) = \sum_\xi b_{\alpha,\xi} y^\xi$, the coefficient of x^α in $Q(\cdot,y)$. With $c_{\alpha,t} = \sigma_s^t (\sum_{|\xi|=t} |b_{\alpha,\xi}|^2)^{1/2}$ we have $\|c_{\alpha,\cdot}\|_{\ell^2} = \|T_{\sigma_s} g_\alpha\|_2 \le \|g_\alpha\|_p$ by Lemma 2.1, so Minkowski's inequality ($p \le 2$) gives

$$\sum_t \sigma_s^{2t} Y_{s,t}^2 = \big\|(\|c_{\cdot,t}\|_{\ell^p})_t\big\|_{\ell^2}^2 \le \big\|(\|c_{\alpha,\cdot}\|_{\ell^2})_\alpha\big\|_{\ell^p}^2 \le \Big(\sum_\alpha \|g_\alpha\|_p^p\Big)^{2/p} = \big(\mathbb{E}_y A_s(Q(\cdot,y))^p\big)^{2/p},$$

which is at most $\mathbb{E}_y A_s(Q(\cdot,y))^2$. Divide by $w(s)^2$ and sum over $2 \le s \le R$. Each section $Q(\cdot,y)$ has degree $\le M$, at most N coordinates and sup norm $\le \|Q\|_\infty$, so $\mathbb{E}_y \sum_{s\le R} A_s(Q(\cdot,y))^2/w(s)^2 \le G_w(R)^2 \|Q\|_\infty^2$. □

Lemma 2.6 (binomial tails)**.** *If $Z \sim \mathrm{Bin}(r, 1/L)$, $\mu = r/L$ and $0 < a < 1$, then $\Pr[|Z - \mu| > a\mu] \le 2e^{-c_a \mu}$, $c_a = a^2(3-a)/6$.*

Proof. $\mathbb{E}e^{\tau Z} \le e^{\mu(e^\tau - 1)}$. With $\tau = \log(1+a)$, resp. $\tau = \log(1-a) < 0$, Markov's inequality for $e^{\tau Z}$ bounds the upper, resp. lower, tail by $e^{-\mu\phi(a)}$, resp. $e^{-\mu\psi(a)}$, where $\phi(a) = (1+a)\log(1+a) - a \ge \frac{a^2}{2} - \frac{a^3}{6}$ and $\psi(a) = (1-a)\log(1-a) + a \ge \frac{a^2}{2}$. □

3. Degree-marginal aggregation and the stage inequality

Fix $L \geq 2$ and $r \geq 4$. A *window* is a nonempty set $\mathcal{C}$ of multidegrees $\mathbf{d} \in \mathbb{N}_0^L$ with $|\mathbf{d}| = r$ and $1 \leq d_j \leq r-2$ for all j. For $\mathbf{d} \in \mathcal{C}$ let $s_j = r - d_j \geq 2$ and $\theta_j = \theta_j(\mathbf{d}) = s_j/((L-1)r)$ as in Lemma 2.3, and $D(\mathbf{d}) = \sum_j \theta_j \log(L\theta_j) \geq 0$ (the relative entropy of θ with respect to the uniform law on $[L]$). Let $T(\mathcal{C}) = \{d_j : \mathbf{d} \in \mathcal{C},\ 1 \leq j \leq L\}$. For a weight w define

$$\mathsf{M}_w(r,\mathbf{d}) = \prod_{j=1}^{L} \Big(\frac{\sigma_{s_j}^{-d_j}\, w(s_j)}{w(r)}\Big)^{\theta_j}, \qquad \kappa_w(r,\mathcal{C})^2 = \eta(\mathcal{C})^{-(1+\frac{1}{r})}\, |T(\mathcal{C})|^{\frac{1}{r}} \max_{\mathbf{d}\in\mathcal{C}} \mathsf{M}_w(r,\mathbf{d})^2 e^{(1+\frac{1}{r})D(\mathbf{d})}.$$

For a splitting $[n] = I \sqcup J$ write $Y_{s,t}(f;J)$ for the quantity $Y_{s,t}$ of Lemma 2.5 with $Q = f$.

Lemma 3.1 (degree-marginal aggregation)**.** *For every $f : \{-1,1\}^n \to \mathbb{C}$, every window $\mathcal{C}$ at level r and every weight w,*

$$\frac{A_r(f)^2}{w(r)^2} \leq \kappa_w(r,\mathcal{C})^2 \cdot \frac{1}{L}\, \mathbb{E}_\omega \sum_{j=1}^{L} \sum_{t\in T(\mathcal{C})} \frac{\sigma_{r-t}^{2t}\, Y_{r-t,t}(f;I_j)^2}{w(r-t)^2}.$$

Proof. Write $p = p_r$ and fix ω. For every $\mathbf{d}$ with $|\mathbf{d}| = r$ and every j with $1 \leq d_j \leq r-2$ put $U_j(\mathbf{d}) = \sigma_{s_j}^{d_j} X_j(\mathbf{d},\omega)/w(s_j)$, where $s_j = r - d_j$. For $\mathbf{d} \in \mathcal{C}$, Lemma 2.3 gives $\|a_{\mathbf{d}}\|_p \leq \prod_j X_j^{\theta_j} = w(r)\mathsf{M}_w(r,\mathbf{d}) \prod_j U_j^{\theta_j}$. By the weighted AM–GM inequality,

$$\prod_j (U_j^p)^{\theta_j} = \prod_j (L\theta_j)^{\theta_j} \prod_j \Big(\frac{U_j^p}{L\theta_j}\Big)^{\theta_j} \leq e^{D(\mathbf{d})} \sum_j \theta_j \frac{U_j^p}{L\theta_j} = \frac{e^{D(\mathbf{d})}}{L} \sum_j U_j^p.$$

Hence, with $\mu = \max_{\mathbf{d}\in\mathcal{C}} \mathsf{M}_w(r,\mathbf{d})^p e^{D(\mathbf{d})}$,

$$\sum_{\mathbf{d}\in\mathcal{C}} \|a_{\mathbf{d}}\|_p^p \leq \frac{\mu\, w(r)^p}{L} \sum_{j=1}^{L} \sum_{t\in T} \sum_{\mathbf{d}\in\mathcal{C},\ d_j = t} U_j(\mathbf{d})^p. \tag{4}$$

Fix j and $t \in T$, and let $s = r - t$; every $\mathbf{d}$ in the inner sum has $s_j = s$. The sets $\alpha \subseteq [n] \setminus I_j$ with $|\alpha| = s$ are partitioned according to the profile $(|\alpha \cap I_k|)_{k\neq j}$, and the part of profile $(d_k)_{k\neq j}$ is exactly the outer index set of $X_j(\mathbf{d},\omega)$. Hence $\sum_{\mathbf{d}:|\mathbf{d}|=r,\, d_j=t} X_j(\mathbf{d},\omega)^{p_s} = Y_{s,t}(f;I_j)^{p_s}$, and since $p_s \leq p$,

$$\sum_{\mathbf{d}\in\mathcal{C},\ d_j=t} U_j(\mathbf{d})^p \leq \Big(\sum_{\mathbf{d}:\,|\mathbf{d}|=r,\, d_j=t} U_j(\mathbf{d})^{p_s}\Big)^{p/p_s} = V_{j,t}(\omega)^p, \qquad V_{j,t} = \frac{\sigma_s^t\, Y_{s,t}(f;I_j)}{w(s)}.$$

Take $\mathbb{E}_\omega$ in (4). By Lemma 2.4 the left side becomes $\eta(\mathcal{C})A_r(f)^p$. On the right, Hölder's inequality for the measure $\Pr \otimes$(counting on $[L]\times T$), of total mass $L|T|$, with exponents $2/p$ and $2/(2-p)$, gives $\mathbb{E}\sum_{j,t} V_{j,t}^p \leq (L|T|)^{1-p/2}(\mathbb{E}\sum_{j,t} V_{j,t}^2)^{p/2}$. Therefore

$$A_r(f)^p \leq \eta^{-1}\mu\, w(r)^p L^{-1}(L|T|)^{1-p/2}\Big(\mathbb{E}\sum_{j,t} V_{j,t}^2\Big)^{p/2}.$$

Raise this to the power $2/p = 1 + \frac{1}{r}$. Then $\mu^{2/p} = \max \mathsf{M}_w^2 e^{(1+1/r)D}$ and $L^{-2/p}(L|T|)^{2/p-1} = L^{-1}|T|^{1/r}$. □

Proposition 3.2 (stage inequality)**.** *Let w be a weight, $4 \leq R \leq R' \leq M$, $L \geq 2$, and for each $r \in [R,R']$ let $\mathcal{C}_r$ be a window at level r with L blocks such that $\kappa_w(r,\mathcal{C}_r) \leq \kappa$. Then*

$\sum_{r=R}^{R'} A_r(f)^2/w(r)^2 \le \kappa^2 G_w(R'-1)^2\|f\|_\infty^2$ *for every* f *with* $\deg f \le M$ *on* $n \le N$ *coordinates. Consequently, if* $\kappa < 1$,

$$G_w(R')^2 \le \frac{G_w(R-1)^2}{1-\kappa^2}.$$

Proof. Sum Lemma 3.1 over $r \in [R, R']$ and exchange the sum with $\mathbb{E}_\omega$ and $\frac{1}{L}\sum_j$. For fixed ω and j, the pairs $(s,t) = (r-t,t)$ with $r \in [R,R']$ and $t \in T(\mathcal{C}_r)$ are distinct and satisfy $2 \le s \le R'-1$. So Lemma 2.5, applied to $Q = f$ with $J = I_j$, bounds their contribution by $G_w(R'-1)^2\|f\|_\infty^2$. For the second claim, take the supremum over g of $\sum_{s<R} A_s(g)^2/w(s)^2 + \sum_{s=R}^{R'} A_s(g)^2/w(s)^2$; this gives $G_w(R')^2 \le G_w(R-1)^2 + \kappa^2 G_w(R'-1)^2 \le G_w(R-1)^2 + \kappa^2 G_w(R')^2$, and $G_w(R') < \infty$. □

4. Windows, fixed levels and the two stages

For $L \ge 3$, $0 < a \le \frac{1}{2}$ and $r \ge 4L$ let $\mathcal{C}_r(L,a) = \{\mathbf{d} \in \mathbb{N}_0^L : |\mathbf{d}| = r,\ |d_j - r/L| \le ar/L\ \forall j\}$.

Lemma 4.1 (the multiplier)**.** *Let* $L \ge 3$, $0 < a \le \frac{1}{2}$, $b \ge 1$, $r \ge 4L$ *with* $ar \ge L$, *and let* w *satisfy* $w(s)/w(r) \le (s/r)^b$ *for* $r/2 \le s \le r$. *Then* $\mathcal{C} = \mathcal{C}_r(L,a)$ *is a window,* $|T(\mathcal{C})| \le 2ar/L + 1$, $\eta(\mathcal{C}) \ge 1 - 2Le^{-c_a r/L}$, *and for* $\mathbf{d} \in \mathcal{C}$

$$2\log \mathsf{M}_w(r,\mathbf{d}) + \big(1+\tfrac{1}{r}\big)D(\mathbf{d}) \le \frac{2}{(L-1)(1-4/r^2)} - 2b\log\frac{L}{L-1} + \big(2b+1+\tfrac{1}{r}\big)\frac{a^2}{(L-1)^2}.$$

Proof. For $\mathbf{d} \in \mathcal{C}$ we have $d_j \ge (1-a)r/L \ge r/(2L) \ge 2$ and $s_j = r - d_j \ge r(1 - \frac{1+a}{L}) \ge \frac{r}{2} \ge 2$. $\mathcal{C}$ is nonempty: it contains a vector with entries $\lfloor r/L \rfloor$ and $\lceil r/L \rceil$, since $ar/L \ge 1$. So $\mathcal{C}$ is a window. The set T consists of integers in an interval of length $2ar/L$. By the union bound and Lemma 2.6, $1 - \eta \le \sum_j \Pr[|d_j - r/L| > ar/L] \le 2Le^{-c_a r/L}$. Next, $\log \mathsf{M}_w = P_1 + P_2$ with $P_1 = \sum_j \theta_j d_j \operatorname{artanh}(1/s_j)$ and $P_2 = \sum_j \theta_j \log(w(s_j)/w(r))$. Using $\operatorname{artanh} u \le u/(1-u^2)$, $s_j \ge r/2$ and $\theta_j/s_j = 1/((L-1)r)$, we get $P_1 \le (1-4/r^2)^{-1}\sum_j d_j/((L-1)r) = (1-4/r^2)^{-1}/(L-1)$. As $s_j/r = (L-1)\theta_j \in [\frac{1}{2}, 1]$, the hypothesis gives $P_2 \le b\sum_j \theta_j \log((L-1)\theta_j) = b(D - \log\frac{L}{L-1})$. Finally $\log x \le x - 1$ gives $D \le \sum_j \theta_j(L\theta_j - 1) = L\sum_j(\theta_j - \frac{1}{L})^2$, and $\theta_j - \frac{1}{L} = -(d_j/r - \frac{1}{L})/(L-1)$, so $D \le \frac{L}{(L-1)^2}\cdot L \cdot \frac{a^2}{L^2} = \frac{a^2}{(L-1)^2}$. Combine: $2P_1 + 2P_2 + (1+\frac{1}{r})D \le 2P_1 - 2b\log\frac{L}{L-1} + (2b+1+\frac{1}{r})D$. □

Proposition 4.2 (stage one)**.** *Let* $L = 3$, $a = \frac{1}{20}$, $b = \frac{5}{4}$, *and let* w *satisfy* $w(s)/w(r) \le (s/r)^{5/4}$ *for* $r/2 \le s \le r$. *Then* $\kappa_w(r, \mathcal{C}_r(3, \frac{1}{20}))^2 \le e^{-1/125}$ *for every* $r \ge R_0 := 25000$.

Proof. Lemma 4.1 applies since $r \ge 25000 \ge \max(4L, L/a) = 60$; it bounds the multiplier term by $(1-4/r^2)^{-1} - \frac{5}{2}\log\frac{3}{2} + (\frac{7}{2} + \frac{1}{r})\frac{1}{1600} \le 1 + 10^{-8} - 1.013662 + 0.002188 \le -0.01147$. Here $c_a = \frac{59}{48000} \ge 0.001229$, so $c_a r/3 \ge 10.24$ and $1 - \eta \le 6e^{-10.24} \le 2.2 \cdot 10^{-4}$; the capture term is $-(1+\frac{1}{r})\log\eta \le (1+\frac{1}{r})\frac{1-\eta}{\eta} \le 2.3 \cdot 10^{-4}$. The counting term is $\frac{1}{r}\log|T| \le \frac{1}{r}\log(\frac{r}{30}+1) \le 2.7 \cdot 10^{-4}$, since $x \mapsto \frac{1}{x}\log(\frac{x}{30}+1)$ is decreasing for $x \ge R_0$. The total is at most $-0.01097 < -\frac{1}{125}$. □

Proposition 4.3 (stage two)**.** *Let* $L \ge 40$, $a = \frac{1}{4}$, $b = 1 + \frac{1}{L}$, *and let* w *satisfy* $w(s)/w(r) \le (s/r)^{1+1/L}$ *for* $r/2 \le s \le r$. *Then* $\kappa_w(r, \mathcal{C}_r(L, \frac{1}{4}))^2 \le e^{-1/(2L^2)}$ *for every* $r \ge R_L := \lceil 20L^2 \log L \rceil$.

Proof. Lemma 4.1 applies since $R_L \ge 4L = L/a$. Let $u = \frac{1}{L-1}$. Since $2u - 2\log(1+u) \le u^2$ and $\frac{2}{L}\log\frac{L}{L-1} \ge \frac{2}{L^2}$, Lemma 4.1 bounds the multiplier term by

$$u^2 + \frac{10u}{r^2} - \frac{2}{L^2} + \Big(3 + \frac{2}{L} + \frac{1}{r}\Big)\frac{u^2}{16} \le 1.1907\,u^2 - \frac{2}{L^2} + \frac{10u}{r^2} \le -\frac{0.747}{L^2} + \frac{10^{-7}}{L^2},$$

using $(1-4/r^2)^{-1} \le 1 + 5/r^2$, $(L/(L-1))^2 \le (40/39)^2 \le 1.052$ and $r^2 \ge 400L^4\log^2 L$. Here $c_{1/4} = \frac{11}{384}$, so $c_a r/L \ge 0.57\,L\log L$ and $1 - \eta \le 2L \cdot L^{-0.57L} \le 10^{-30}/L^2$; the capture term is

below $10^{-29}/L^2$. The counting term is $\frac{1}{r}\log|T| \le \frac{\log r}{r} \le \frac{\log R_L}{R_L}$, and $\log R_L \le \log(21L^2\log L) \le 3.045 + 2\log L + \log\log L$, so

$$\frac{\log R_L}{R_L} \le \frac{1}{20L^2}\Big(2 + \frac{3.045 + \log\log L}{\log L}\Big) \le \frac{0.16}{L^2}.$$

The total is at most $-0.58/L^2 < -1/(2L^2)$. □

Remark 4.4. The proofs of Propositions 4.2 and 4.3 use only Lemma 4.1. For the pure power $w(s) = s^b$, the objective $2\log \mathsf{M}_w(r,\mathbf{d}) + (1+\frac{1}{r})D(\mathbf{d})$ is $\sum_j \chi(s_j)$ with χ convex, so it is largest at a vertex of the window, but that observation is not required above. Explicitly, with $c = 1/((L-1)r)$ and $\Psi(s) = s\,\mathrm{artanh}(1/s) = \sum_{k\ge 0} s^{-2k}/(2k+1)$, one has $\chi(s) = c[2(r-s)\Psi(s) + 2bs\log(s/r) + (1+\frac{1}{r})s\log(Lcs)]$ and $\chi'' = c[(2b+1+\frac{1}{r})/s + 2(r-s)\Psi'' - 4\Psi'] > 0$, since $\Psi' < 0 < \Psi''$. Any weight satisfying the hypothesis of Lemma 4.1 has no larger objective than this pure power, because $s_j \ge r/2$ on the window.

Proposition 4.5 (fixed levels)**.** *Let $S(M) \ge \frac{\pi}{2}M$ satisfy $\Phi(M,2) \le S(M)$. Define $D_1 = D_2 = 1$, $D_3 = (9/2)^{2/3}\sqrt{3}$, and $D_r = \pi_r^{-1/p_r}(\sigma_2\sigma_{r-2})^{-2(r-2)/r} D_{r-2}^{(r-2)/r}$ for $r \ge 4$, where $\pi_r = \binom{r}{2}2^{-r}$. Then $A_r(f) \le D_r S(M)\|f\|_\infty$ for all $r \ge 1$ and $\deg f \le M$.*

Proposition 2.6 of [1] is the case of a level-two bound of order $M^{3/2}$, for functions on a finite local system. The statement here is the Boolean case with an arbitrary level-two bound $S(M)$.

Proof. Let $\|f\|_\infty \le 1$. The cases $r = 1, 2$ are Lemma 2.2 and the hypothesis. For $r = 3$ take $L = 3$ and $\mathbf{d} = (1,1,1)$, so $s_j = 2$ and $\theta_j = \frac{1}{3}$. Fix j, and for each $\mathbf{S} = (S_k)_{k\ne j}$ with $|S_k| = 1$ let $G_{\mathbf{S}}(x_j) = \sum_{S\subseteq I_j} \widehat{f}(S \cup \bigcup_{k\ne j} S_k) x_j^S$. By (3), $\sigma_2 X_j \le (\sum_{\mathbf{S}} \|G_{\mathbf{S}}\|_{p_2}^{p_2})^{1/p_2}$. For fixed x_j the numbers $G_{\mathbf{S}}(x_j)$ are level-two coefficients of a section of f, so $\sum_{\mathbf{S}} |G_{\mathbf{S}}(x_j)|^{p_2} \le S(M)^{p_2}$ and $X_j \le \sqrt{3}S(M)$. Lemmas 2.3 and 2.4 (with $\eta = \frac{2}{9}$) give $A_3 \le (9/2)^{1/p_3}\sqrt{3}S(M)$. For $r \ge 4$ take $L = 2$ and $\mathbf{d} = (2, r-2)$. The same argument bounds the two Blei factors by $\sigma_{r-2}^{-2} D_{r-2} S(M)$ (sections in the variables of I_2, by induction) and $\sigma_2^{-(r-2)} S(M)$ (sections in the variables of I_1); their geometric mean with weights $\frac{r-2}{r}, \frac{2}{r}$ and the capture $\eta = \pi_r$ give the recursion. □

5. Proof of Theorem 1.2

Let $S(M) = C_0' M^\beta (1+\log M)^{K_0}$; then $S(M) \ge \frac{\pi}{2}M$ and $\Phi(M,2) \le S(M)$. A function on the cube has integer degree, so $\Phi(t,r)$ depends on t only through $\lfloor t \rfloor$. Replace M by $\lfloor M \rfloor$ and take M to be a positive integer. In the application to $\mathrm{BH}(d)$ one has $M = d$, already an integer, so this replacement does not change the bounds below. Put $H^2 = \sum_{r<R_0} D_r^2 r^{-5/2}$, an absolute constant. Fix $N \ge 1$ and let $L_* = \lfloor 1 + \log M \rfloor$.

The weight. If $L_* \le 39$, let $R_* = M$ and $w(r) = r^{5/4}$. If $L_* \ge 40$, let $R_* = R_{L_*} = \lceil 20L_*^2 \log L_* \rceil$; then $118000 < R_* \le 21L_*^2\log L_* < e^{L_*-1} \le M$. Let

$$w(r) = r^{5/4}\ (r \le R_*), \qquad w(r) = R_*^{5/4}(r/R_*)^{1+1/L_*}\ (r > R_*).$$

For $L_* \ge 40$, $\log w$ is a piecewise linear function of $\log r$ with slopes $\frac{5}{4} \ge 1 + \frac{1}{L_*}$, so $w(s)/w(r) \le (s/r)^{1+1/L_*}$ for all $s \le r$. In both cases w is increasing and $w(s)/w(r) = (s/r)^{5/4}$ for $s \le r \le R_*$.

The stages. By Proposition 4.5, $G_w(\min(M, R_0-1))^2 \le \sum_{r<R_0} D_r^2 S(M)^2/r^{5/2} = H^2 S(M)^2$. If $R_* \ge R_0$, Proposition 3.2 with three blocks and Proposition 4.2 on the levels $[R_0, R_*]$ give $G_w(R_*)^2 \le H^2 S(M)^2/(1-e^{-1/125}) \le 126 H^2 S(M)^2$. If $L_* \ge 40$, Proposition 3.2 with L_* blocks and Proposition 4.3 with $L = L_*$ on the levels $[R_*+1, M]$ give

$$G_w(M)^2 \le \frac{G_w(R_*)^2}{1 - e^{-1/(2L_*^2)}} \le 4L_*^2 G_w(R_*)^2 \le 504\, L_*^2 H^2 S(M)^2.$$

In all cases $G_w(M) \le 23\, L_* HS(M)$.

Completion. Let $\deg f \le d$, $\|f\|_\infty \le 1$, and take $M = d$ and $N = n$. Since $p_r \le p_d$ for $r \le d$, $\sum_{S\neq\emptyset} |\widehat{f}(S)|^{p_d} \le \sum_{r=1}^{d} A_r(f)^{p_d}$. Hölder's inequality over the d levels and the monotonicity of w then give

$$\|\widehat{f}\|_{p_d} \le 1 + d^{\frac{1}{2d}} \Big(\sum_{r=1}^{d} A_r(f)^2 \Big)^{1/2} \le 1 + e^{1/(2e)} w(d)\, G_w(d).$$

If $L_* \le 39$ then $w(d) \le e^{10} d$. If $L_* \ge 40$ then $w(d) \le R_*^{1/4} d \cdot d^{1/L_*} \le e\, R_*^{1/4} d$, since $L_* \ge \log d$, and $R_* \le 21 L_*^2 \log L_*$. With $L_* \le 1 + \log d$,

$$\mathrm{BH}(d) \le 1 + C\, d\, L_*^{3/2} (\log(1+L_*))^{1/4} S(d) \le 1 + C\, C_0'\, d^{1+\beta} (1+\log d)^{K_0+3/2} \big(1 + \log(1+\log d)\big)^{1/4}.$$

This is the first bound of Theorem 1.2.

Three stages. Let $L_3 = L_*$. If $L_3 \ge 40$, let $R_3 = R_{L_3}$, $L_2 = \lceil \log R_3 \rceil$ and $R_2 = R_{L_2}$. Suppose $L_3 \ge 40$ and $L_2 \ge 40$; otherwise $\log d < e^{20}$, and the first bound already implies the second. Then $L_2 < L_3$, $R_0 < R_2 < R_3 < M$, and we use

$$w(r) = \begin{cases} r^{5/4} & (r \le R_2), \\ R_2^{5/4} (r/R_2)^{1+1/L_2} & (R_2 < r \le R_3), \\ w(R_3)(r/R_3)^{1+1/L_3} & (r > R_3). \end{cases}$$

The slopes $\frac{5}{4} \ge 1 + \frac{1}{L_2} \ge 1 + \frac{1}{L_3}$ decrease. So Proposition 3.2 applies with Proposition 4.2 on $[R_0, R_2]$, with Proposition 4.3 for $L = L_2$ on $[R_2 + 1, R_3]$, and with Proposition 4.3 for $L = L_3$ on $[R_3 + 1, M]$. Hence $G_w(M) \le \sqrt{126}\, HS(M) \cdot 2L_2 \cdot 2L_3$, and $w(d) \le d\, R_2^{1/4} R_3^{1/L_2} d^{1/L_3} \le e^2 R_2^{1/4} d$. Since $L_3 \le 1 + \log d$, $L_2 \le 1 + \log(21 L_3^2 \log L_3) \le 5 + 3 \log L_3 \le 5\lambda(d)$ and $R_2 \le 21 L_2^2 \log L_2 \le C\lambda(d)^2(1 + \log \lambda(d))$, this gives the second bound of Theorem 1.2. □

6. The seed: proof of Theorem 1.3

Symbols introduced in this section are local to it. The block-constant estimate is proved directly, by Bernstein–Walsh on complex lines; polarization is not used. All functions in this section are real-valued. For complex f, apply the result to $\mathrm{Re} f$ and $\mathrm{Im} f$, whose Walsh coefficients are $\mathrm{Re}\widehat{f}$ and $\mathrm{Im}\widehat{f}$, so $\Phi_{\mathbb{C}}(M, 2) \le 2\Phi_{\mathbb{R}}(M, 2)$, where $\Phi_{\mathbb{C}}$ and $\Phi_{\mathbb{R}}$ denote Φ with the supremum over complex, resp. real, f. For real g with $|g| \le 1$ and $\deg g \le M$ we use three facts.

(a) $\sum_i |\hat{g}(\{i\})| \le M$ (Lemma 2.2).

(b) (Level-one inequality.) If $\alpha = \|g\|_1 > 0$, then $\|P_1 g\|_2 \le C_1 \alpha \sqrt{\log(e/\alpha)}$ with $C_1 = 1 + \sqrt{2}$, where $P_1 g = \sum_i \hat{g}(\{i\}) x_i$. The same shape of bound is classical [12, 4], [9, Sec. 5.4]; the constant used here is proved as follows. If $h = P_1 g = 0$ there is nothing to prove. Otherwise put $Z = |h|/\|h\|_2$. Hoeffding's inequality [7, Theorem 2] says that if Y_i are independent and $a_i \le Y_i \le b_i$, then $\Pr[S - \mathbb{E}S \ge t] \le \exp\big(-2t^2/\sum_i (b_i - a_i)^2\big)$ for $t > 0$. Write $h = \sum_i c_i x_i$ with $x_i \in \{-1, 1\}$. Each summand lies in $[-|c_i|, |c_i|]$, so $\sum_i (b_i - a_i)^2 = 4\|h\|_2^2$. The one-sided bound and the same bound for $-h$ give $\Pr[|h| \ge u\|h\|_2] \le 2e^{-u^2/2}$, that is $\Pr[Z > u] \le 2e^{-u^2/2}$. Since $0 \le |g| \le 1$ and $\mathbb{E}|g| = \alpha$,

$$\|h\|_2^2 = \mathbb{E}[gh] \le \mathbb{E}[|g||h|] = \|h\|_2\, \mathbb{E}[|g|Z],$$

so $\|h\|_2 \le \mathbb{E}[|g|Z] \le \int_0^\infty \min(\alpha, 2e^{-u^2/2})\, du$. With $u_0 = (2\log(2/\alpha))^{1/2}$ the integral up to u_0 is at most αu_0. The tail satisfies $\int_{u_0}^\infty 2e^{-u^2/2}\, du \le 2u_0^{-1} e^{-u_0^2/2} = \alpha/u_0$: indeed $\frac{d}{du} e^{-u^2/2} = -ue^{-u^2/2}$, so $\int_{u_0}^\infty u e^{-u^2/2}\, du = e^{-u_0^2/2}$, and $u \ge u_0$ on the range gives $\int_{u_0}^\infty e^{-u^2/2}\, du \le u_0^{-1} e^{-u_0^2/2}$, while $e^{-u_0^2/2} = \alpha/2$. For $0 < \alpha \le 1$ one has $u_0 \ge 1$ and $\log(e/\alpha) \ge 1$,

and $u_0^2 = 2\log(2/\alpha) \le 2\log(e/\alpha)$, so $u_0 + 1/u_0 \le u_0 + 1 \le (1+\sqrt{2})\sqrt{\log(e/\alpha)}$. Thus $\|h\|_2 \le C_1\alpha\sqrt{\log(e/\alpha)}$.

(c) (Blei–Littlewood; Lemma 2.3 with $L = 2$, $\mathbf{d} = (1,1)$.) For every finite array, $\|a\|_{4/3} \le (\sum_u \|a_{u\cdot}\|_2)^{1/2}(\sum_i \|a_{\cdot i}\|_2)^{1/2}$.

Lemma 6.1 (Bernstein–Walsh on complex lines)**.** *Let $F : \{-1,1\}^J \to \mathbb{C}$, $\deg F \le M$, $\|F\|_\infty \le 1$, and let F also denote its multi-affine extension to $\mathbb{C}^J$. Let $z = a + ib$ with $a \in [-1,1]^J$, $b \in \mathbb{R}^J \setminus \{0\}$, and suppose that $\Lambda = \min_{i:b_i\neq 0}(1-|a_i|)/|b_i| > 0$. Then $|F(z)| \le e^{M/\Lambda}$.*

Proof. $F_{a,b}(\zeta) = F(a+\zeta b)$ is a polynomial of degree $\le M$ in $\zeta \in \mathbb{C}$. For real $|\zeta| \le \Lambda$ the point $a + \zeta b$ lies in $[-1,1]^J$, where F is an average of its values on the cube; so $|F_{a,b}| \le 1$ on $[-\Lambda, \Lambda]$. By the Bernstein–Walsh inequality [10, Thm. 5.5.7], $|F_{a,b}(\zeta)| \le |\omega + \sqrt{\omega^2-1}|^M$ with $\omega = \zeta/\Lambda$ and the branch of modulus ≥ 1. (Apply the maximum principle to $F_{a,b}(\Lambda\omega)(\omega+\sqrt{\omega^2-1})^{-M}$ on $(\mathbb{C}\cup\{\infty\})\setminus[-1,1]$; this holds for complex coefficients.) At $\zeta = i$ this is $(\Lambda^{-1} + \sqrt{1+\Lambda^{-2}})^M = e^{M\,\mathrm{arsinh}(1/\Lambda)} \le e^{M/\Lambda}$. □

Theorem 6.2 (block-constant test forms)**.** *Let $|B| \le 1$ on $\{-1,1\}^J$, $\deg B \le M$, $\alpha = \|B\|_1$. Let $G_1, \dots, G_m \subseteq J$ be disjoint with $|G_j| = n_j \ge n_0$, let $\varepsilon_i \in \{\pm 1\}$ and $\lambda_j \ge 0$ with $\sum_j \lambda_j^2 = 1$, and let $\mathcal{L} = \sum_j \lambda_j \sum_{i\in G_j} \varepsilon_i x_i/\sqrt{n_j}$. Put $\gamma = \mathbb{E}[B\mathcal{L}]/\alpha$ and $C' = 2e$. If $\alpha > 0$ and $K \ge 2$ is an integer with $\gamma \ge 18\sqrt{K}$ and $Kn_0 \ge 4C'^2M^2$, then $\alpha \le 2\,(C'M/\sqrt{Kn_0})^K$.*

Proof. Replacing x_i by $\varepsilon_i x_i$, we may assume $\varepsilon \equiv 1$. Let $\bar{B}$ be the average of B over $\prod_j \mathrm{Sym}(G_j)$, acting on the coordinates. Then $|\bar{B}| \le 1$, $\deg \bar{B} \le M$, $\|\bar{B}\|_1 \le \alpha$, and $\mathbb{E}[\bar{B}\mathcal{L}] = \mathbb{E}[B\mathcal{L}] = \gamma\alpha$, since $\mathcal{L}$ is invariant. Fix the coordinates outside the blocks in $\bar{B}$ and call the resulting function g of the block variables; g is invariant, $|g| \le 1$ and $\deg g \le M$. So its level-k part is $Q_k = \sum_{|\beta|=k} a_\beta \prod_j e_{\beta_j}(x_{G_j})$, where $\beta \in \mathbb{N}_0^m$ and e_b is the elementary symmetric polynomial; hence $\|Q_k\|_2^2 = \sum_\beta |a_\beta|^2 \prod_j \binom{n_j}{\beta_j}$.

For $t \in \mathbb{C}$ and $y \in \mathbb{T}^m$ let $z(t,y)$ have coordinates $z_i = ty_j$ for $i \in G_j$. Since $e_b(ty_j, \dots, ty_j) = \binom{n_j}{b} t^b y_j^b$, the multi-affine extension satisfies

$$g(z(t,y)) = \sum_{k\le M} t^k h_k(y), \qquad h_k(y) = \sum_{|\beta|=k} c_\beta y^\beta, \qquad c_\beta = a_\beta \prod_j \binom{n_j}{\beta_j}.$$

If $|t| = \delta \le \frac{1}{2}$, each coordinate of $z(t,y)$ has real and imaginary parts of modulus $\le \delta$. So either $z(t,y)$ is real and $|g(z(t,y))| \le 1$, or Lemma 6.1 applies with $\Lambda \ge (1-\delta)/\delta$ and gives $|g(z(t,y))| \le e^{M\delta/(1-\delta)} \le e^{2M\delta}$. Cauchy's estimate in t with $\delta = k/(2M) \le \frac{1}{2}$ gives $|h_k(y)| \le (2M/k)^k e^k = (C'M/k)^k$ for $y \in \mathbb{T}^m$ and $1 \le k \le M$, and Parseval on $\mathbb{T}^m$ gives $\sum_\beta |c_\beta|^2 \le (C'M/k)^{2k}$. Since $\prod_j \binom{n_j}{\beta_j} \ge \prod_j (n_j/\beta_j)^{\beta_j} \ge (n_0/k)^k$,

$$\|Q_k\|_2^2 = \sum_\beta \frac{|c_\beta|^2}{\prod_j \binom{n_j}{\beta_j}} \le \Big(\frac{k}{n_0}\Big)^k \Big(\frac{C'M}{k}\Big)^{2k}, \qquad \text{i.e.} \qquad \|Q_k\|_2 \le \Big(\frac{C'M}{\sqrt{kn_0}}\Big)^k.$$

Now $\mathcal{L}$ has level one and $\|\mathcal{L}\|_2 = 1$. Let $q = C'M/\sqrt{Kn_0} \le \frac{1}{2}$. For $\deg\psi < K$, Hölder's inequality and Lemma 2.1 give $\|\psi\|_{1+1/K} \le \|\psi\|_1^{(K-1)/(K+1)} \|\psi\|_2^{2/(K+1)} \le 9\|\psi\|_1$, and Lemma 2.1 gives $\|\mathcal{L}\|_{K+1} \le \sqrt{K}$. Hence

$$\mathbb{E}[g\mathcal{L}] = \mathbb{E}[g^{<K}\mathcal{L}] \le 9\sqrt{K}\,\|g^{<K}\|_1 \le 9\sqrt{K}\Big(\|g\|_1 + \sum_{k\ge K} \|Q_k\|_2\Big) \le 9\sqrt{K}(\|g\|_1 + 2q^K).$$

Averaging over the fixed coordinates gives $\gamma\alpha = \mathbb{E}[\bar{B}\mathcal{L}] \le 9\sqrt{K}(\|\bar{B}\|_1 + 2q^K) \le 9\sqrt{K}(\alpha + 2q^K)$, and $\gamma \ge 18\sqrt{K}$ yields $\alpha \le 2q^K$. □

Theorem 6.3 (decomposition of the level-one tail)**.** *Let $M \ge 1$ and $K \ge 2$ be integers and $B : \{-1,1\}^J \to \mathbb{R}$ with $|B| \le 1$, $\deg B \le M$ and $\alpha = \|B\|_1 > 0$. Let v be the level-one coefficient vector of B and $s_v = \|v\|_2$. The set $\mathcal{T}$ of coordinates outside a fixed choice of the $\min(M^2, |J|)$ largest $|v_i|$ splits as $\mathcal{T} = T' \sqcup T''$ with*

$$\|v|_{T'}\|_2 \le 36\sqrt{K}\,\alpha, \qquad \|v|_{T''}\|_{4/3}^{4/3} \le C_2 M^{2/3} s_v^{4/3}\Big(1 + \frac{1}{K}\Big(\frac{2}{\alpha}\Big)^{2/K}\Big), \qquad C_2 = 24C'^2.$$

Proof. If $|J| \le M^2$, then $\mathcal{T} = \emptyset$ and there is nothing to prove. Otherwise every entry of $v|_{\mathcal{T}}$ is $\le s_v/M$, since the M^2 top entries dominate it. Let $V_k = \{i \in \mathcal{T} : 2^{-k-1} < |v_i| \le 2^{-k}\}$ and $m_k = |V_k|$. Only k with $2^{-k} < 2s_v/M$ occur, $\|v|_{V_k}\|_{4/3}^{4/3} \le m_k 2^{-4k/3}$, and $\sum_{2^{-k}<2s_v/M} 2^{-4k/3} \le 6(s_v/M)^{4/3}$. Coordinates with $v_i = 0$ go to T''. Let $n_* = 4C'^2M^2$. Levels with $m_k < n_*$ go to T'', at total cost $\le 6n_*(s_v/M)^{4/3} = 24C'^2M^{2/3}s_v^{4/3}$. For integers $\nu \ge n_*$, let $W(\nu)$ be the union of the levels with $m_k \ge \nu$ and $\varphi(\nu) = \|v|_{W(\nu)}\|_2/\alpha$, which is non-increasing in ν and vanishes for large ν. Let n^* be the least $\nu \ge n_*$ with $\varphi(\nu) \le 36\sqrt{K}$, and put $T' = W(n^*)$. Every remaining level has $n_* \le m_k < n^*$, hence $\varphi(m_k) > 36\sqrt{K}$. For such k apply Theorem 6.2 to the blocks $\{V_l : m_l \ge m_k\}$, all of size $\ge n_0 := m_k$, with $\varepsilon_i = \operatorname{sgn} v_i$ and $\lambda_l = \|v|_{V_l}\|_2/\|v|_{W(m_k)}\|_2$. The entries of a level agree within a factor 2, so $\|v|_{V_l}\|_1 \ge \frac12\sqrt{m_l}\,\|v|_{V_l}\|_2$ and

$$\mathbb{E}[B\mathcal{L}] = \sum_l \lambda_l \frac{\|v|_{V_l}\|_1}{\sqrt{m_l}} \ge \frac12\|v|_{W(m_k)}\|_2 > 18\sqrt{K}\,\alpha;$$

also $Km_k \ge n_*$. Hence $\alpha \le 2(C'M/\sqrt{Km_k})^K$, i.e. $m_k \le (C'^2M^2/K)(2/\alpha)^{2/K}$, and these levels cost at most $6(C'^2M^2/K)(2/\alpha)^{2/K}(s_v/M)^{4/3}$ in total. □

Lemma 6.4 (summing the rows)**.** *Let B_u ($u \in U$) be real with $|B_u| \le 1$, $\alpha_u = \|B_u\|_1$ and $s_u = \|P_1 B_u\|_2$, and suppose $\sum_u s_u^2 \le 1$ and $\sum_u \alpha_u \le \Gamma$, where $\Gamma \ge 2$. Then for every integer $K \ge 7$,*

$$\sum_{u:\alpha_u>0} s_u^{4/3}\Big(1 + \frac{1}{K}\Big(\frac{2}{\alpha_u}\Big)^{2/K}\Big) \le C_3\,\Gamma^{2/3}\sqrt{\log\Gamma}\,\Big(1 + \frac{1}{K}(2\Gamma^{10})^{2/K}\Big).$$

Proof. By (b), $s_u \le C_1\alpha_u\sqrt{\ell_u}$ with $\ell_u = \log(e/\alpha_u)$; note $\alpha_u \le 1$.

Rows with $0 < \alpha_u < \Gamma^{-10}$. Here $1 \le (2/\alpha_u)^{2/K}$, so the summand is at most $2s_u^{4/3}(2/\alpha_u)^{2/K} \le C\alpha_u^{4/3-2/K}\ell_u^{2/3} \le C\alpha_u \cdot \alpha_u^{1/21}\ell_u^{2/3}$, as $\frac13 - \frac{2}{K} \ge \frac{1}{21}$. The function $t \mapsto t^{1/21}\log(e/t)^{2/3}$ is increasing on $(0, e^{-13}]$ and bounded by 3.2 on $(0,1]$. So on $(0,\Gamma^{-10})$ it is at most $C\Gamma^{-10/21}(\log\Gamma)^{2/3}$ (use the bound 3.2 when $\Gamma < e^{1.3}$), and these rows contribute $\le C\Gamma^{11/21}(\log\Gamma)^{2/3} \le C\Gamma^{2/3}$.

Rows with $\alpha_u \ge \Gamma^{-10}$. Here $(2/\alpha_u)^{2/K} \le (2\Gamma^{10})^{2/K}$, and it remains to bound $\sum s_u^{4/3}$. Rows with $s_u \ge 1/\Gamma$ give $\sum s_u^{4/3} \le \Gamma^{2/3}\sum s_u^2 \le \Gamma^{2/3}$. Rows with $s_u < 1/\Gamma$ give $s_u^{4/3} \le \Gamma^{-1/3}s_u \le \Gamma^{-1/3}C_1\alpha_u\sqrt{\log(e\Gamma^{10})}$, which sums to $\le C\Gamma^{2/3}\sqrt{\log\Gamma}$. □

Proof of Theorem 1.3. Degree is an integer, so $\Phi(M,2) = \Phi(\lfloor M\rfloor, 2)$. The map $u \mapsto u\sqrt{\log(u+1)}$ is increasing on $[1,\infty)$, and $\lfloor M\rfloor \le M$, so the bound at $\lfloor M\rfloor$ implies the bound at M. Replace M by $\lfloor M\rfloor$ and assume that M is a positive integer. Fix a bipartition $I \sqcup J$ of the coordinates and a real f with $|f| \le 1$ and $\deg f \le M$. Write $b_{ui} = \widehat{f}(\{u,i\})$ for $u \in I$, $i \in J$. The rows $B_u(y) = \mathbb{E}_x[x_u f(x,y)]$ and the columns $B^i(x) = \mathbb{E}_y[y_i f(x,y)]$ are bounded by 1, have degree $\le M$, and have level-one vectors $b_u = (b_{ui})_i$ and $b^i = (b_{ui})_u$. For each y the section $f(\cdot,y)$ has level-one coefficients $B_u(y)$, so by (a) $\sum_u |B_u(y)| \le M$; integrating, $\sum_u \|B_u\|_1 \le M$, and likewise $\sum_i \|B^i\|_1 \le M$. By Parseval, $\sum_u \|b_u\|_2^2 = \sum_i \|b^i\|_2^2 \le 1$.

Let $\Gamma = \max(2, M)$ and $K = \max(7, \lceil \log\Gamma\rceil)$. For every row with $\alpha_u = \|B_u\|_1 > 0$ fix a set top_u of $\min(M^2, |J|)$ largest entries, apply Theorem 6.3, and write $b_u = b_u|_{\operatorname{top}_u} + t'_u + t''_u$ with $t'_u = b_u|_{T'_u}$

and $t''_u = b_u|_{T''_u}$, where $\mathrm{top}_u, T'_u, T''_u$ partition J. A row with $\alpha_u = 0$ vanishes; put $\mathrm{top}_u = T'_u = \emptyset$ and $T''_u = J$. Do the same for the columns, with sets $\mathrm{top}^i, T'^i, T''^i$ partitioning I and vectors t'^i, t''^i. Partition the entries (u,i) into

$$E_1 = \{(u,i) : i \in \mathrm{top}_u \cup T''_u\}, \qquad E_2 = \{(u,i) : i \in T'_u,\ u \in \mathrm{top}^i \cup T''^i\},$$
$$E_3 = \{(u,i) : i \in T'_u,\ u \in T'^i\}.$$

E_1. Hölder on at most M^2 entries gives $\|b_u|_{\mathrm{top}_u}\|_{4/3}^{4/3} \le M^{2/3} s_u^{4/3}$, where $s_u = \|b_u\|_2$. With Theorem 6.3 and Lemma 6.4,

$$\sum_{E_1} |b_{ui}|^{4/3} \le (1 + C_2) M^{2/3} \sum_{u:\alpha_u > 0} s_u^{4/3} \Big(1 + \frac{1}{K}\Big(\frac{2}{\alpha_u}\Big)^{2/K}\Big) \le C M^{2/3} \Gamma^{2/3} \sqrt{\log \Gamma},$$

because $(2\Gamma^{10})^{2/K} \le \exp(2(\log 2 + 10 \log \Gamma)/K) \le e^{21}$.

E_2. Every entry of E_2 lies in $\mathrm{top}^i \cup T''^i$ for its column i, so the same argument applied to the columns gives the same bound.

E_3. Let $a = b\,\mathbf{1}_{E_3}$. Its row u is a sub-vector of t'_u and its column i a sub-vector of t'^i, so $\sum_u \|a_{u\cdot}\|_2 \le 36\sqrt{K} \sum_u \alpha_u \le 36\sqrt{K} M$, and likewise for the columns. By (c), $\sum_{E_3} |b_{ui}|^{4/3} \le (36\sqrt{K} M)^{4/3}$.

Altogether $\|b\|_{4/3}^{4/3} \le C M^{4/3} (\log(M+1))^{2/3}$ for every bipartition. For a uniformly random bipartition each pair $\{i,j\}$ is split with probability $\frac{1}{2}$, so $A_2(f)^{4/3} = 2\,\mathbb{E}_{(I,J)} \|b^{(I,J)}\|_{4/3}^{4/3}$ and $A_2(f) \le C M \sqrt{\log(M+1)}$. □

7. Remarks

1. Where the length went. The fixed levels are the recursion of Proposition 4.5. Above the constant level R_0, Proposition 3.2 is applied on two or three ranges of degrees. Lemma 3.1 is stated for an arbitrary window and weight, so the two contracting ranges differ only in the values fed to Lemma 4.1.

2. The exponent (heuristic). The top stage needs $L \gtrsim \log d$ blocks, since its weight $r^{1+1/L}$ costs $d^{1/L}$ in the completion. Lemma 4.1 then gives only $1 - \kappa^2 \asymp L^{-2}$, which costs a factor $\asymp \log d$. The stage can start only at a level $R \gtrsim L^2 \log L$, where the counting term $(\log r)/r$ drops below L^{-2}, and the weight below R costs $R^{1/4}$. A third stage with $L_2 \approx \log R_3$ blocks covers the levels between $R_2 \approx L_2^2 \log L_2$ and R_3. This turns the cost $(\log d)^{1/2+o(1)}$ into $(\log\log d)^{3/2+o(1)}$, which is (2). More stages improve only the iterated logarithms: $K_0 + 1$ is the exponent of $\log d$ that this kind of stacking gives.

3. The seed. By Theorem 1.2, any seed $\Phi(M,2) \le C M (1 + \log M)^{K_0}$ with $K_0 < 1$ (two stages) or $K_0 < \frac{3}{2}$ (three stages) already gives $\mathrm{BH}(d) \le C d^2 (1 + \log d)^{5/2}$. In particular a cruder seed than Theorem 1.3 would suffice for the target. The bootstrap cannot start from level one alone: at $s = 1$ the extraction (3) is void ($\sigma_1 = 0$), so a bound for $\Phi(M,2)$ is the input this bootstrap needs.

4. The constant. The constant C is absolute. It depends on $H^2 = \sum_{r<R_0} D_r^2 r^{-5/2}$, a finite sum up to the fixed cutoff $R_0 = 25000$, and no attempt is made to optimize it. The inequalities in Propositions 4.2 and 4.3 are proved in the text; in particular Proposition 4.3 uses the factor $e^{-1/(2L^2)}$.

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