THE MAUREY–PISIER PROBLEM FOR BMW TYPE

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Abstract. We resolve the Maurey–Pisier problem for Bourgain–Milman–Wolfson (BMW) type in the negative. For every $p \in (1,2)$, we construct a complete separable metric space with supremal BMW type exponent p for which the distortions of the discrete ℓ_p cubes tend to infinity. We also characterize uniform containment of these cubes in an arbitrary metric space: the least uniform distortion equals the infimum of the BMW p-type constants over subsets whose supremal BMW exponent is p. Whenever finite, this infimum is attained by a countable closed subset. The finite critical BMW constant criterion follows as a corollary. Finally, the counterexample contains almost isometric critical cubes in finite ℓ_2 products, giving a negative answer to the corresponding product question.

1. Introduction

The Maurey–Pisier theorem [4] states that if r is the supremal Rademacher type exponent of an infinite-dimensional Banach space E, then E contains almost isometric linear copies of ℓ_r^n for every n. Bourgain, Milman, and Wolfson introduced a metric notion of type and investigated an analogous question for general metric spaces [1]. We give a negative answer at every exponent in $(1,2)$ and an exact characterization of the metric spaces that contain their critical cubes with uniformly bounded distortion.

We equip $\{0,1\}^n$ with the uniform probability measure. For $x = (x_1, \dots, x_n) \in \{0,1\}^n$, define

$$x^{(i)} = (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n), \qquad \bar{x} = (1 - x_1, \dots, 1 - x_n).$$

Thus $x^{(i)}$ flips coordinate i, and $\bar{x}$ flips every coordinate. Addition of cube vertices is coordinatewise modulo two. In particular, $x + e_i = x^{(i)}$ and $x + \mathbf{1} = \bar{x}$. For a metric space (X, d_X) with at least two points and $q \in [1,2]$, let $\mathrm{BMW}_q(X)$ be the least constant T such that

$$\mathbb{E}_x d_X(f(x), f(\bar{x}))^2 \le T^2 n^{2/q-1} \sum_{i=1}^{n} \mathbb{E}_x d_X(f(x), f(x^{(i)}))^2 \tag{1.1}$$

for every $n \ge 1$ and every map $f : \{0,1\}^n \to X$. If no finite constant exists, set $\mathrm{BMW}_q(X) = \infty$. For a singleton, the inequalities hold with constant zero. The supremal BMW exponent of a space with at least two points is

$$p_X = \sup\{q \in [1,2] : \mathrm{BMW}_q(X) < \infty\}. \tag{1.2}$$

We say that this exponent is attained when $\mathrm{BMW}_{p_X}(X) < \infty$. The equality $p_X = p$ alone does not include this endpoint inequality.

Write $Q_p^n = (\{0,1\}^n, \|\cdot\|_p)$. Two vertices differing in k coordinates have distance $k^{1/p}$. For a finite metric space (Z, d_Z), the distortion $c_X(Z)$ is the infimum of the numbers $D \ge 1$ for which there exist $F : Z \to X$ and $s > 0$ satisfying

$$s\, d_Z(u,v) \le d_X(F(u), F(v)) \le D s\, d_Z(u,v) \qquad (u, v \in Z). \tag{1.3}$$

We set $c_X(Z) = \infty$ if no such embedding exists. In particular, $c_X(Z) = 1$ means that Z embeds with distortion arbitrarily close to one; the infimum need not be attained by one map. Put

$$C_p(X) = \sup_{n \ge 1} c_X(Q_p^n). \tag{1.4}$$

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The Maurey–Pisier problem for BMW type originates in [1, Sections 1.10 and 4] and is stated explicitly in [5, Section 1.3.2, p. 9]. It asks whether

$$1 < p_X < 2 \quad \Longrightarrow \quad C_{p_X}(X) < \infty. \tag{1.5}$$

Bourgain, Milman, and Wolfson proved a conditional embedding result under an additional restriction on the largest edge of a cube map [1, Theorem 4.1, pp. 307–309]. Throughout the paper log is the natural logarithm; base-two logarithms are written $\log_2$.

The unrestricted implication (1.5) fails at every exponent in $(1, 2)$. To state an explicit example, let

$$\tau(v) = \inf_{k\in\mathbb{Z}} |v - 2k|, \qquad H(u) = \sum_{j=1}^{\infty} 4^j \tau(16^{-j}u). \tag{1.6}$$

The function τ is the triangular wave of period two. The series defining H converges locally uniformly, as will be checked in Section 3. For $1 < p < 2$, define

$$\phi_p(t) = t^{1/p} 2^{(1-1/p)H(\log_2 t)} \quad (t > 0), \qquad \phi_p(0) = 0, \tag{1.7}$$

and equip the real sequence space ℓ_1 with the distance

$$d_{X_p}(x, y) = \phi_p(\|x - y\|_1). \tag{1.8}$$

Theorem 1.1. *For every $p \in (1, 2)$, the space X_p defined in* (1.8) *is a complete separable metric space such that*

$$p_{X_p} = p, \qquad \mathrm{BMW}_p(X_p) = \infty, \qquad \lim_{n\to\infty} c_{X_p}(Q_p^n) = \infty. \tag{1.9}$$

More precisely, there are constants $c_p, C_p > 0$ such that

$$e^{c_p\sqrt{\log n}} \le c_{X_p}(Q_p^n) \le e^{C_p\sqrt{\log n}}$$

for all sufficiently large n.

Our second theorem characterizes uniform cube containment by attainment of the critical exponent on a subset. Every subset below carries the inherited metric; no linear structure is required.

Theorem 1.2. *For every metric space X and every $1 < p < 2$,*

$$C_p(X) = \inf_{\substack{Y\subseteq X \\ p_Y = p}} \mathrm{BMW}_p(Y), \tag{1.10}$$

where the infimum of an empty family is ∞. The infimum may be restricted to countable subsets that are closed in X. If $C_p(X) < \infty$, there is such a subset Y satisfying

$$p_Y = p, \qquad \mathrm{BMW}_p(Y) = C_p(Y) = C_p(X). \tag{1.11}$$

Thus the finite infimum in (1.10) is a minimum. The theorem applies without completeness or separability assumptions on X. At the critical exponent it gives the following criterion, which also contains the finite-endpoint implication.

Corollary 1.3. *Let X be a metric space with $1 < p_X < 2$. Then*

$$C_{p_X}(X) < \infty \quad \Longleftrightarrow \quad \exists\, Y \subseteq X :\ p_Y = p_X \text{ and } \mathrm{BMW}_{p_X}(Y) < \infty. \tag{1.12}$$

Whenever these conditions hold, Y can be chosen countable and closed in X. In particular, if $\mathrm{BMW}_{p_X}(X) < \infty$, then

$$c_X(Q_{p_X}^n) \le \mathrm{BMW}_{p_X}(X) \qquad (n \ge 1). \tag{1.13}$$

Proof. Apply Theorem 1.2 with $p = p_X$. For (1.13), use $Y = X$ in (1.10). □

The embeddings in (1.13) take values in X itself. For comparison, Naor [5, Theorem 1.15] proved that each critical cube embeds into a sufficiently large finite ℓ_2 product of X, with infimum distortion at most $\mathrm{BMW}_{p_X}(X)^2$. Here $\ell_2^N(X)$ denotes X^N with distance

$$d_{\ell_2^N(X)}(x,y) = \left(\sum_{i=1}^N d_X(x_i,y_i)^2\right)^{1/2}.$$

Attainment of the critical exponent on all of X is sufficient but not necessary. This already follows from the Kalton–Peck examples [3, Theorems 6.1–6.2] and the BMW comparison [1, Corollary 5.10]; Theorem 1.2 gives the exact condition on a subset.

The same space X_p also answers the finite-product question in [5, Section 1.3.2, p. 10]. In Section 5, we prove that every Q_p^n embeds almost isometrically into a sufficiently large finite ℓ_2 product of X_p, although its distortions in X_p itself tend to infinity.

The counterexample uses an elementary metric transform and the triangle inequality along a path in the cube. The proof of Theorem 1.2 has two parts. Random coordinate labels and a scalar variance estimate give the embedding implication from an attained exponent: for $p < 2$, fluctuations of order $N^{1/2}$ are negligible compared with distances of order $N^{1/p}$. Conversely, we select finite cube patterns with small diameters relative to their separations. A decomposition of squared distances then joins these patterns without increasing the BMW p-type constant.

2. Preliminaries

We use two elementary facts about metric transforms.

Lemma 2.1. *Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism with $\phi(0) = 0$. Suppose that $\phi(t)/t$ is nonincreasing on $(0,\infty)$. For every metric space (Z,d), the function $d_\phi(x,y) = \phi(d(x,y))$ is a metric. It has the same topology and Cauchy sequences as d. In particular, completeness and separability are preserved.*

Proof. For $s,t > 0$,

$$\phi(s+t) = s\frac{\phi(s+t)}{s+t} + t\frac{\phi(s+t)}{s+t} \le \phi(s) + \phi(t).$$

The inequality also holds when $st = 0$. Subadditivity and monotonicity give the triangle inequality for d_ϕ. Continuity of ϕ and ϕ^{-1} at zero shows that the two metrics have the same topology and Cauchy sequences. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, suppose that Z has at least two points and that $1 \le q \le 2$ and $B < \infty$ satisfy*

$$\phi(rt) \le Br^{1/q}\phi(t) \qquad (r \ge 1,\ t > 0). \tag{2.1}$$

Then $\mathrm{BMW}_q(Z,d_\phi) \le B$.

Proof. Let $t_1,\dots,t_n \ge 0$ and $L = \sum_i t_i > 0$. By (2.1),

$$\sum_{i=1}^n \phi(t_i)^q \ge B^{-q}\sum_{i=1}^n \frac{t_i}{L}\phi(L)^q = B^{-q}\phi(L)^q.$$

Thus

$$\phi\Big(\sum_{i=1}^n t_i\Big)^2 \le B^2\Big(\sum_{i=1}^n \phi(t_i)^q\Big)^{2/q} \le B^2 n^{2/q-1}\sum_{i=1}^n \phi(t_i)^2. \tag{2.2}$$

The case $L = 0$ is immediate. For $f : \{0,1\}^n \to Z$, apply this inequality along the path $x^{[i]} = x + e_1 + \cdots + e_i$, $0 \le i \le n$, with $t_i = d(f(x^{[i-1]}), f(x^{[i]}))$. Since $x^{[i-1]}$ is uniform when x is,

$$\mathbb{E}_x d_\phi(f(x), f(\bar{x}))^2 \le B^2 n^{2/q-1}\sum_{i=1}^n \mathbb{E}_x d_\phi(f(x), f(x^{(i)}))^2.$$ □

Taking $\phi(t) = t^{1/q}$ in Lemma 2.2 gives

$$\mathrm{BMW}_q(Z, d^{1/q}) \le 1 \qquad (1 \le q \le 2). \tag{2.3}$$

We will also use the immediate comparison $\mathrm{BMW}_q(Z,d) \le (b/a)\,\mathrm{BMW}_q(Z,\rho)$ whenever $a\rho \le d \le b\rho$.

3. Counterexample

Fix $p \in (1,2)$ and, throughout this section, set

$$\alpha = 1/p, \qquad \lambda = 1-\alpha, \qquad a_j = 4^j \quad (j \ge 1).$$

We use the function H in (1.6) and write $\phi = \phi_p$ for the transform in (1.7).

Lemma 3.1. *The series $H(u) = \sum_{j\ge1} a_j \tau(u/a_j^2)$ converges locally uniformly, $H \ge 0$, $H(0) = 0$, and*

$$\mathrm{Lip}(H) \le \frac{1}{3}, \qquad |H(u+v) - H(u)| \le \frac{8}{3}\sqrt{|v|}. \tag{3.1}$$

Every interval of length $4a_j^2$ contains $u < v$ with

$$v - u = a_j^2, \qquad H(u) - H(v) \ge \frac{2}{3}a_j. \tag{3.2}$$

Moreover,

$$H(a_j^2) = \frac{4}{3}a_j. \tag{3.3}$$

Proof. For every j,

$$0 \le a_j\tau(u/a_j^2) \le \frac{|u|}{a_j}, \qquad \mathrm{Lip}\left(a_j\tau(\,\cdot\,/a_j^2)\right) \le a_j^{-1}, \qquad \sum_{j\ge1} a_j^{-1} = \frac{1}{3}.$$

These bounds give local uniform convergence and $\mathrm{Lip}(H) \le 1/3$. If $|v| \ge 16$, choose J with $a_J \le \sqrt{|v|} < 4a_J$. Then

$$\begin{aligned} |H(u+v) - H(u)| &\le \sum_{j\ge1} \min\{a_j, |v|/a_j\} \\ &\le \sum_{j\le J} a_j + |v| \sum_{j>J} a_j^{-1} \\ &\le \frac{4}{3}a_J + \frac{|v|}{3a_J} \le \frac{8}{3}\sqrt{|v|}. \end{aligned}$$

For $|v| < 16$, use $|H(u+v) - H(u)| \le |v|/3 \le (4/3)\sqrt{|v|}$.

In any interval of length $4a_j^2$, choose an odd multiple $u = (2k+1)a_j^2$ in its first $2a_j^2$, and set $v = u + a_j^2$. The summands with index $i < j$ vanish at both points, since $a_j^2/a_i^2 = 16^{j-i}$ is even. Therefore

$$H(u) - H(v) \ge a_j - a_j^2 \sum_{i>j} a_i^{-1} = \frac{2}{3}a_j.$$

At $u = a_j^2$, the tail terms are linear, so

$$H(a_j^2) = a_j + a_j^2 \sum_{i>j} a_i^{-1} = \frac{4}{3}a_j.$$

□

Lemma 3.2. *The distance in* (1.8) *makes X_p complete and separable. Its supremal BMW exponent equals p, and $\mathrm{BMW}_p(X_p) = \infty$. For $1 \le q < p$,*

$$\mathrm{BMW}_q(X_p) \le 2^{16\lambda^2/[9(1/q-1/p)]}. \tag{3.4}$$

Proof. By $\mathrm{Lip}(H) \le 1/3$,

$$r^{(4-p)/(3p)} \le \frac{\phi(rt)}{\phi(t)} \le r^{(p+2)/(3p)} \qquad (r \ge 1,\ t > 0). \tag{3.5}$$

Both exponents lie in $(0,1)$. Thus ϕ is an increasing homeomorphism of $[0,\infty)$ and $\phi(t)/t$ is decreasing. Lemma 2.1 gives the metric, completeness, and separability assertions.

For $1 \le q < p$, (3.1) gives

$$\frac{\phi(rt)}{\phi(t)} \le r^{1/p} 2^{(8\lambda/3)\sqrt{\log_2 r}} \le r^{1/q} 2^{16\lambda^2/[9(1/q-1/p)]},$$

using $a\sqrt{v} - bv \le a^2/(4b)$ for $v \ge 0$ and $a, b > 0$. Apply Lemma 2.2.

The coordinate map $x \mapsto x$ has edge length $\phi(1) = 1$ and antipodal distance $\phi(n)$. Hence, for every $q \in [1,2]$,

$$\mathrm{BMW}_q(X_p) \ge \frac{\phi(n)}{n^{1/q}} = n^{1/p-1/q} 2^{\lambda H(\log_2 n)}.$$

Since $H \ge 0$, the right-hand side tends to infinity for $q > p$. For $q = p$, take $n_j = 2^{a_j^2}$ and use (3.3):

$$\mathrm{BMW}_p(X_p) \ge 2^{(4\lambda/3)a_j} \longrightarrow \infty.$$

Together with (3.4), this proves $p_{X_p} = p$. □

We next prove a lower bound for every embedding.

Let $P_p^n = (\{0,\dots,n\}, |i-j|^{1/p})$. The vertices $0, e_1, e_1+e_2, \dots, e_1+\cdots+e_n$ form an isometric copy of P_p^n in Q_p^n.

Proposition 3.3. *For every $n \ge 2$,*

$$2^{-2/3} 2^{\frac{\lambda}{12\sqrt{3}}\sqrt{\log_2 n}} \le c_{X_p}(P_p^n) \le c_{X_p}(Q_p^n) \le 2^{\frac{8\lambda}{3}\sqrt{\log_2 n}}. \tag{3.6}$$

Proof. Let $z_0, \dots, z_n \in \ell_1$ and $\sigma > 0$ satisfy

$$\sigma |i-j|^\alpha \le \phi(\|z_i - z_j\|_1) \le D\sigma |i-j|^\alpha.$$

Set $s_k = \phi^{-1}(\sigma k^\alpha)$, $1 \le k \le n$. By (3.5),

$$s_{|i-j|} \le \|z_i - z_j\|_1 \le D^{3p/(4-p)} s_{|i-j|} \qquad (i \ne j).$$

Partitioning a segment of length l into $\lceil l/k \rceil$ pieces of length at most k gives

$$\frac{s_l}{l} \le \frac{\lceil l/k \rceil}{l} D^{3p/(4-p)} s_k \le 2D^{3p/(4-p)} \frac{s_k}{k} \qquad (1 \le k \le l \le n). \tag{3.7}$$

Put $u_k = \log_2 s_k$. Since

$$\alpha(u_k - \log_2 k) + \lambda H(u_k) = \log_2 \sigma,$$

taking logarithms in (3.7) yields

$$\lambda\big(H(u_k) - H(u_l)\big) \le \frac{3}{4-p} \log_2 D + \alpha. \tag{3.8}$$

The power bounds also give

$$0 < u_{k+1} - u_k \le \frac{3}{4-p} \log_2 \Big(1 + \frac{1}{k}\Big) \le \frac{3}{4-p}, \qquad u_n - u_1 \ge \frac{3}{p+2} \log_2 n \ge \frac{3}{4} \log_2 n.$$

For $u_1 \le u \le v \le u_n$, choose left mesh points u_k, u_l. Then $k \le l$, and each approximation error is at most $3/(4-p)$. Using $\mathrm{Lip}(H) \le 1/3$ in (3.8),

$$\begin{aligned} \lambda\big(H(u) - H(v)\big) &\le \frac{3}{4-p} \log_2 D + \alpha + \frac{2\lambda}{4-p} \\ &\le \frac{3}{2} \log_2 D + 1. \end{aligned} \tag{3.9}$$

Choose the largest $j \geq 1$ with $4a_j^2 \leq (3/4)\log_2 n$. Then

$$4a_j^2 \leq \frac{3}{4}\log_2 n < 4a_{j+1}^2 = 64a_j^2, \qquad a_j > \frac{\sqrt{3}}{16}\sqrt{\log_2 n}.$$

Applying (3.2) inside $[u_1, u_n]$ and then (3.9) gives

$$\log_2 D \geq \frac{4\lambda}{9}a_j - \frac{2}{3} > \frac{\lambda}{12\sqrt{3}}\sqrt{\log_2 n} - \frac{2}{3}.$$

If no such j exists, then $\log_2 n < 256/3$ and the last lower bound is negative, so $D \geq 1$ suffices. Taking the infimum over embeddings proves the lower bound in (3.6).

The path P_p^n is an isometric subset of Q_p^n. For the upper bound, the coordinate embedding satisfies

$$1 \leq \frac{d_{X_p}(x,y)}{\|x-y\|_p} = 2^{\lambda H(\log_2 \|x-y\|_1)} \leq 2^{(8\lambda/3)\sqrt{\log_2 n}} \qquad (x \neq y).$$

□

Theorem 1.1 follows from Lemma 3.2 and Proposition 3.3. The lower bound in (3.6) also shows that the obstruction already occurs for the one-dimensional paths P_p^n.

4. The characterization

For $n \geq 1$, write

$$\mathrm{BMW}_q^n(Z) = \sup_{f \text{ nonconstant}} \frac{\left(\mathbb{E}_x d_Z(f(x), f(\bar{x}))^2\right)^{1/2}}{n^{1/q-1/2}\left(\sum_{i=1}^n \mathbb{E}_x d_Z(f(x), f(x^{(i)}))^2\right)^{1/2}}, \qquad f : \{0,1\}^n \to Z.$$

Thus $\mathrm{BMW}_q(Z) = \sup_n \mathrm{BMW}_q^n(Z)$.

Lemma 4.1. *If $1 < p = p_Z < 2$ and $\mathrm{BMW}_p(Z) < \infty$, then*

$$c_Z(Q_p^d) \leq \limsup_{n\to\infty} \mathrm{BMW}_p^n(Z) \leq \mathrm{BMW}_p(Z) \qquad (d \geq 1),$$

where $\mathrm{BMW}_p^n(Z)$ is the optimal constant in (1.1) at dimension n.

Proof. Set $B_n = \mathrm{BMW}_p^n(Z)$ and $C = \limsup_n B_n$. Partition mn coordinates into m blocks of size n, and let $x^{[j]}$ denote the flip of the jth block. For $f : \{0,1\}^{mn} \to Z$, the block argument from [5, Section 6.1] gives

$$\begin{aligned}\mathbb{E}d_Z(f(x), f(\bar{x}))^2 &\leq B_m^2 m^{2/p-1} \sum_{j=1}^m \mathbb{E}d_Z(f(x), f(x^{[j]}))^2 \\ &\leq B_m^2 B_n^2 (mn)^{2/p-1} \sum_{i=1}^{mn} \mathbb{E}d_Z(f(x), f(x^{(i)}))^2.\end{aligned}$$

The first inequality applies BMW to the block flips; the second applies it on each n-dimensional face. Extending a map on $\{0,1\}^n$ to $\{0,1\}^N$ by ignoring $N-n$ coordinates also gives

$$B_{mn} \leq B_m B_n, \qquad B_n \leq (N/n)^{1/p-1/2} B_N \quad (N \geq n). \tag{4.1}$$

These inequalities hold for every exponent in $[1,2]$. Since $B_1 = 1$, if $B_m < 1$ then $m \geq 2$ and some $q \in (p,2)$ satisfies

$$\mathrm{BMW}_q^m(Z) = m^{1/p-1/q} B_m < 1.$$

For $m^{j-1} < n \leq m^j$, (4.1) yields

$$\mathrm{BMW}_q^n(Z) \leq m^{1/q-1/2}\left(\mathrm{BMW}_q^m(Z)\right)^j \leq m^{1/q-1/2},$$

contrary to $p_Z = p$. Hence $B_n \geq 1$ and

$$1 \leq C \leq \mathrm{BMW}_p(Z) < \infty. \tag{4.2}$$

Rescale the metric for a nonconstant $f : \{0,1\}^N \to Z$ so that

$$\sum_{i=1}^{N} \mathbb{E}_x d_Z(f(x), f(x^{(i)}))^2 = N. \tag{4.3}$$

The sum is positive, and this rescaling preserves distortion and BMW constants. Let U be uniform on $\{0,1\}^N$, obtain V by independently flipping its coordinates with probability $\theta \in [0,1]$, and put

$$W = d_Z(f(U), f(V)), \qquad R(\theta) = (\mathbb{E}W^2)^{1/2}.$$

For a fixed i, independently resample the pair (U_i, V_i), obtaining $(\widetilde{U}, \widetilde{V})$ and $W_i' = d_Z(f(\widetilde{U}), f(\widetilde{V}))$. Since each endpoint bit changes with probability $1/2$,

$$\begin{aligned}\mathbb{E}|W - W_i'|^2 &\le 2\mathbb{E}d_Z(f(U), f(\widetilde{U}))^2 + 2\mathbb{E}d_Z(f(V), f(\widetilde{V}))^2 \\ &= 2\mathbb{E}_x d_Z(f(x), f(x^{(i)}))^2.\end{aligned}$$

Efron–Stein applied to the independent pairs (U_i, V_i), together with $\mathbb{E}W \le R(\theta)$, now gives

$$\begin{aligned}\operatorname{Var} W &\le \frac{1}{2}\sum_{i=1}^{N} \mathbb{E}|W - W_i'|^2 \le N, \\ \mathbb{E}|W - R(\theta)|^2 &= 2R(\theta)(R(\theta) - \mathbb{E}W) \le 2\operatorname{Var} W \le 2N.\end{aligned} \tag{4.4}$$

Choose independent uniform labels $J_i \in \{1, \dots, m\}$ and an independent uniform $x \in \{0,1\}^N$. Apply BMW to

$$u \longmapsto f\big((x_i + u_{J_i})_{i=1}^N\big), \qquad u \in \{0,1\}^m.$$

An antipodal pair flips all coordinates; an edge flips each coordinate independently with probability $1/m$. Averaging over x and the labels therefore gives

$$R(1)^2 \le B_m^2 m^{2/p-1} \sum_{j=1}^{m} R(1/m)^2 = B_m^2 m^{2/p} R(1/m)^2. \tag{4.5}$$

Conversely, condition on the number $K \sim \mathrm{Bin}(N, \theta)$ of flipped coordinates. BMW on the random k-dimensional face gives

$$\begin{aligned}\mathbb{E}[W^2 \mid K = k] &\le B_k^2 k^{2/p-1} \frac{k}{N} \sum_{i=1}^{N} \mathbb{E}_x d_Z(f(x), f(x^{(i)}))^2 \\ &= B_k^2 k^{2/p} \qquad (k \ge 1).\end{aligned}$$

Thus, with the $K = 0$ term interpreted as zero,

$$R(\theta)^2 \le \mathbb{E}[B_K^2 K^{2/p}]. \tag{4.6}$$

For fixed $\theta > 0$ and $\eta > 0$, choose k_0 so that $B_k \le C + \eta$ for $k \ge k_0$. Using

$$\mathbb{E}K^{2/p} \le (\mathbb{E}K^2)^{1/p} = (N\theta)^{2/p}\left(1 + \frac{1-\theta}{N\theta}\right)^{1/p},$$

we obtain

$$\frac{R(\theta)^2}{(N\theta)^{2/p}} \le (C+\eta)^2\left(1 + \frac{1-\theta}{N\theta}\right)^{1/p} + \mathrm{BMW}_p(Z)^2\left(\frac{k_0}{N\theta}\right)^{2/p}.$$

Letting $N \to \infty$ and then $\eta \downarrow 0$ shows that, for fixed $\theta > 0$ and $\varepsilon > 0$,

$$R(\theta) \le (C + \varepsilon)(N\theta)^{1/p} \tag{4.7}$$

for all sufficiently large N, uniformly over normalized f.

Fix $d \ge 1$ and $0 < \varepsilon < 1/4$. Choose M divisible by $1, \dots, d$ and large enough that

$$B_{M/r} \le (1+\varepsilon)C \qquad (1 \le r \le d).$$

Since $C = \limsup B_N$, arbitrarily large N admit a normalized f with

$$R(1) \geq (1-\varepsilon)CN^{1/p}.$$

For these maps, (4.5) with $m = M/r$ and (4.7) yield, once N is sufficiently large,

$$(1-2\varepsilon)(Nr/M)^{1/p} \leq R(r/M) \leq (C+\varepsilon)(Nr/M)^{1/p} \qquad (1 \leq r \leq d), \tag{4.8}$$

where $(1-\varepsilon)/(1+\varepsilon) \geq 1-2\varepsilon$.

Now choose independent uniform labels $J_i \in \{1,\dots,M\}$ and an independent uniform x. Extend $u \in \{0,1\}^d$ by $u_j = 0$ for $j > d$ and set

$$G(u) = f((x_i + u_{J_i})_{i=1}^N).$$

For $u, v \in \{0,1\}^d$, the pairs $(x_i + u_{J_i}, x_i + v_{J_i})$ are independent. The first bit is uniform independently of J_i, and the flip has probability $\|u-v\|_1/M$. Hence (4.4) gives

$$\mathbb{E} \sum_{\{u,v\}} \left| d_Z(G(u),G(v)) - R(\|u-v\|_1/M) \right|^2 \leq 2N \binom{2^d}{2} < 4^d N.$$

Some realization therefore satisfies, simultaneously for all pairs,

$$\left| d_Z(G(u),G(v)) - R(\|u-v\|_1/M) \right| \leq 2^d \sqrt{N}.$$

Because $p < 2$, we may choose N from the preceding unbounded sequence so large that

$$\frac{2^d\sqrt{N}}{(N/M)^{1/p}} = 2^d M^{1/p} N^{1/2-1/p} \leq \varepsilon.$$

Together with (4.8) and $\|u-v\|_p \geq 1$ for $u \neq v$, this gives

$$\begin{aligned}(1-3\varepsilon)(N/M)^{1/p}\|u-v\|_p &\leq d_Z(G(u),G(v)) \\ &\leq (C+2\varepsilon)(N/M)^{1/p}\|u-v\|_p.\end{aligned}$$

Thus G is injective with distortion at most $(C+2\varepsilon)/(1-3\varepsilon)$. Let $\varepsilon \downarrow 0$. □

We next record a finite Hamming cube construction.

Lemma 4.2. *If $1 \leq p < q$, then for every k and every $\varepsilon > 0$ there are an M and a $(1+\varepsilon)$-distortion embedding $F : Q_q^k \to Q_p^M$ with $F(0) = 0$.*

Proof. Put $\beta = p/q \in (0,1)$ and

$$c_\beta^{-1} = \int_0^\infty (1-e^{-t}) t^{-1-\beta}\, dt \in (0,\infty).$$

Substitution gives

$$h^\beta = c_\beta \int_0^\infty (1-e^{-th}) t^{-1-\beta}\, dt \qquad (h \geq 0).$$

Under $\mathbb{P}_t$, include each coordinate in a random subset $S \subseteq \{1,\dots,k\}$ independently with probability $(1-e^{-t})/2$. For the parity $\chi_S(x) = \sum_{i\in S} x_i \pmod 2$,

$$\mathbb{P}_t(\chi_S(x) \neq \chi_S(y)) = \frac{1-e^{-t\|x-y\|_1}}{2}.$$

Hence

$$\|x-y\|_1^\beta = \sum_{\varnothing \neq S \subseteq \{1,\dots,k\}} w_S |\chi_S(x) - \chi_S(y)|, \qquad w_S = 2c_\beta \int_0^\infty \mathbb{P}_t(S) t^{-1-\beta}\, dt.$$

Each w_S is positive and finite: the integrand is $O(t^{|S|-1-\beta})$ at zero and $O(t^{-1-\beta})$ at infinity. Choose positive integers B, b_S such that

$$w_S \leq \frac{b_S}{B} \leq (1+\varepsilon)^p w_S \qquad (S \neq \varnothing).$$

Let F repeat the coordinate χ_S exactly b_S times, so that $M = \sum_{S\neq\varnothing} b_S$ and $F(0) = 0$. Then

$$B\|x-y\|_1^{p/q} \le \|F(x)-F(y)\|_1 \le B(1+\varepsilon)^p\|x-y\|_1^{p/q},$$
$$B^{1/p}\|x-y\|_q \le \|F(x)-F(y)\|_p \le (1+\varepsilon)B^{1/p}\|x-y\|_q.$$

□

We next select separated copies of finite patterns; related selection arguments appear in [2]. Recall that an ultrametric satisfies $R(i,j) \le \max\{R(i,k), R(k,j)\}$.

Lemma 4.3. *Fix $p \ge 1$ and $D \ge 1$. Suppose that, for every $N \ge 1$, there are $f_N : \{0,1\}^N \to X$ and $s_N > 0$ such that*

$$s_N\|u-v\|_p \le d_X(f_N(u), f_N(v)) \le Ds_N\|u-v\|_p.$$

For each $k \ge 1$, prescribe a positive integer M_k, a subset $P_k \subseteq \{0,1\}^{M_k}$ containing 0, and an integer b_k. For every $0 < \delta < 1/10$, there are strictly increasing integers $N_k \ge \max\{M_k, b_k\}$ and vertices $v_k \in \{0,1\}^{N_k}$ such that the sets

$$A_k = f_{N_k}(v_k + (P_k \times \{0\}^{N_k - M_k}))$$

are pairwise disjoint. Addition here is coordinatewise modulo two. Their union, with at most one additional singleton block, is a countable closed subset $Y \subseteq X$. The blocks admit an ultrametric R on their labels such that, for distinct labels i, j and $a \in A_i$, $b \in A_j$,

$$\begin{aligned}(1-3\delta)R(i,j) \le d_X(a,b) \le (1+3\delta)R(i,j),\\ (\operatorname{diam} A_i)^2 + (\operatorname{diam} A_j)^2 \le 2\delta^2 R(i,j)^2.\end{aligned} \tag{4.9}$$

Proof. For $N \ge M_k$ and any vertex v, the image of the prescribed pattern satisfies

$$\operatorname{diam} f_N(v + (P_k \times \{0\}^{N-M_k})) \le Ds_N M_k^{1/p}.$$

We first choose centers $y_N = f_N(v_N)$. Pass to a subsequence on which either $s_N \ge a > 0$ or $s_N \to 0$.

If $s_N \ge a > 0$, fix $z \in X$ and choose $y_N \in \{f_N(0), f_N(\mathbf{1})\}$ so that

$$r_N = d_X(y_N, z) \ge \tfrac{1}{2}s_N N^{1/p} \longrightarrow \infty, \qquad \frac{r_N}{s_N} \ge \tfrac{1}{2}N^{1/p} \longrightarrow \infty.$$

If $s_N \to 0$ and $(z_N) = (f_N(0))$ is unbounded, take $y_N = z_N$ along a subsequence with $r_N = d_X(z_N, z) \to \infty$ for a fixed $z \in X$. Again $r_N/s_N \to \infty$.

Suppose that $s_N \to 0$ and (z_N) is bounded. It has either a Cauchy subsequence or an a-separated subsequence. In the separated case, $a \le d_X(z_N, z_{N'}) \le b$ for some $b < \infty$ and all $N \ne N'$. Color pairs according to a finite partition of $[a,b]$ into multiplicative intervals of ratio at most $1+\delta$. The infinite Ramsey theorem gives $R_0 > 0$ and a subsequence on which

$$R_0 \le d_X(z_N, z_{N'}) \le (1+\delta)R_0 \qquad (N \ne N').$$

Take $y_N = z_N$.

In the Cauchy case, let z be the limit in the completion of X, and put

$$L_N = \min\{N^{1/p}/4, s_N^{-1/2}\} \longrightarrow \infty, \qquad L_N s_N \le \sqrt{s_N} \longrightarrow 0.$$

If $d_X(z_N, z) \ge L_N s_N$, take $y_N = z_N$. Otherwise,

$$d_X(f_N(\mathbf{1}), z) \ge s_N N^{1/p} - d_X(z_N, z) > 3L_N s_N.$$

Choose the first vertex on a coordinate path from 0 to $\mathbf{1}$ whose image has distance at least $L_N s_N$ from z. Each step has length at most Ds_N, so its image y_N satisfies

$$L_N s_N \le d_X(y_N, z) \le (L_N + D)s_N.$$

In both branches,

$$r_N = d_X(y_N, z) \longrightarrow 0, \qquad r_N/s_N \ge L_N \longrightarrow \infty.$$

In either case with radii r_N, choose strictly increasing $N_k \geq \max\{M_k, b_k\}$ so that

$$Ds_{N_k} M_k^{1/p} \leq \delta r_{N_k}, \qquad \begin{cases} r_{N_{k+1}} \geq \delta^{-1} r_{N_k}, & r_N \to \infty, \\ r_{N_{k+1}} \leq \delta r_{N_k}, & r_N \to 0 \end{cases} \quad (k \geq 1).$$

These choices are possible since $r_N/s_N \to \infty$. In the separated case, instead require $Ds_{N_k} M_k^{1/p} \leq \delta R_0$. Set $v_k = v_{N_k}$, $y_k = y_{N_k}$, and, in the radii cases, $r_k = r_{N_k}$. Then $y_k \in A_k$ because $0 \in P_k$, and

$$\operatorname{diam} A_k \leq \begin{cases} \delta r_k, & \text{in the radii cases,} \\ \delta R_0, & \text{in the separated case.} \end{cases}$$

In the radii cases, set $R(i,j) = \max\{r_i, r_j\}$ for $i \neq j$ and $R(i,i) = 0$. For distinct indices, interchange them so that $r_j \leq \delta r_i$. The triangle inequality gives, for $a \in A_i$ and $b \in A_j$,

$$\begin{aligned} |d_X(a,b) - R(i,j)| &\leq r_j + \operatorname{diam} A_i + \operatorname{diam} A_j \\ &\leq (2\delta + \delta^2) R(i,j) \leq 3\delta R(i,j), \\ (\operatorname{diam} A_i)^2 + (\operatorname{diam} A_j)^2 &\leq \delta^2 (r_i^2 + r_j^2) \leq 2\delta^2 R(i,j)^2. \end{aligned}$$

In the separated case, set $R(i,j) = R_0$ for $i \neq j$ and $R(i,i) = 0$; then

$$(1 - 2\delta) R_0 \leq d_X(a,b) \leq (1 + 3\delta) R_0, \qquad (\operatorname{diam} A_i)^2 + (\operatorname{diam} A_j)^2 \leq 2\delta^2 R_0^2.$$

Thus R is an ultrametric and (4.9) holds. Its positive lower bound also shows that the blocks are disjoint.

The blocks are finite. In the increasing-radii and separated cases, the lower bounds above exclude convergence of a sequence visiting infinitely many blocks, so their union is closed in X. In the decreasing-radii case,

$$\sup_{a \in A_k} d_X(a,z) \leq (1 + \delta) r_k \longrightarrow 0,$$

so z is the only possible accumulation point. If $z \notin X$, the union is closed in X. If $z \in X$, adjoin $A_0 = \{z\}$ and set $R(0,i) = r_i$. The resulting label metric is still an ultrametric, and

$$(1 - \delta) r_i \leq d_X(z,a) \leq (1 + \delta) r_i, \qquad (\operatorname{diam} A_i)^2 \leq \delta^2 r_i^2,$$

so (4.9) also holds for the added block. The resulting subset is countable and closed in X. □

The following lemma controls the BMW constant of the union.

Lemma 4.4. *Fix $1 < p < 2$ and $C \geq 1$. Let a metric space be the disjoint union of blocks A_i with $\mathrm{BMW}_p(A_i) \leq C$. Suppose there is an ultrametric R on their labels such that, for $i \neq j$, $a \in A_i$, and $b \in A_j$,*

$$\begin{aligned} (1 - 3\delta) R(i,j) \leq d(a,b) \leq (1 + 3\delta) R(i,j), \\ (\operatorname{diam} A_i)^2 + (\operatorname{diam} A_j)^2 \leq 2\delta^2 R(i,j)^2. \end{aligned}$$

If $\delta > 0$ is sufficiently small depending only on p, then the union, with its actual metric d, has BMW p-type constant at most C.

Proof. Choose $o_i \in A_i$, and let π_i be the identity on A_i and equal to o_i elsewhere. Set

$$I(a,b) = \sum_i d(\pi_i(a), \pi_i(b))^2, \qquad V(a,b) = d(a,b)^2 - I(a,b).$$

The sum has at most two nonzero terms, and

$$I(a,b) = \begin{cases} d(a,b)^2, & a, b \in A_i, \\ d(a,o_i)^2 + d(o_j,b)^2, & a \in A_i,\ b \in A_j,\ i \neq j. \end{cases}$$

Writing $\ell(a) = i$ for $a \in A_i$, we obtain, for all a, b,

$$a_\delta R(\ell(a), \ell(b))^2 \leq V(a,b) \leq b_\delta R(\ell(a), \ell(b))^2,$$

where

$$a_\delta = (1-3\delta)^2 - 2\delta^2, \qquad b_\delta = (1+3\delta)^2.$$

Since $p < 2$, choose δ so that $a_\delta > 0$ and

$$b_\delta / a_\delta \le 2^{2/p-1}. \tag{4.10}$$

Fix $m \ge 2$ and $f : \{0,1\}^m \to \bigcup_i A_i$. Applying the block BMW inequalities to $\pi_i \circ f$ and summing gives

$$\mathbb{E} I(f(x), f(\bar{x})) \le C^2 m^{2/p-1} \sum_{j=1}^m \mathbb{E} I(f(x), f(x^{(j)})).$$

Only finitely many terms are nonzero. Put $g = \ell \circ f$ and $x^{[j]} = x + e_1 + \cdots + e_j$ for $0 \le j \le m$. The ultrametric inequality gives

$$R(g(x), g(\bar{x}))^2 \le \max_{1 \le j \le m} R(g(x^{[j-1]}), g(x^{[j]}))^2 \le \sum_{j=1}^m R(g(x^{[j-1]}), g(x^{[j]}))^2.$$

Each $x^{[j-1]}$ is uniform when x is uniform. Thus

$$\begin{aligned} \mathbb{E} V(f(x), f(\bar{x})) &\le b_\delta \mathbb{E} R(g(x), g(\bar{x}))^2 \\ &\le b_\delta \sum_{j=1}^m \mathbb{E} R(g(x), g(x^{(j)}))^2 \\ &\le \frac{b_\delta}{a_\delta} \sum_{j=1}^m \mathbb{E} V(f(x), f(x^{(j)})). \end{aligned}$$

Since $b_\delta / a_\delta \le 2^{2/p-1} \le C^2 m^{2/p-1}$, adding the estimates for I and V proves the BMW inequality for $d^2 = I + V$. The case $m = 1$ holds with constant one. □

Proof of Theorem 1.2. For $Y \subseteq X$ with $p_Y = p$ and $\mathrm{BMW}_p(Y) < \infty$, Lemma 4.1 gives

$$C_p(X) \le C_p(Y) \le \mathrm{BMW}_p(Y). \tag{4.11}$$

If $C_p(X) = \infty$, both infima in the statement are therefore infinite.

Suppose $C = C_p(X) < \infty$. Choose $f_N : Q_p^N \to X$ and $s_N > 0$ with

$$s_N \|u - v\|_p \le d_X(f_N(u), f_N(v)) \le D_N s_N \|u - v\|_p, \qquad D_N < C + 1/N.$$

Let $q_k \in (p, 2)$ decrease to p and put $t_k = 1/p - 1/q_k > 0$. Lemma 4.2 gives embeddings

$$F_k : Q_{q_k}^k \longrightarrow Q_p^{M_k}, \qquad F_k(0) = 0,$$

with distortions $h_k < 2^{t_k/2}$. Choose b_k with $(C + 1/b_k) h_k < C 2^{t_k}$. Then

$$D_N h_k < (C + 1/b_k) h_k < C 2^{t_k} \qquad (N \ge b_k). \tag{4.12}$$

Apply Lemma 4.3 with uniform distortion bound $C + 1$, patterns $P_k = F_k(\{0,1\}^k)$, thresholds b_k, and δ as in Lemma 4.4. Each selected block A_k is a copy of $Q_{q_k}^k$ with distortion at most $D_{N_k} h_k$. By (2.3), for every $m \ge 2$,

$$\begin{aligned} \mathrm{BMW}_p^m(A_k) &= m^{1/q_k - 1/p} \, \mathrm{BMW}_{q_k}^m(A_k) \\ &\le D_{N_k} h_k \, m^{-t_k} \le D_{N_k} h_k \, 2^{-t_k} < C. \end{aligned}$$

Since $\mathrm{BMW}_p^1(A_k) = 1$, we have $\mathrm{BMW}_p(A_k) \le C$. The possible singleton block also satisfies this bound. Their union Y is countable and closed in X, and Lemma 4.4 gives $\mathrm{BMW}_p(Y) \le C$.

Fix $p < q \le 2$. Testing the BMW q-inequality on the copy of $Q^k_{q_k}$ in A_k, and using $D_{N_k} h_k < 2(C+1)$, gives, for all sufficiently large k,

$$\mathrm{BMW}_q(Y) \ge \frac{k^{1/q_k - 1/q}}{D_{N_k} h_k} \ge \frac{k^{(1/p-1/q)/2}}{2(C+1)} \longrightarrow \infty.$$

Thus $p_Y = p$. Finally, (4.11) gives

$$C = C_p(X) \le C_p(Y) \le \mathrm{BMW}_p(Y) \le C,$$

which proves (1.10) and (1.11). □

The restriction $p < 2$ is necessary. A two-point metric space has $p_X = 2$ and $\mathrm{BMW}_2(X) = 1$, but does not contain an injective image of Q^2_2. In Lemma 4.1, the error term $N^{1/2-1/p}$ ceases to tend to zero at $p = 2$; in Lemma 4.4, the strict inequality $2^{2/p-1} > 1$ is lost.

5. Finite ℓ_2 products

Naor [5, Section 1.3.2, p. 10] asks whether, for every metric space X and every $p \in (1,2)$,

$$\sup_{n\ge 1} \inf_{N\ge 1} c_{\ell_2^N(X)}(Q^n_p) < \infty \quad \Longrightarrow \quad \sup_{n\ge 1} c_X(Q^n_p) < \infty. \tag{5.1}$$

The number of factors N may depend on n, but the distortion bound must be independent of n. The following result, together with Theorem 1.1, gives a negative answer for every $p \in (1,2)$.

Theorem 5.1. *For every $p \in (1,2)$, the space X_p defined in* (1.8) *satisfies*

$$\inf_{N\ge 1} c_{\ell_2^N(X_p)}(Q^n_p) = 1 \qquad (n \ge 1). \tag{5.2}$$

Proof. Fix $p \in (1,2)$ and $n \ge 1$. Write $\alpha = 1/p$, $\lambda = 1-\alpha$, $a_j = 4^j$, and $\phi = \phi_p$. Choose J with $\log_2 n \le 14 a_J^2$ and put

$$H_J(u) = \sum_{j=1}^{J} a_j \tau(u/a_j^2), \qquad L_J = 2a_J^2, \qquad \beta_J = \alpha + \frac{\lambda}{3a_J}.$$

The function H_J is L_J-periodic. For $0 \le u \le L_J + \log_2 n$, we have $u \le a^2_{J+1}$, so

$$H(u) - H_J(u) = u \sum_{j>J} a_j^{-1} = \frac{u}{3a_J}.$$

Consequently, for $1 \le k \le n$,

$$\begin{aligned}\int_0^{L_J} \phi(2^u k)^2\, 2^{-2\beta_J u}\, du &= k^{2\beta_J} \int_0^{L_J} 2^{2\lambda H_J(u+\log_2 k)}\, du \\ &= C_J k^{2\beta_J}, \qquad C_J = \int_0^{L_J} 2^{2\lambda H_J(u)}\, du > 0.\end{aligned}$$

Fix $0 < \varepsilon < 1$. Take a common tagged partition of $[0, L_J]$, with tags u_i and interval lengths Δu_i, and approximate its weights $\Delta u_i\, 2^{-2\beta_J u_i}$ by positive rationals b_i/B, where $b_i, B \in \mathbb{N}$. These choices can be made so that

$$\left| \sum_{i=1}^{m} \frac{b_i}{B} \phi(2^{u_i} k)^2 - C_J k^{2\beta_J} \right| \le \varepsilon C_J k^{2\beta_J} \qquad (1 \le k \le n).$$

Define

$$F(x) = \big(\underbrace{2^{u_i}x, \dots, 2^{u_i}x}_{b_i \text{ copies}} \big)_{i=1}^{m} \in \ell_2^N(X_p), \qquad N = \sum_{i=1}^{m} b_i.$$

For $k = \|x-y\|_1 \geq 1$,

$$\frac{1}{B} d_{\ell_2^N(X_p)}(F(x),F(y))^2 = \sum_{i=1}^{m} \frac{b_i}{B} \phi(2^{u_i}k)^2 \in \big[(1-\varepsilon)C_J k^{2\beta_J}, (1+\varepsilon)C_J k^{2\beta_J}\big].$$

Thus

$$c_{\ell_2^N(X_p)}(Q_p^n) \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\, n^{\beta_J - \alpha}.$$

First let $\varepsilon \downarrow 0$, and then let $J \to \infty$. Since $\beta_J \downarrow \alpha$, this proves (5.2). □

Thus the hypothesis of (5.1) holds with supremum equal to one, whereas Theorem 1.1 gives $c_{X_p}(Q_p^n) \to \infty$.

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