

An explicit Bellman function for disjoint partitions: the sharp logarithmic gain for three families

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Abstract

For families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of subsets of $[m] = \{1, \dots, m\}$ let $N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be the number of ordered partitions $[m] = T_1 \sqcup T_2 \sqcup T_3$ with $T_j \in \mathcal{F}_j$. Every function $B \geq 0$ on $[1, \infty)^3$ with $B(1, 1, 1) \geq 1$ that satisfies $B(x_0 + x_1, y_0 + y_1, z_0 + z_1) \geq B(x_1, y_0, z_0) + B(x_0, y_1, z_0) + B(x_0, y_0, z_1)$ for all $x_i, y_i, z_i \geq 0$ (with $B := 0$ when an argument is less than 1), a *Bellman function*, gives $N \leq B(|\mathcal{F}_1|, |\mathcal{F}_2|, |\mathcal{F}_3|)$. The classical choice $B = (xyz)^c$, $c = \ln 3 / \ln(27/4)$, yields the Kane–Tao bound $N_3(X) \leq |X|^{3c}$ for the number $N_3(X)$ of ordered pairs of disjoint members of a family X whose union also belongs to X . We give an explicit closed-form Bellman function that captures the sharp logarithmic gain,

$$B(x, y, z) = (xyz)^c \left(\frac{G}{M} \right)^\kappa \left(1 + \frac{\ln(xyz)}{2000} \right)^{-\gamma},$$

where G is the geometric mean of x, y, z , M is their power mean of order $\alpha / \ln(xyz)$, $\gamma = \ln(27/16) / (2 \ln(27/4))$, and α, κ are explicit constants built from $\ln 2$ and $\ln 3$. Consequently $N_3(X) \leq 2.44 |X|^{3c} (1 + \ln |X|)^{-\gamma}$, and the exponent γ is sharp. The proof combines the Gibbs variational principle with an exact decomposition of the slack; finitely many explicit numerical inequalities are verified rigorously in interval arithmetic. High-precision numerical tests confirm the result and indicate that the shift 2000 can be lowered to about 29.

Keywords: Bellman function, disjoint partitions, Kane–Tao bound, logarithmic gain, power mean, entropy, computer-assisted proof.

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1 Introduction

For finite families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of subsets of $[m]$ let $N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be the number of triples $(T_1, T_2, T_3) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3$ with $T_1 \sqcup T_2 \sqcup T_3 = [m]$, and for a family X of subsets of $[m]$ let $N_3(X)$ be the number of triples $(T_1, T_2, T) \in X^3$ with $T = T_1 \sqcup T_2$; thus $N_3(X) = N(X, X, X^c)$ with $X^c = \{[m] \setminus T : T \in X\}$. Kane and Tao [3] proved

$$N_3(X) \leq |X|^{c(3)}, \quad c(3) = \frac{\ln 27}{\ln(27/4)} = 1.72598\dots,$$

with a sharp exponent, and the bound was extended to partitions into any number of parts in [1]. Whether a power of $\ln |X|$ can be gained was asked in the discussion of [4]. The companion paper [5] proves the sharp refinement $N_3(X) \leq C_3 |X|^{c(3)} (1 + \ln |X|)^{-\gamma_3}$ with $\gamma_3 = \frac{2-c(3)}{2} = 0.13700\dots$ and $C_3 \leq 359$, together with its analogue for any number of parts; its Bellman function has the form $V_0 \min\{1, \Theta\}$, with an envelope V_0 of weighted bounds and an explicit correction Θ . In this paper we show that for three parts a single closed-form Bellman function gives the sharp bound, with the constant 2.44 in place of C_3 .

The main inequality. Sorting the partitions according to the part that contains the element m leads to the following inequality for a function $B: [1, \infty)^3 \rightarrow [0, \infty)$, extended by $B(x, y, z) := 0$ if $\min\{x, y, z\} < 1$: for all $x_0, x_1, y_0, y_1, z_0, z_1 \geq 0$,

$$B(x_0 + x_1, y_0 + y_1, z_0 + z_1) \geq B(x_1, y_0, z_0) + B(x_0, y_1, z_0) + B(x_0, y_0, z_1). \quad (1.1)$$

We call (1.1) the *main inequality*, and a function that satisfies it and $B(1, 1, 1) \geq 1$ a *Bellman function*. Every Bellman function bounds the number of partitions (Lemma 2.1):

$$N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \leq B(|\mathcal{F}_1|, |\mathcal{F}_2|, |\mathcal{F}_3|), \quad \text{in particular} \quad N_3(X) \leq B(|X|, |X|, |X|). \quad (1.2)$$

The classical Bellman function. The function

$$B(x, y, z) = (xyz)^c, \quad c = \frac{c(3)}{3} = \frac{\ln 3}{\ln(27/4)} = 0.57532\dots, \quad (1.3)$$

satisfies the main inequality [3, 1] (it is also the case $\pi = U$ of Remark A.5), and $B(x, x, x) = x^{c(3)}$; by (1.2) this gives the Kane–Tao bound.

The new Bellman function. Let $\lambda = \ln 2$, $\mu = \ln \frac{27}{4}$, and

$$\begin{aligned} \gamma &= \frac{\ln(27/16)}{2 \ln(27/4)} = 0.13700\dots, \\ \alpha &= \frac{3\mu}{2\lambda} \left(\frac{9}{\lambda} - 1 - \frac{1}{\gamma} \right) = 19.3618\dots, \quad \kappa = \frac{9\gamma\mu}{\lambda^2\alpha} = 0.25311\dots; \end{aligned} \quad (1.4)$$

note that $\gamma = \frac{2-3c}{2} = \gamma_3$. For $x, y, z \geq 1$ let

$$G = (xyz)^{1/3}, \quad M = \left(\frac{x^t + y^t + z^t}{3} \right)^{1/t}, \quad t = \frac{\alpha}{\ln(xyz)}$$

be the geometric mean and the power mean of order t (with $G/M := 1$ at $x = y = z = 1$), and put

$$\boxed{B(x, y, z) = (xyz)^c \left(\frac{G}{M} \right)^\kappa \left(1 + \frac{\ln(xyz)}{2000} \right)^{-\gamma}} \quad (1.5)$$

More generally, for $0 < A \leq \infty$ let B_A be defined by (1.5) with 2000 replaced by A ; thus $B = B_{2000}$, and $B_\infty = (xyz)^c (G/M)^\kappa$. On the diagonal $G = M$, so

$$B(x, x, x) = \frac{x^{3c}}{\left(1 + \frac{3}{2000} \ln x\right)^\gamma} \leq 2.44 \frac{x^{3c}}{(1 + \ln x)^\gamma}, \quad (1.6)$$

because $1 + \frac{3}{2000} \ln x \geq \frac{3}{2000} (1 + \ln x)$ and $(2000/3)^\gamma = 2.4373\dots$

Theorem 1.1. *For every $A \in [2000, \infty]$ the function B_A satisfies the main inequality (1.1), and $B_A(1, 1, 1) = 1$. Consequently, for all finite families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and all nonempty finite families X of subsets of $[m]$,*

$$N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \leq B_A(|\mathcal{F}_1|, |\mathcal{F}_2|, |\mathcal{F}_3|) \quad \text{and} \quad N_3(X) \leq \frac{|X|^{3c}}{\left(1 + \frac{3}{A} \ln |X|\right)^\gamma} \leq \left(\frac{A}{3}\right)^\gamma \frac{|X|^{3c}}{(1 + \ln |X|)^\gamma}.$$

In particular (1.5) is a Bellman function, and $N_3(X) \leq 2.44 |X|^{c(3)} (1 + \ln |X|)^{-\gamma_3}$.

The exponent γ_3 is sharp: for the two-layer families $X_m = \{T \subseteq [m] : |T| \in \{\frac{m}{3}, \frac{2m}{3}\}\}$ one has $N_3(X_m) \asymp |X_m|^{c(3)} (\ln |X_m|)^{-\gamma_3}$ [5, Thm. 1.1(b)]. Off the diagonal $G < M$ by the power-mean inequality, so already $B_\infty = (xyz)^c (G/M)^\kappa$ improves on the classical function (1.3), and the logarithmic factor gains the extra $(\ln x)^{-\gamma}$ on the diagonal. The constants α and κ are forced by the behaviour of the main inequality near the balanced configuration (Section 2.2). The shift 2000 comes from the explicit constants of the proof and is far from optimal: numerically (Section 3), B_A satisfies the main inequality if and only if $A \geq A_* \approx 29.05$, the binding sizes being $\ln(xyz) \approx 40$. With $A = A_*$ the diagonal constant would be $(A_*/3)^\gamma \approx 1.365$ (for $A = 30$ it is $10^\gamma = 1.371$).

Outline. Section 2 derives the counting bound (1.2) from the main inequality, explains the choice of the constants in logarithmic coordinates, and outlines the proof of Theorem 1.1. Section 3 reports high-precision numerical tests. The proof is given in Appendix A, and the computer-assisted steps, together with the scripts that perform them, are described in Appendix B.

2 The Bellman function

2.1 From the main inequality to the counting bound

Lemma 2.1. *Let $B: [1, \infty)^3 \rightarrow [0, \infty)$, extended by 0 as in Section 1, satisfy the main inequality and $B(1, 1, 1) \geq 1$. Then (1.2) holds for all $m \geq 0$, all families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and all families X of subsets of $[m]$.*

Proof. We use induction on m . For $m = 0$ the only subset of $[m]$ is $\emptyset$, so $N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = 1 \leq B(1, 1, 1)$ if $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \{\emptyset\}$, and $N(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = 0$ otherwise. Let $m \geq 1$. Sorting the partitions by the part that contains m gives $N(\mathcal{F}) = N(\mathcal{F}^{(1)}) + N(\mathcal{F}^{(2)}) + N(\mathcal{F}^{(3)})$, where $\mathcal{F}^{(i)}$ consists of the sets of $\mathcal{F}_i$ that contain m (with m removed) and the sets of $\mathcal{F}_j$, $j \neq i$, that do not; these are families of subsets of $[m-1]$. If x_1, y_1, z_1 are the numbers of sets in $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ that contain m and x_0, y_0, z_0 the numbers of the other sets, then the triples $\mathcal{F}^{(1)}, \mathcal{F}^{(2)}, \mathcal{F}^{(3)}$ have sizes (x_1, y_0, z_0) , (x_0, y_1, z_0) , (x_0, y_0, z_1) . By the induction hypothesis and the main inequality,

$$\begin{aligned} N(\mathcal{F}) &\leq B(x_1, y_0, z_0) + B(x_0, y_1, z_0) + B(x_0, y_0, z_1) \\ &\leq B(x_0 + x_1, y_0 + y_1, z_0 + z_1) = B(|\mathcal{F}_1|, |\mathcal{F}_2|, |\mathcal{F}_3|). \end{aligned}$$

Finally $N_3(X) = N(X, X, X^c)$ and $|X^c| = |X|$. $\square$

2.2 Logarithmic coordinates and the choice of the constants

Throughout, $\lambda = \ln 2$, $\mu = \ln \frac{27}{4}$, $\nu = \ln \frac{27}{16}$, and $c, \gamma, \alpha, \kappa$ are the constants of (1.3) and (1.4); thus $3c = c(3)$ and $\gamma = \frac{\nu}{2\mu} = \frac{2-3c}{2} = \gamma_3$. In logarithmic coordinates $\ell = (\ln x, \ln y, \ln z)$, $s = \ell_1 + \ell_2 + \ell_3$,

$$\begin{aligned} \ln B_A &= F(\ell) - \gamma \ln \left(1 + \frac{s}{A}\right), & F(\ell) &= c s - K s L\left(\frac{\ell}{s}\right), \\ L(w) &= \ln \left(\frac{1}{3} \sum_{j=1}^3 e^{\alpha(w_j - \frac{1}{3})}\right), & K &= \frac{\kappa}{\alpha}, \end{aligned}$$

because $\ln M = \frac{s}{\alpha} \ln \left(\frac{1}{3} \sum_j e^{\alpha \ell_j / s}\right)$ and $\ln G = \frac{s}{3}$. The function e^F is the lower envelope of an explicit family of weighted bounds $\prod_j |\mathcal{F}_j|^{b_j(\pi)}$, $\pi \in \Delta$, each of which satisfies the main inequality (Lemma A.3 and Remark A.5); this plays the role of V_0 in [5], and the single factor $(1 + s/A)^{-\gamma}$

plays the role of $\min\{1, \Theta\}$. Near the diagonal, $K L(\frac{1}{3}\mathbf{1} + y) = \beta|y|^2 + \kappa_3 y_1 y_2 y_3 + O(|y|^4)$ with $\beta = \frac{K\alpha^2}{6}$ and $\kappa_3 = \frac{K\alpha^3}{6}$, and both numbers are forced. (i) Per step, the logarithmic factor loses $\gamma\mu/s$ to leading order, and the curvature of the profile gains $\frac{2}{3}\beta\lambda^2/s$; the two balance exactly for $\beta = \frac{3\nu}{4\lambda^2} = 0.8168\dots$ (a larger β destroys admissibility, a smaller one loses at order $1/s$). (ii) The entropic (“layer”) profile has the cubic term $\kappa_3 = \frac{3\mu^2}{\lambda^3}(\frac{27\gamma}{4\lambda} - \frac{3(1+\gamma)}{4}) = 15.81\dots$, and admissibility forces the profile to match it (otherwise the admissibility gap Q of Proposition A.6 changes sign at the balanced point). The log-sum-exp profile is the simplest one with independent quadratic and cubic coefficients; matching β and κ_3 gives $\alpha = \kappa_3/\beta$ and $\kappa = 6\beta/\alpha$, i.e. the values (1.4). The only remaining freedom is the shift A in the logarithmic factor, which absorbs the errors of order s^{-2} .

2.3 Structure of the proof

The proof is given in Appendix A. By the Gibbs variational principle the main inequality is equivalent to the nonnegativity of an explicit slack $E(\ell, u, q)$, where $u \in [0, 1]^3$ are the split fractions and q is a probability vector on the three children (Lemma A.2); E is increasing in A , so $A = 2000$ suffices. Two exact identities do most of the work: the envelope representation of F (Lemma A.3) and a decomposition of E into five explicit terms (Lemma A.4). Four of them are nonnegative as soon as the *admissibility gap* $Q(q) = \frac{I(q)}{S(q)} - K L(\frac{h(q)}{S(q)})$ is nonnegative (Proposition A.6); the fifth is the loss caused by the logarithmic factor. Away from the balanced configuration the gap, a Kullback–Leibler term in the split, or a Kullback–Leibler term in the shape absorbs the loss (Proposition A.10). Near it, a second-order expansion with explicit constants reduces everything to an inequality in the single variable s (Proposition A.16). The finitely many numerical facts used are verified rigorously in ball arithmetic, or exactly in symbolic arithmetic (Appendix B).

3 Numerical verification

For the main inequality we write $\delta = \ln \text{LHS} - \ln \text{RHS}$ (so the relative slack is $1 - \text{RHS}/\text{LHS} = 1 - e^{-\delta}$; $\delta = +\infty$ if the right side is 0), parametrised by the parent sizes $X = e^{\ell_1}$, $Y = e^{\ell_2}$, $Z = e^{\ell_3}$ and the split fractions $u = (x_1/X, y_1/Y, z_1/Z)$. Equality holds at the *trivial splits* $u \in \{e_1, e_2, e_3\}$ (e.g. $x_0 = y_1 = z_1 = 0$), where a single child survives and equals the parent; it was not observed anywhere else. As u approaches a trivial split the slack vanishes linearly, and near the balanced configuration it is of order s^{-2} ; the worst-case slack is therefore reported in these normalisations. All values are computed with `mpmath` at 50 significant digits from the logarithmic form of Section 2.2 (the closed form (1.5) agrees with it to a relative error of $5 \cdot 10^{-50}$ at 300 random points). Script: `verification/numerics.py` (6.5 minutes on 8 cores); raw output: `verification/results/`.

Integer grid. All integer parents $1 \leq X, Y, Z \leq 24$ and all splits $0 \leq x_1 \leq X$, $0 \leq y_1 \leq Y$, $0 \leq z_1 \leq Z$: 34 012 224 configurations for each A , of which 41 472 are trivial splits (where $|\delta| < 10^{-25}$ was checked) and 546 048 have a vanishing right side.

A	$\min \delta$ (non-trivial splits)	relative slack $1 - e^{-\delta}$	attained at $(x_1, x_0, y_1, y_0, z_1, z_0)$
2000	$7.1444 \cdot 10^{-3}$	0.71%	(1, 4, 1, 4, 14, 9)
30	$7.6306 \cdot 10^{-4}$	0.076%	(1, 4, 1, 4, 12, 8)
∞	$7.2582 \cdot 10^{-3}$	0.72%	(1, 4, 1, 4, 14, 9)

Random samples. 400 000 real configurations per class and per value of A (seeded; the same samples for each A), in five classes: *generic* ($s = 10^{U(-3,5)}$, shape ℓ/s uniform on the simplex, u uniform in $[0, 1]^3$); *balanced* ($s = 10^{U(0,6)}$, shape within $\approx s^{-1/2}$ of $\frac{1}{3}\mathbf{1}$, $u_j = \frac{1}{3} + N(0, \sigma^2)$, $\sigma = 10^{U(-4, -0.7)}$); *faces* (one or two parent sizes in $[1, 1.1]$ or exactly 1, and $u_j \in \{\varepsilon, 1 - \varepsilon, U(0, 1)\}$, $\varepsilon = 10^{U(-12, -1)}$); *near-trivial* ($u = e_k + O(10^{U(-10, -0.5)})$); *child at threshold* (an active child one of whose sizes is e^ε , $\varepsilon = 10^{U(-12, -1)}$, i.e. just above the jump of B from 0 to ≥ 1). The table gives the number of samples with a positive right side (for the others the inequality is trivial), and the minima of δ and of the normalised slack appropriate to the class: $s^2\delta$ (over the samples with $s \geq 10$), or $\delta/\text{dist}(u)$ with $\text{dist}(u) = \min_k |u - e_k|_1$.

class	RHS > 0	minimum of	$A = 2000$	$A = 30$	$A = \infty$
generic	214 515	δ	$1.02 \cdot 10^{-3}$	$4.92 \cdot 10^{-4}$	$1.13 \cdot 10^{-3}$
		$s^2\delta$	0.976	0.258	0.990
balanced	368 294	δ	$2.33 \cdot 10^{-9}$	$6.36 \cdot 10^{-10}$	$2.69 \cdot 10^{-7}$
		$s^2\delta$	0.981	0.181	0.994
faces	68 463	δ	$2.46 \cdot 10^{-12}$	$2.44 \cdot 10^{-12}$	$2.46 \cdot 10^{-12}$
		$\delta/\text{dist}(u)$	0.0100	0.0086	0.0101
near-trivial	384 027	δ	$1.64 \cdot 10^{-10}$	$1.64 \cdot 10^{-10}$	$1.64 \cdot 10^{-10}$
		$\delta/\text{dist}(u)$	0.0219	0.0167	0.0221
child at threshold	399 707	δ	$6.64 \cdot 10^{-4}$	$6.62 \cdot 10^{-4}$	$6.64 \cdot 10^{-4}$
		$s^2\delta$	1.036	0.425	1.048

Worst case at fixed size. For each s the minimum of $s^2\delta$ over the shape ℓ/s and over $u \in [0.02, 0.98]^3$ (the near-trivial region is covered by the samples above) was searched by Nelder–Mead from 16 starting points, with every evaluation in 50-digit arithmetic. Negative entries are violations.

$A \setminus s$	10	30	10^2	300	10^3	10^4	10^5	10^6
2000	0.687	3.779	18.89	61.57	167.8	429.8	506.9	516.2
30	0.105	0.076	0.309	0.857	1.326	1.695	1.795	1.824
25	0.012	-0.278	-0.489	-0.237	0.089	0.394	0.487	0.515
22	-0.058	-0.523	-1.000	-0.909	-0.659	-0.387	-0.297	-0.269

For large s the minimal $s^2\delta$ approaches $\gamma\mu(A - A_\infty)$ with $\gamma\mu = \frac{\nu}{2} = 0.2616$ and $A_\infty \approx 23.0$ (interpolating linearly between the rows $A = 22$ and $A = 25$, the zero crossings in A at $s = 10^4, 10^5, 10^6$ are 23.49, 23.14, 23.03; the direct computation below gives $A_*(10^4) = 23.486$); for $A = 2000$ this is ≈ 517 .

The smallest admissible shift. For fixed $(s, \ell/s, u)$ the slack is non-decreasing in A (Lemma A.2), so it changes sign at a critical value $A_0(s, \ell/s, u)$, and B_A satisfies the main inequality iff $A \geq A_*(s)$ for every s , where $A_*(s)$ is the supremum of A_0 over the shape and the split. The curve $A_*(s)$ was computed by maximising A_0 (root-finding in A ; 20 000 random seeds and 24 Nelder–Mead runs per size; `verification/threshold.py`); at every maximiser the sign change of δ between $A_0 - 0.01$ and $A_0 + 0.01$ was confirmed in 50-digit arithmetic. The maximum is $A_* = 29.05$, attained at $s \approx 40$ (Figure 1).

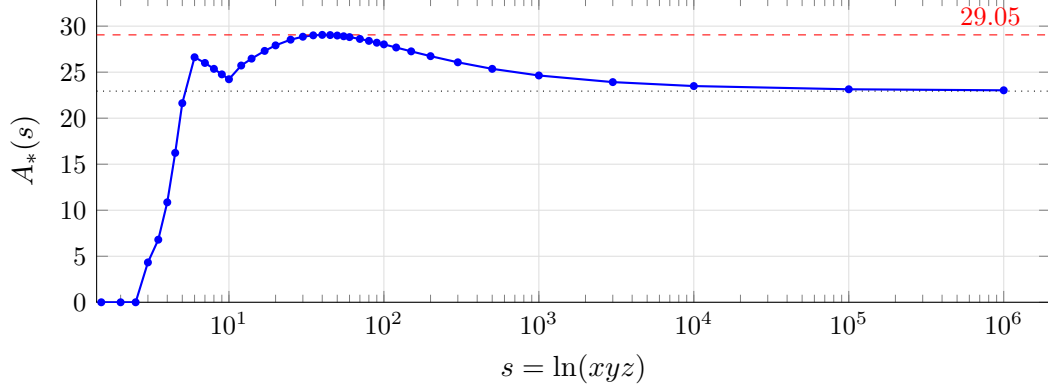


Figure 1: The smallest admissible shift $A_*(s)$ at each size s . Dashed: its maximum 29.05; dotted: the large- s limit ≈ 23 . The two rightmost points are the zero crossings in A of the worst-case values in the table above (linear interpolation between $A = 22$ and $A = 25$); the others are computed directly.

Summary. No violation was found for $A = 2000$, $A = 30$ or $A = \infty$: $3.4 \cdot 10^7$ integer configurations and $2 \cdot 10^6$ random real configurations, each evaluated for the three values of A , including parents on the faces of $[1, \infty)^3$, splits near the trivial ones, and children at the activation threshold. For $A = 2000$ the worst-case slack is: $\delta \geq 7.14 \cdot 10^{-3}$ (relative slack 0.71%) on the integer grid; $\min s^2 \delta = 0.687$ at $s = 10$, increasing with s to 516 at $s = 10^6$ (asymptotically ≈ 517); and $\delta \geq 0.0100 \text{ dist}(u)$ near the trivial splits. In the proof itself the final one-variable inequality holds with ratio deficit/gain at most 0.644 (Appendix B).

A Proof of Theorem 1.1

A.1 Notation and reduction

Let $\Delta = \{q \in [0, 1]^3 : \sum_j q_j = 1\}$, $U = \frac{1}{3}\mathbf{1}$, e_1, e_2, e_3 the unit vectors (the vertices of Δ), $\mathbf{1}^\perp = \{v \in \mathbb{R}^3 : \sum_j v_j = 0\}$, $|\cdot|$ the Euclidean norm, and P the orthogonal projection onto $\mathbf{1}^\perp$. We use $0 \ln 0 = 0$,

$$h(t) = -t \ln t - (1-t) \ln(1-t), \quad H(q) = -\sum_j q_j \ln q_j,$$

$$d(x \| y) = x \ln \frac{x}{y} + (1-x) \ln \frac{1-x}{1-y}, \quad D(p \| p') = \sum_j p_j \ln \frac{p_j}{p'_j},$$

apply h and d coordinatewise ($h(q) = (h(q_j))_j$, $d(q \| u) = (d(q_j \| u_j))_j$), and put $S(q) = \sum_j h(q_j)$. For $p \in \Delta$ and $f, g \in \mathbb{R}^3$ (functions on $\{1, 2, 3\}$), $\mathbb{E}_p f = \sum_j p_j f_j$, $\tilde{f} = f - \mathbb{E}_p f$, $\text{Cov}_p(f, g) = \mathbb{E}_p \tilde{f} \tilde{g}$, $\text{Var}_p(f) = \text{Cov}_p(f, f)$, and $\text{osc } f = \max_j f_j - \min_j f_j$. For $\zeta \in \mathbb{R}^3$ let $\pi(\zeta) = (e^{\alpha \zeta_j} / \sum_k e^{\alpha \zeta_k})_j$. For $u \in [0, 1]^3$ and $i \in \{1, 2, 3\}$ let $D^{(i)}(u) \in [-\infty, 0]^3$,

$$D_i^{(i)}(u) = \ln u_i, \quad D_j^{(i)}(u) = \ln(1 - u_j) \ (j \neq i), \quad d_i(u) = -\sum_j D_j^{(i)}(u) \in [0, \infty].$$

For $\ell \in [0, \infty)^3$ let $s = \ell_1 + \ell_2 + \ell_3$, $\xi = \ell/s \in \Delta$ (if $s > 0$), $\ell^{(i)} = \ell + D^{(i)}(u)$ (the children), $s_i = s - d_i(u) \leq s$, $\mathcal{A}(\ell, u) = \{i : \ell^{(i)} \in [0, \infty)^3\}$ (the active children), and $\Delta(\mathcal{A}) = \{q \in \Delta : q_i =$

0 for $i \notin \mathcal{A}$. Finally $K = \kappa/\alpha$, $b_{\min} = c - \frac{2}{3}\kappa = 0.40658\dots$,

$$L(w) = \ln\left(\frac{1}{3} \sum_j e^{\alpha(w_j - \frac{1}{3})}\right), \quad F(\ell) = cs - KsL(\xi) \quad (s > 0), \quad F(0) = 0,$$

$$\Phi_A(\ell) = F(\ell) - \gamma \ln\left(1 + \frac{s}{A}\right),$$

so that $\ln B_A(x, y, z) = \Phi_A(\ln x, \ln y, \ln z)$ (Section 2.2). Decimal values are rounded; the computer-assisted steps use rigorous enclosures of all constants.

Lemma A.1 (Gibbs principle). (a) For a finite nonempty set $\mathcal{A}$ and $a \in \mathbb{R}^{\mathcal{A}}$, $\ln \sum_{i \in \mathcal{A}} e^{a_i} = \max_{q \in \Delta(\mathcal{A})} (H(q) + \sum_i q_i a_i)$.

(b) If $q \in \Delta(\mathcal{A}(\ell, u))$, then $D^{(i)}(u)$ is finite whenever $q_i > 0$, $\sum_i q_i D^{(i)}(u) = -h(q) - d(q \| u)$ (all entries finite), and hence $\sum_i q_i d_i(u) = S(q) + \sum_j d(q_j \| u_j)$.

(c) $d(x \| y) \geq 2(x - y)^2$ for $x, y \in [0, 1]$ (Pinsker).

Proof. (a) Let $\pi_i = e^{a_i} / \sum_k e^{a_k}$. For $q \in \Delta(\mathcal{A})$, Jensen's inequality gives $H(q) + \sum_i q_i a_i - \ln \sum_k e^{a_k} = \sum_{q_i > 0} q_i \ln \frac{\pi_i}{q_i} \leq \ln \sum_{q_i > 0} \pi_i \leq 0$, with equality for $q = \pi$. (b) If $q_i > 0$ then $i \in \mathcal{A}(\ell, u)$, so $D^{(i)}(u)$ is finite. The j -th entry of $\sum_i q_i D^{(i)}(u)$ is $q_j \ln u_j + (1 - q_j) \ln(1 - u_j) = -h(q_j) - d(q_j \| u_j)$. It is finite: $q_j > 0$ forces $u_j > 0$, and $q_j < 1$ forces $q_i > 0$ for some $i \neq j$, hence $u_j < 1$. Summing over j gives the last identity. (c) is Pinsker's inequality for two-point distributions. $\square$

Lemma A.2 (reduction). Let $0 < A \leq \infty$. The function B_A satisfies the main inequality if and only if

$$E_A(\ell, u, q) := \Phi_A(\ell) - H(q) - \sum_i q_i \Phi_A(\ell^{(i)}) \geq 0 \quad \text{for all } \ell \in [0, \infty)^3, u \in [0, 1]^3, q \in \Delta(\mathcal{A}(\ell, u)). \quad (\text{A.1})$$

For fixed (ℓ, u, q) , E_A is non-decreasing in A . Hence, if B_{2000} satisfies the main inequality, then so does B_A for every $A \in [2000, \infty]$.

Proof. Let $x_0, \dots, z_1 \geq 0$ and $X = x_0 + x_1$, $Y = y_0 + y_1$, $Z = z_0 + z_1$. Each child (x_1, y_0, z_0) , (x_0, y_1, z_0) , (x_0, y_0, z_1) is coordinatewise at most (X, Y, Z) , so if $\min\{X, Y, Z\} < 1$ the right side vanishes. Otherwise let $\ell = (\ln X, \ln Y, \ln Z) \in [0, \infty)^3$ and $u = (x_1/X, y_1/Y, z_1/Z) \in [0, 1]^3$. The i -th child is $e^{\ell^{(i)}}$ (coordinatewise, $e^{-\infty} = 0$), and all its coordinates are ≥ 1 iff $i \in \mathcal{A}(\ell, u)$. Thus the main inequality says $\sum_{i \in \mathcal{A}(\ell, u)} e^{\Phi_A(\ell^{(i)})} \leq e^{\Phi_A(\ell)}$, and every $(\ell, u) \in [0, \infty)^3 \times [0, 1]^3$ arises. By Lemma A.1(a) this is (A.1) (nothing is to be proved if $\mathcal{A}(\ell, u) = \emptyset$). Next, $E_A = E_\infty - \gamma \sum_i q_i \ln \frac{A+s}{A+s_i}$, and since $s_i \leq s$ each logarithm is non-increasing in A and tends to 0 as $A \rightarrow \infty$. $\square$

From now on $A = 2000$, and we write $\Phi = \Phi_{2000}$, $E = E_{2000}$.

A.2 The function F

Lemma A.3. For $\pi \in \Delta$ let $b(\pi) = c\mathbf{1} - \kappa(\pi - U) + K D(\pi \| U) \mathbf{1}$, where $D(\pi \| U) = \ln 3 - H(\pi)$. For all $\ell' \in [0, \infty)^3$ with $s' = \sum_j \ell'_j > 0$ and all $\pi \in \Delta$,

$$F(\ell') = \langle b(\pi), \ell' \rangle - K s' D(\pi \| \pi(\ell'/s')). \quad (\text{A.2})$$

Consequently:

- (a) $F(\ell') = \min_{\pi \in \Delta} \langle b(\pi), \ell' \rangle$ on $[0, \infty)^3$, attained at $\pi = \pi(\ell'/s')$; F is concave and positively homogeneous of degree one.
- (b) If $s > 0$ and $\pi = \pi(\xi)$, then $F(\ell') = F(\ell) + \langle b(\pi), \ell' - \ell \rangle - K s' D(\pi \| \pi(\ell'/s'))$ for all $\ell' \in [0, \infty)^3$ (the last term read as 0 if $s' = 0$). In particular $\nabla F(\ell) = b(\pi(\xi))$.
- (c) $b_j(\pi) \geq b_{\min}$ for all π, j , and $\Phi(\ell') - \Phi(\ell'') \geq (b_{\min} - \gamma/2000)(s' - s'') \geq 0$ whenever $\ell' \geq \ell''$ coordinatewise.

Proof. Let $w = \ell'/s' \in \Delta$. Since $\ln \sum_k e^{\alpha w_k} = L(w) + \ln 3 + \frac{\alpha}{3}$, we have $\ln \pi_j(w) = \alpha(w_j - \frac{1}{3}) - \ln 3 - L(w)$, and since $\sum_j \pi_j(w_j - \frac{1}{3}) = \langle \pi - U, w \rangle$ (because $\langle U, w \rangle = \frac{1}{3}$),

$$D(\pi \| \pi(w)) = \sum_j \pi_j \ln \pi_j + \ln 3 - \alpha \langle \pi - U, w \rangle + L(w) = D(\pi \| U) - \alpha \langle \pi - U, w \rangle + L(w).$$

Multiplying by $K s'$ and using $K\alpha = \kappa$ gives $K s' D(\pi \| \pi(w)) = \langle b(\pi), \ell' \rangle - c s' + K s' L(w)$, which is (A.2). (a) $D(\pi \| \pi(w)) \geq 0$ with equality iff $\pi = \pi(w)$; for $s' = 0$ both sides vanish. A minimum of linear functions is concave and homogeneous. (b) By (a), $F(\ell) = \langle b(\pi), \ell \rangle$; subtract this from (A.2). If $s' = 0$ then $\ell' = 0$ and both sides vanish. Since $D(\pi \| \pi(\ell'/s'))$ vanishes to second order at $\ell' = \ell$, $\nabla F(\ell) = b(\pi)$. (c) $b_j(\pi) \geq c - \kappa(\pi_j - \frac{1}{3}) \geq c - \frac{2}{3}\kappa$, because $D(\pi \| U) \geq 0$ and $\pi_j \leq 1$. If $\ell' \geq \ell''$ and $s' > 0$, take $\pi' = \pi(\ell'/s')$; by (a), $F(\ell') - F(\ell'') \geq \langle b(\pi'), \ell' - \ell'' \rangle \geq b_{\min}(s' - s'')$, while $\ln(1 + \frac{s'}{2000}) - \ln(1 + \frac{s''}{2000}) \leq \frac{s' - s''}{2000}$. (If $s' = 0$ then $\ell' = \ell'' = 0$.) $\square$

A.3 The exact slack

For $q \in \Delta \setminus \{e_1, e_2, e_3\}$ we have $S(q) > 0$, and we put

$$w(q) = \frac{h(q)}{S(q)} \in \Delta, \quad I(q) = c S(q) - H(q), \quad Q(q) = \frac{I(q)}{S(q)} - K L(w(q)), \quad r(q) = |q - U|.$$

Q is the *admissibility gap*; S, H, I, Q are invariant under permutations of the coordinates, and w is equivariant.

Lemma A.4 (slack identity). *Let $\ell \in [0, \infty)^3$ with $s > 0$, $u \in [0, 1]^3$, and $q \in \Delta(\mathcal{A}(\ell, u)) \setminus \{e_1, e_2, e_3\}$. Put $\pi = \pi(\xi)$, $b = b(\pi)$, $\pi_w = \pi(w(q))$, and $\pi_i = \pi(\ell^{(i)}/s_i)$ for $q_i > 0$, $s_i > 0$. Then*

$$E = S(q) Q(q) + K S(q) D(\pi \| \pi_w) + \sum_j b_j d(q_j \| u_j) + K \sum_i q_i s_i D(\pi \| \pi_i) - \gamma \sum_i q_i \ln \frac{2000 + s}{2000 + s_i}, \quad (\text{A.3})$$

where terms with $s_i = 0$ are read as 0, and the last sum (the loss) satisfies

$$0 \leq \gamma \sum_i q_i \ln \frac{2000 + s}{2000 + s_i} \leq \gamma \frac{S(q) + \sum_j d(q_j \| u_j)}{2000 + s - \max_{i: q_i > 0} d_i(u)}. \quad (\text{A.4})$$

Proof. By Lemma A.3(b) with $\ell' = \ell^{(i)}$, $F(\ell^{(i)}) = F(\ell) + \langle b, D^{(i)}(u) \rangle - K s_i D(\pi \| \pi_i)$. Averaging with weights q_i and using Lemma A.1(b),

$$E = \langle b, h(q) \rangle - H(q) + \langle b, d(q \| u) \rangle + K \sum_i q_i s_i D(\pi \| \pi_i) - \gamma \sum_i q_i \ln \frac{2000 + s}{2000 + s_i}.$$

By (A.2) with $\ell' = w(q)$ ($s' = 1$), $\langle b, w(q) \rangle = F(w(q)) + K D(\pi \| \pi_w) = c - K L(w(q)) + K D(\pi \| \pi_w)$; multiplying by $S(q)$, $\langle b, h(q) \rangle - H(q) = S(q) Q(q) + K S(q) D(\pi \| \pi_w)$. For (A.4) use $\ln \frac{2000+s}{2000+s_i} = \ln(1 + \frac{d_i}{2000+s-d_i}) \leq \frac{d_i}{2000+s-d_i}$ and $\sum_i q_i d_i = S(q) + \sum_j d(q_j \| u_j)$. $\square$

Remark A.5. By the same computation with $V = e^{\langle b(\pi), \ell \rangle}$ in place of e^Φ , the slack of a single weighted bound is $S(q)Q(q) + KS(q)D(\pi \parallel \pi_w) + \langle b(\pi), d(q \parallel u) \rangle$. So, by Proposition A.6 below, every weighted bound $\prod_j |\mathcal{F}_j|^{b_j(\pi)}$, $\pi \in \Delta$, satisfies the main inequality, and e^F is their lower envelope.

A.4 The admissibility gap

Proposition A.6. Let $c_4 = 0.2$ and $r_q = 0.13605$. For $q \in \Delta \setminus \{e_1, e_2, e_3\}$:

- (i) $Q(q) \geq c_4 r(q)^4$ if $r(q) \leq r_q$;
- (ii) $Q(q) \geq \gamma/2000 = 6.8504 \dots \cdot 10^{-5}$ if $r(q) \geq r_q$.

In particular $Q \geq 0$, with equality only at $q = U$.

The proof combines an exact expansion at U , an analytic bound near the vertices, and a rigorous cover of the rest.

Lemma A.7 (expansion at U). For $v \in \mathbf{1}^\perp$ and $t \rightarrow 0$, with $\sum v^3 = \sum_j v_j^3$,

$$\begin{aligned} S(U + tv) &= \mu - \frac{9}{4}t^2|v|^2 + \frac{9}{8}t^3 \sum v^3 + O(t^4), & H(U + tv) &= \ln 3 - \frac{3}{2}t^2|v|^2 + \frac{3}{2}t^3 \sum v^3 + O(t^4), \\ I(U + tv) &= \frac{3\gamma}{2}t^2|v|^2 - \frac{3}{4}(1 + \gamma)t^3 \sum v^3 + O(t^4), \\ w(U + tv) - U &= \frac{\lambda}{\mu}tv + \frac{9}{4\mu}t^2\left(\frac{|v|^2}{3}\mathbf{1} - v \circ v\right) + O(t^3), \end{aligned}$$

and for $y \in \mathbf{1}^\perp$, $KL(U + y) = \beta|y|^2 + \kappa_3 y_1 y_2 y_3 + O(|y|^4)$ with $\beta = \frac{K\alpha^2}{6}$, $\kappa_3 = \frac{K\alpha^3}{6}$. The constants (1.4) satisfy $\beta = \frac{3\nu}{4\lambda^2}$ and $\kappa_3 = \frac{3\mu^2}{\lambda^3}\left(\frac{27\gamma}{4\lambda} - \frac{3(1+\gamma)}{4}\right)$, and

$$Q(U + tv) = q_4|v|^4 t^4 + O(t^5), \quad q_4 = 0.3480572556 \dots > 0.$$

Proof. At $\frac{1}{3}$: $h = \ln 3 - \frac{2}{3}\lambda$, $h' = \lambda$, $h'' = -\frac{9}{2}$, $h''' = \frac{27}{4}$, and for $e(t) = -t \ln t$: $e = \frac{1}{3} \ln 3$, $e' = \ln 3 - 1$, $e'' = -3$, $e''' = 9$. Summing the Taylor expansions over j (with $\sum_j v_j = 0$, $3h(\frac{1}{3}) = \mu$) gives S and H ; $I = cS - H$ follows from $c\mu = \ln 3$, $\frac{3}{2} - \frac{9}{4}c = \frac{3}{2}\gamma$ and $\frac{9}{8}c - \frac{3}{2} = -\frac{3}{4}(1 + \gamma)$ (recall $\frac{3}{2}c = 1 - \gamma$); w follows by dividing. Next, $\frac{1}{3} \sum_j e^{\alpha y_j} = 1 + \frac{\alpha^2}{6}|y|^2 + \frac{\alpha^3}{18} \sum y^3 + O(|y|^4)$ and $\sum y^3 = 3y_1 y_2 y_3$ on $\mathbf{1}^\perp$. With $y = w - U$: $|y|^2 = \frac{\lambda^2}{\mu^2}t^2|v|^2 - \frac{9\lambda}{2\mu^2}t^3 \sum v^3 + O(t^4)$ and $y_1 y_2 y_3 = \frac{\lambda^3}{3\mu^3}t^3 \sum v^3 + O(t^4)$, while $I/S = I/\mu + O(t^4)$. Hence Q has no terms of order t^0, t^1 ; the t^2 coefficient is $(\frac{3\gamma}{2\mu} - \beta \frac{\lambda^2}{\mu^2})|v|^2$, and the t^3 coefficient is $(-\frac{3(1+\gamma)}{4\mu} + \frac{9\lambda\beta}{2\mu^2} - \frac{\kappa_3\lambda^3}{3\mu^3}) \sum v^3$. Both vanish iff $\beta = \frac{3\gamma\mu}{2\lambda^2} = \frac{3\nu}{4\lambda^2}$ and κ_3 has the stated value; since $\alpha = \kappa_3/\beta$ and $K = 6\beta/\alpha^2$ by the definitions (1.4) of α and κ , this is the case. By symmetry the t^4 coefficient is an S_3 -invariant quartic form on $\mathbf{1}^\perp$, hence a multiple of $|v|^4$ (the invariant ring is generated by $|v|^2$ and $v_1 v_2 v_3$). Its value q_4 , and independently the vanishing of the coefficients of order ≤ 3 , were computed exactly, with $\ln 2, \ln 3$ as independent symbols (claim (C1) of Appendix B). $\square$

Lemma A.8 (near the vertices). Let $q \in \Delta$ be no vertex, and $\sigma = 1 - \max_j q_j \leq \sigma_V = 10^{-3}$. Then, with $\varepsilon_V = (\ln 2 + \frac{\sigma_V}{4(1-\sigma_V)})/(4 \ln \frac{1}{\sigma_V})$,

$$Q(q) \geq c - \frac{1}{2} - \frac{\ln 2}{4 \ln(1/\sigma_V)} - KL(\frac{1}{2} - \varepsilon_V, \frac{1}{2} + \varepsilon_V, 0) = 0.011869 \dots > \frac{\gamma}{2000}.$$

Proof. Let $q_1 = 1 - \sigma$ be the largest coordinate, so $q_2 + q_3 = \sigma$. Write $h = e + \psi$ with $e(t) = -t \ln t$, $\psi(t) = -(1-t) \ln(1-t)$; both are concave and vanish at 0, hence subadditive, and $e(1-t) = \psi(t)$. Since $S = H + \sum_j \psi(q_j)$,

$$2H - S = H - \sum_j \psi(q_j) = (e(q_2) + e(q_3) - e(\sigma)) + (\psi(\sigma) - \psi(q_2) - \psi(q_3)) \leq \sigma \ln 2,$$

by the grouping property $e(q_2) + e(q_3) = e(\sigma) + \sigma h(q_2/\sigma)$ and subadditivity of ψ . Also $S = h(\sigma) + h(q_2) + h(q_3) \geq 2h(\sigma) \geq 2\sigma \ln \frac{1}{\sigma}$ by subadditivity of h . Hence $\frac{H}{S} \leq \frac{1}{2} + \frac{\ln 2}{4 \ln(1/\sigma)}$. Next, $w_1 = h(\sigma)/S \leq \frac{1}{2}$, and, by concavity, $h(q_2) + h(q_3) \leq 2h(\frac{\sigma}{2}) = h(\sigma) + \sigma \ln 2 + 2\psi(\frac{\sigma}{2}) - \psi(\sigma)$, where $2\psi(\frac{\sigma}{2}) - \psi(\sigma) = \sum_{k \geq 2} \frac{1-2^{1-k}}{k(k-1)} \sigma^k \leq \frac{\sigma^2}{4(1-\sigma)}$. Using $\frac{1}{1+y} \geq 1-y$ and $h(\sigma) \geq \sigma \ln \frac{1}{\sigma}$ we get $w_1 \geq \frac{1}{2} - \varepsilon(\sigma)$, $\varepsilon(\sigma) = (\ln 2 + \frac{\sigma}{4 \ln(1-\sigma)}) / (4 \ln \frac{1}{\sigma})$. So $w(q)$ lies in the polygon $\{w \in \Delta : \frac{1}{2} - \varepsilon \leq w_1 \leq \frac{1}{2}\}$; the convex symmetric function L attains its maximum there at a vertex, and the largest vertex value is $L(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 0)$ (the vertex $(\frac{1}{2}, \frac{1}{2}, 0)$ is the midpoint of $(\frac{1}{2} \mp \varepsilon, \frac{1}{2} \pm \varepsilon, 0)$). Therefore $Q = c - \frac{H}{S} - KL(w) \geq c - \frac{1}{2} - \frac{\ln 2}{4 \ln(1/\sigma)} - KL(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 0)$; the right side decreases in σ , and its value at σ_V is enclosed in (C2). $\square$

Proof of Proposition A.6. For $r(q) \leq r_0 = 0.003$: by Taylor's formula along the ray, $Q(U + tv) = q_4 t^4 + R_5$ ($|v| = 1$) with $|R_5| \leq M_5 t^5$, where $M_5 = 9.374$ bounds the fifth Taylor coefficient of $\tau \mapsto Q(U + t'v + \tau v)$ for all $t' \leq r_0$ and all directions (C2). Hence $Q \geq (q_4 - r_0 M_5) t^4 \geq 0.3199 t^4 \geq c_4 t^4$. In the vertex regions $\{\max_j q_j \geq 1 - \sigma_V\}$, where $r(q) > 0.8 > r_q$, Lemma A.8 applies. On the rest of Δ the inequalities $Q \geq c_4 r^4$ (for $r \leq r_q$) and $Q \geq \gamma/2000$ (for $r \geq r_q$) are verified on a finite cover by boxes (C2). The last statement follows since $c_4 r^4 > 0$ for $0 < r \leq r_q$. $\square$

A.5 Far from the balanced configuration

In this and the next subsection, $\ell \in [0, \infty)^3$ with $s > 0$, $u \in [0, 1]^3$, $q \in \Delta(\mathcal{A}(\ell, u)) \setminus \{e_1, e_2, e_3\}$, with the notation of Lemma A.4. Since $b_j \geq b_{\min}$ and $2000 + s - d_i = 2000 + s_i \geq 2000$, (A.3) and (A.4) give

$$E \geq S(q) \left(Q(q) - \frac{\gamma}{2000} \right) + \left(b_{\min} - \frac{\gamma}{2000} \right) \sum_j d(q_j \| u_j) + K S(q) D(\pi \| \pi_w) + K \sum_i q_i s_i D(\pi \| \pi_i). \quad (\text{A.5})$$

Lemma A.9 (divergence in the shape). *Let $w \in \Delta$, let $e \in \mathbf{1}^\perp$ be a unit vector, and $\pi_\tau = \pi(w + \tau e)$. Then, for $\rho \geq 0$,*

$$D(\pi_\rho \| \pi_0) = \alpha^2 \int_0^\rho \tau \operatorname{Var}_{\pi_\tau}(e) d\tau,$$

which is non-decreasing in ρ . For $p \in \Delta$ the quadratic form $v \mapsto \operatorname{Var}_p(v)$ on $\mathbf{1}^\perp$ has smallest eigenvalue $\lambda_{\min}(p) = \frac{6P}{T + \sqrt{T^2 - 12P}}$, where $T = 1 - |p|^2$ and $P = p_1 p_2 p_3$. Consequently, if $\lambda_{\min}(\pi_\tau) \geq \Lambda$ for $0 \leq \tau \leq \rho_0$, then $D(\pi_\rho \| \pi_0) \geq \frac{1}{2} \alpha^2 \Lambda \min\{\rho, \rho_0\}^2$ for all $\rho \geq 0$.

Proof. $\pi_\tau(j) = \pi_0(j) e^{\alpha \tau e_j} / Z(\tau)$ with $\psi(\tau) = \ln Z(\tau)$, $\psi' = \alpha \mathbb{E}_{\pi_\tau} e$, $\psi'' = \alpha^2 \operatorname{Var}_{\pi_\tau}(e)$. Hence $D(\pi_\rho \| \pi_0) = \rho \psi'(\rho) - \psi(\rho) + \psi(0)$, whose derivative in ρ is $\rho \psi''(\rho)$. The form is $v^\top (\operatorname{diag} p - p p^\top) v$; the matrix annihilates $\mathbf{1}$, has trace T , and each of its three principal 2×2 minors equals $p_1 p_2 p_3$, so its eigenvalues on $\mathbf{1}^\perp$ are the roots of $t^2 - Tt + 3P$. $\square$

Let $R_w = 0.0607$, $r_x = 0.017$, $R_X = R_w + r_x = 0.0777$, $\Lambda = 0.1$. By (C4.1), $|w(q) - U| \leq R_w$ whenever $r(q) \leq r_q$, and by (C4.2), $\lambda_{\min}(\pi(\zeta)) \geq \Lambda$ whenever $\sum_j \zeta_j = 1$ and $|\zeta - U| \leq R_X$. Put

$$r_u = \left(\frac{\gamma \mu}{2 \cdot 2000 (b_{\min} - \gamma/2000)} \right)^{1/2} = 0.012684 \dots$$

Proposition A.10 (far cases). $E \geq 0$ in each of the cases (F1) $r(q) \geq r_q$; (F2) $|u - q| \geq r_u$; (F3) $r(q) < r_q$ and $|\xi - w(q)| \geq r_x$.

Proof. All terms of (A.5) other than the first are ≥ 0 , and $Q \geq 0$ (Proposition A.6). (F1) By Proposition A.6(ii) the first term is ≥ 0 as well. (F2) S is concave and symmetric, so $S \leq S(U) = \mu$; with Pinsker, $E \geq -\frac{\gamma\mu}{2000} + 2(b_{\min} - \frac{\gamma}{2000})|u - q|^2 \geq 0$. (F3) $E \geq S(q)(KD(\pi(\xi) \parallel \pi(w)) - \frac{\gamma}{2000})$ with $w = w(q)$. Apply Lemma A.9 with $e = (\xi - w)/|\xi - w|$ and $\rho_0 = r_x$ (the points $w + \tau e$, $\tau \leq r_x$, satisfy $|w + \tau e - U| \leq R_X$): $KD(\pi(\xi) \parallel \pi(w)) \geq \frac{K\alpha^2}{2}\Lambda r_x^2 = 7.0817 \cdot 10^{-5} \geq \frac{\gamma}{2000}$ (C4.2). $\square$

A.6 Near the balanced configuration

It remains to treat *case N*: $r(q) < r_q$, $|u - q| < r_u$, $|\xi - w(q)| < r_x$. We record some consequences, with

$$\delta_u = \sqrt{\frac{2}{3}}r_q + r_u, \quad t_0 = \frac{1}{3} - \sqrt{\frac{2}{3}}r_q, \quad S_{\min} = \mu - \frac{r_q^2}{2t_0(1-t_0)},$$

$$\bar{d} = -\ln(\frac{1}{3} - \delta_u) - 2\ln(\frac{2}{3} - \delta_u), \quad \underline{d} = -\ln(\frac{1}{3} + \delta_u) - 2\ln(\frac{2}{3} + \delta_u);$$

numerically $\delta_u = 0.12377$, $\bar{d} = 2.78439$, $\underline{d} = 1.25319$, $S_{\min} = 1.85600$.

- (a) $q_j \geq t_0 > 0.22$ for all j (as $|q_j - \frac{1}{3}| \leq \sqrt{2/3}|q - U|$ on $\mathbf{1}^\perp$); hence $\mathcal{A}(\ell, u) = \{1, 2, 3\}$ and $s_i \geq 0$ for all i .
- (b) $|u_j - \frac{1}{3}| < \delta_u$, so $\underline{d} \leq d_i(u) \leq \bar{d}$ for all i , and $s \geq \underline{d}$ by (a); all entries of $D^{(i)}(u)$ lie in $[\ln(\frac{1}{3} - \delta_u), \ln(\frac{2}{3} + \delta_u)]$.
- (c) $S_{\min} \leq S(q) \leq \mu$: by Taylor's formula, $S(q) = \mu - \frac{1}{2} \sum_j \frac{(q_j - 1/3)^2}{\vartheta_j(1 - \vartheta_j)}$ with ϑ_j between q_j and $\frac{1}{3}$, and $\vartheta_j(1 - \vartheta_j) \geq t_0(1 - t_0)$.
- (d) $|\xi - U| < R_X$, and $\text{osc } \xi \leq \sqrt{2}|\xi - U| \leq \sqrt{2}R_X$.

For $\zeta \in \mathbb{R}^3$ with $\sum_j \zeta_j = 1$, $u \in (0, 1)^3$ and $q \in \Delta$ let

$$g_i(\zeta, u) = D^{(i)}(u) + d_i(u)\zeta \in \mathbb{R}^3, \quad V(\zeta, u, q) = \frac{K\alpha^2}{2} \sum_i q_i \text{Var}_{\pi(\zeta)}(g_i(\zeta, u)),$$

$$\omega(q) = V(w(q), q, q) - \gamma S(q).$$

A direct computation gives $\ell^{(i)}/s_i - \xi = g_i(\xi, u)/s_i$ if $s_i > 0$, and $g_i(\xi, u) = 0$ if $s_i = 0$.

Lemma A.11 (gain). For $p \in \Delta$ with positive entries and $v \in \mathbb{R}^3$, $\ln \mathbb{E}_p e^v - \mathbb{E}_p v \geq \phi(\text{osc } v) \text{Var}_p(v)$, where $\phi(t) = \frac{t-1+e^{-t}}{t^2} = \int_0^1 (1-\tau)e^{-\tau t} d\tau$. Moreover ϕ is decreasing and $t \mapsto t\phi(t) = 1 - \frac{1-e^{-t}}{t}$ is increasing on $[0, \infty)$.

Proof. Let $p_\tau \propto p e^{\tau v}$. The function $\tau \mapsto \ln \mathbb{E}_p e^{\tau v} - \tau \mathbb{E}_p v$ vanishes with its derivative at 0 and has second derivative $\text{Var}_{p_\tau}(v)$, so the left side is $\int_0^1 (1-\tau) \text{Var}_{p_\tau}(v) d\tau$. Since the density of p_τ with respect to p is at least $e^{-\tau \text{osc } v}$, $\text{Var}_{p_\tau}(v) = \min_m \mathbb{E}_{p_\tau}(v - m)^2 \geq e^{-\tau \text{osc } v} \text{Var}_p(v)$. The monotonicity statements are clear from the integral and from $\frac{1-e^{-t}}{t} = \int_0^1 e^{-\tau t} d\tau$. $\square$

Corollary A.12. In case N, with $a = \alpha(\ln \frac{2/3+\delta_u}{1/3-\delta_u} + \sqrt{2}\bar{d}R_X) = 31.628$ and $m(s) = \frac{2}{s} \phi(\frac{a}{s})$,

$$K \sum_i q_i s_i D(\pi \parallel \pi_i) \geq m(s) V(\xi, u, q).$$

The function m is decreasing, and $\frac{1}{s} - \frac{a}{3s^2} \leq m(s) \leq \frac{1}{s}$.

Proof. If $s_i = 0$ the i -th term on the right vanishes. Otherwise $\pi_i(j) = \pi_j e^{v_j} / \mathbb{E}_\pi e^v$ with $v = \alpha g_i / s_i$, so $D(\pi \parallel \pi_i) = \ln \mathbb{E}_\pi e^v - \mathbb{E}_\pi v \geq \phi(\text{osc } v) \frac{\alpha^2}{s_i^2} \text{Var}_\pi(g_i)$. By (b) and (d), $\alpha \text{osc } g_i \leq \alpha(\text{osc } D^{(i)} + d_i \text{osc } \xi) \leq a$, so $K s_i D(\pi \parallel \pi_i) \geq K \alpha^2 \text{Var}_\pi(g_i) \phi(a/s_i)/s_i \geq K \alpha^2 \text{Var}_\pi(g_i) \phi(a/s)/s$, because $\tau \mapsto \phi(a/\tau)/\tau = \frac{1}{a}(t\phi(t))|_{t=a/\tau}$ is decreasing and $s_i \leq s$. Multiply by q_i and sum. The same monotonicity shows that m is decreasing; $\phi \leq \frac{1}{2}$, and $\phi(t) \geq \int_0^1 (1-\tau)(1-\tau t) d\tau = \frac{1}{2} - \frac{t}{6}$. $\square$

Lemma A.13 (the function ω). $\omega(U) = 0$, and $\omega(q) \geq -C_\omega r(q)^2$ for $r(q) \leq r_q$, with $C_\omega = 4.13$.

Proof. At $q = U$: $w = \pi = U$, $d_i = \mu$, and g_i takes the value $\ln \frac{1}{3} + \frac{\mu}{3}$ once and $\ln \frac{2}{3} + \frac{\mu}{3}$ twice, so $\text{Var}_U(g_i) = \frac{2}{9}\lambda^2$ and $V(U, U, U) = \frac{K\alpha^2}{9}\lambda^2 = \frac{2}{3}\beta\lambda^2 = \frac{\nu}{2} = \gamma\mu = \gamma S(U)$ by Lemma A.7. By symmetry $\omega(U + tv) = \omega_2 |v|^2 t^2 + O(t^3)$ (no invariant linear forms), and $\omega_2 = -3.6926 \dots$ was computed exactly (C1). The bound is verified in (C3): on $r \leq 0.001$ from ω_2 and an enclosure of the third Taylor coefficient, and on $0.001 \leq r \leq r_q$ on a cover by boxes. $\square$

Lemma A.14 (variation in the split). Let $r(q) \leq r_q$, $|u - q| < r_u$, $\sum_j \zeta_j = 1$ and $|\zeta - U| \leq R_X$. Then

$$V(\zeta, u, q) - V(\zeta, q, q) \geq \nabla_u V(\zeta, q, q) \cdot (u - q) - \frac{1}{2} C_2 |u - q|^2,$$

where $C_2 = 0.9682$ bounds $\max_k K\alpha^2 \left(\frac{A_k^+}{(q_k - r_u)^2} + \frac{B_k^+}{(1 - q_k - r_u)^2} \right)$ over this region (C4.4), with $A_k = q_k \chi_{kk}$, $B_k = \sum_{i \neq k} q_i \chi_{ik}$, $\chi_{ik} = \text{Cov}_{\pi(\zeta)}(e_k - \zeta, g_i(\zeta, q))$.

Proof. $\text{Var}(f) - \text{Var}(f') = 2 \text{Cov}(f - f', f') + \text{Var}(f - f') \geq 2 \text{Cov}(f - f', f')$. With $f = g_i(\zeta, u)$, $f' = g_i(\zeta, q)$ and $\varepsilon^{(i)} = D^{(i)}(u) - D^{(i)}(q)$ we have $f - f' = \varepsilon^{(i)} - (\sum_k \varepsilon_k^{(i)})\zeta = \sum_k \varepsilon_k^{(i)}(e_k - \zeta)$, hence

$$V(\zeta, u, q) - V(\zeta, q, q) \geq K\alpha^2 \sum_i q_i \sum_k \varepsilon_k^{(i)} \chi_{ik} = K\alpha^2 \sum_k \left(A_k \ln \frac{u_k}{q_k} + B_k \ln \frac{1 - u_k}{1 - q_k} \right).$$

Taylor's formula with $|u_k - q_k| < r_u$ gives $A_k \ln \frac{u_k}{q_k} \geq A_k \frac{u_k - q_k}{q_k} - \frac{A_k^+ (u_k - q_k)^2}{2(q_k - r_u)^2}$ and $B_k \ln \frac{1 - u_k}{1 - q_k} \geq -B_k \frac{u_k - q_k}{1 - q_k} - \frac{B_k^+ (u_k - q_k)^2}{2(1 - q_k - r_u)^2}$. The linear part is $\nabla_u V(\zeta, q, q) \cdot (u - q)$, since $\partial_{u_k} g_i = \eta_{ik}(e_k - \zeta)$ with $\eta_{kk} = \frac{1}{q_k}$, $\eta_{ik} = -\frac{1}{1 - q_k}$ ($i \neq k$), so $\partial_{u_k} V(\zeta, q, q) = K\alpha^2 \sum_i q_i \eta_{ik} \chi_{ik} = K\alpha^2 \left(\frac{A_k}{q_k} - \frac{B_k}{1 - q_k} \right)$. $\square$

Lemma A.15 (variation in the shape). Let $r(q) \leq r_q$, $w = w(q)$, $\sum_j \zeta_j = 1$ and $\rho = |\zeta - w| \leq r_x$. Then

$$V(\zeta, q, q) - V(w, q, q) \geq -G_x \rho - \frac{1}{2} M_{xx} \rho^2, \quad |\nabla_u V(\zeta, q, q)| \leq G_u + M_{xu} \rho,$$

with $G_x = 1.95$, $G_u = 2.0$ bounding $|P \nabla_\zeta V(w(q), q, q)|$, $|\nabla_u V(w(q), q, q)|$ over $r(q) \leq r_q$ (C4.3), and $M_{xx} = 191.13$, $M_{xu} = 28.844$ bounding the operator norms of $P \nabla_{\zeta\zeta}^2 V(\zeta, q, q) P$ and $\nabla_{u\zeta}^2 V(\zeta, q, q) P$ over $r(q) \leq r_q$, $|\zeta - U| \leq R_X$ (C4.4).

Proof. Taylor's formula along the segment from w to ζ , which lies in $\{|\cdot - U| \leq R_X\}$ and in the direction $\zeta - w \in \mathbf{1}^\perp$. $\square$

For the record, the derivatives used in (C4) are, for $e, f \in \mathbf{1}^\perp$, $g = g_i(\zeta, q)$, $d = d_i(q)$, $\dot{g} = \eta_{im}(e_m - \zeta)$, and $\mathbb{E}, \tilde{\cdot}, \text{Cov}, \text{Var}$ taken with respect to $\pi(\zeta)$:

$$\partial_e \text{Var}(g) = \alpha \mathbb{E}[\tilde{e} \tilde{g}^2] + 2d \mathbb{E}[\tilde{e} \tilde{g}], \quad \partial_{u_m} \text{Var}(g) = 2\eta_{im} \chi_{im},$$

$$\partial_{u_m} \partial_e \text{Var}(g) = 2(\alpha \mathbb{E}[\tilde{e} \tilde{g} \tilde{g}] + d \mathbb{E}[\tilde{e} \tilde{g}] - \eta_{im} \text{Cov}(e, g)),$$

$$\partial_f \partial_e \text{Var}(g) = \alpha^2 (\mathbb{E}[\tilde{e} \tilde{f} \tilde{g}^2] - \text{Cov}(e, f) \text{Var}(g) - 2 \text{Cov}(e, g) \text{Cov}(f, g)) + 4\alpha d \mathbb{E}[\tilde{e} \tilde{f} \tilde{g}] + 2d^2 \text{Cov}(e, f),$$

which follow from $\partial_e \pi_j = \alpha \pi_j \tilde{e}_j$ and $\partial_e g = d e$.

Proposition A.16 (case N). *In case N, $E > 0$. More precisely, with $\theta = r_u/r_x$, $c_x = \frac{K\alpha^2}{2}S_{\min}\Lambda = 0.45480$, $b' = b_{\min} - \frac{\gamma}{2000+\underline{d}-\bar{d}} = 0.40651$, and for $m \geq 0$*

$$\begin{aligned}\text{Def}_\omega(m) &= \max_{0 \leq r \leq r_q} (mC_\omega r^2 - S_{\min}c_4r^4), \\ \text{Def}_x(m) &= \max_{0 \leq \rho \leq r_x} (mG_x\rho - (c_x - \frac{m}{2}(M_{xx} + \theta M_{xu}))\rho^2), \\ \text{Def}_u(m) &= \max_{0 \leq \rho \leq r_u} (mG_u\rho - (2b' - \frac{m}{2}(C_2 + M_{xu}/\theta))\rho^2),\end{aligned}$$

we have, for every $s \geq \underline{d}$,

$$\gamma S_{\min}\left(m(s) - \frac{1}{2000 + s - \bar{d}}\right) > \text{Def}_\omega(m(s)) + \text{Def}_x(m(s)) + \text{Def}_u(m(s)), \quad (\text{A.6})$$

and E is at least the difference of the two sides.

Proof. Inequality (A.6) is (C4.5). Let $\varepsilon_s = 1/(2000 + s - \bar{d})$, $\rho_\xi = |\xi - w(q)|$, $\rho_u = |u - q|$, $d_j = d(q_j \| u_j)$, and split $V(\xi, u, q) = \gamma S + \omega + \Delta_\xi + \Delta_u$ with $\Delta_\xi = V(\xi, q, q) - V(w, q, q)$, $\Delta_u = V(\xi, u, q) - V(\xi, q, q)$. By (A.3), (A.4) with $\max_i d_i \leq \bar{d}$, and Corollary A.12 (with $m = m(s)$),

$$E \geq \gamma S(m - \varepsilon_s) + [SQ + m\omega] + [KSD(\pi \| \pi_w) + m\Delta_\xi] + \left[\sum_j (b_j - \gamma\varepsilon_s)d_j + m\Delta_u\right].$$

By Proposition A.6(i), Lemma A.13 and (c), the first bracket is $\geq S_{\min}c_4r^4 - mC_\omega r^2 \geq -\text{Def}_\omega(m)$. By Lemma A.9 (with $\rho_0 = r_x$) and Lemma A.15, the second is $\geq c_x\rho_\xi^2 - mG_x\rho_\xi - \frac{m}{2}M_{xx}\rho_\xi^2$. Since $s \geq \underline{d}$, $b_j - \gamma\varepsilon_s \geq b'$, so by Pinsker and Lemmas A.14, A.15 the third bracket is $\geq 2b'\rho_u^2 - m(G_u + M_{xu}\rho_\xi)\rho_u - \frac{m}{2}C_2\rho_u^2$. With $\rho_\xi\rho_u \leq \frac{1}{2}(\theta\rho_\xi^2 + \rho_u^2/\theta)$ the last two brackets together are $\geq -\text{Def}_x(m) - \text{Def}_u(m)$. Finally (A.6) shows $m - \varepsilon_s > 0$, so $\gamma S(m - \varepsilon_s) \geq \gamma S_{\min}(m - \varepsilon_s)$. $\square$

A.7 Conclusion of the proof

By Lemma A.2 it suffices to show $E(\ell, u, q) \geq 0$ for $A = 2000$. If $q = e_k$ is a vertex, then $k \in \mathcal{A}(\ell, u)$, $H(q) = 0$ and $E = \Phi(\ell) - \Phi(\ell^{(k)}) \geq 0$ by Lemma A.3(c), since $\ell^{(k)} \leq \ell$. If $s = 0$, then $|\mathcal{A}(\ell, u)| \leq 1$, so q is a vertex. Otherwise Lemma A.4 applies, and one of the cases (F1)–(F3) of Proposition A.10 or case N of Proposition A.16 occurs. Hence B_{2000} , and by Lemma A.2 every B_A with $A \in [2000, \infty]$, satisfies the main inequality. Clearly $B_A(1, 1, 1) = 1$, and the consequences follow from Lemma 2.1, since $B_A(x, x, x) = x^{3c}(1 + \frac{3}{A}\ln x)^{-\gamma}$ and $1 + \frac{3}{A}\ln x \geq \frac{3}{A}(1 + \ln x)$. $\square$

B The computer-assisted steps

B.1 Method

Claim (C1) is an exact computation (**sympy**): truncated power series in t whose coefficients are polynomials in (v_1, v_2) , $v_3 = -v_1 - v_2$, over the field $\mathbb{Q}(L_2, L_3)$ with independent symbols L_2, L_3 ; an identity in this field remains true when $L_2 = \ln 2$, $L_3 = \ln 3$ are substituted. Claims (C2)–(C4) are verified in ball arithmetic with Arb [2] (**python-flint** 0.9.0, 128-bit working precision): every real number is represented by a ball guaranteed to contain it, all operations including $\exp, \log, \sqrt{}$ are rounded outwards, and the constants enter as balls computed from their closed forms. Points of Δ near U are written $q = U + y_1E_1 + y_2E_2$ with $E_1 = \frac{1}{\sqrt{2}}(1, -1, 0)$, $E_2 = \frac{1}{\sqrt{6}}(1, 1, -2)$; by the permutation symmetry it suffices to cover the chamber $q_1 \geq q_2 \geq q_3$, i.e. the sector $30^\circ \leq \arg y \leq 90^\circ$, except in (C3) where the whole disk is covered. Two kinds of enclosures are used on a box $Y = c + [-w_1, w_1] \times [-w_2, w_2]$ in the y -plane:

- a *direct* evaluation of the formula with the three coordinates q_j as balls (an enclosure over a superset of Y);
- a *Taylor form*: $f(c+\eta) = f(c) + \nabla f(c)\eta + \frac{1}{2}\eta^\top \nabla^2 f(c)\eta + R$, where the derivatives at c are enclosed from truncated Taylor series along $(1, 0)$, $(0, 1)$, $(1, 1)$, and $|R| \leq \sum_{|\beta|=3} |T_\beta| w^\beta$, with the third-order Taylor coefficients T_β enclosed over all of Y from series along $(1, 0)$, $(0, 1)$, $(1, 1)$, $(1, -1)$; the quadratic part is minimised over Y by fixing every coordinate in which its partial derivative has constant sign at the corresponding endpoint.

A box on which the required inequality is not established is bisected (quadtrees). For the inner disks $r \leq r_0$ the needed Taylor coefficient is enclosed on polar boxes $\{U + [0, r_0] v(\vartheta) : \vartheta \in \text{interval}\}$.

B.2 The claims

- (C1) *Exact expansions at U* (`taylor_symbolic.py`). The coefficients of $t^0, \dots, t^3$ of $Q(U + tv)$ vanish identically, the coefficient of t^4 equals $q_4(v_1^2 + v_2^2 + v_3^2)^2$ with $q_4 \in \mathbb{Q}(L_2, L_3)$, $q_4 = 0.348057255679\dots$; the identities for β and κ_3 of Lemma A.7 hold; the coefficients of t^0, t^1 of $\omega(U + tv)$ vanish and that of t^2 is $\omega_2|v|^2$, $\omega_2 = -3.69264165895\dots$
- (C2) *Proposition A.6* (`check_Q.py`). Inner disk $r \leq 0.003$: $M_5 \leq 9.3740$ (120 polar boxes), $q_4 - 0.003M_5 \geq 0.3199$. Vertex constant of Lemma A.8: 0.0118696. Rest of the chamber: 16 219 boxes (initial half-width $\frac{1}{64}$, depth ≤ 11 ; 9 134 settled by direct evaluation and 2 818 by Taylor forms; 15 boxes inside the inner disk and 34 inside a vertex region skipped), no failure. A box meeting $\{r < r_q\}$ is checked against $c_4 r^4$ (which also gives $\geq c_4 r_q^4 = 6.8521 \cdot 10^{-5} \geq \gamma/2000$ on its part with $r \geq r_q$), the others against $\gamma/2000$.
- (C3) *Lemma A.13* (`check_omega.py`). Inner disk $r \leq 0.001$: the third Taylor coefficient is bounded by 169.88 (720 polar boxes), so $-\omega/r^2 \leq -\omega_2 + 0.001 \cdot 169.88 = 3.8625 < C_\omega$. Annulus $0.001 \leq r \leq r_q$: $\omega + C_\omega r^2 \geq 0$ on 4 090 boxes (Taylor forms), no failure.
- (C4) *Case N* (`check_near.py`).
- (1) $|w(q) - U| \leq R_w = 0.0607$ for $r(q) \leq r_q$ (126 boxes).
 - (2) $\lambda_{\min}(\pi(\zeta)) \geq \Lambda = 0.1$ for $|\zeta - U| \leq R_X$ (1411 boxes of half-width $\frac{1}{256}$; π computed from the differences $\zeta_j - \zeta_k$, formula of Lemma A.9); and $\frac{K\alpha^2}{2} r_x^2 \Lambda = 7.0817 \cdot 10^{-5} \geq \gamma/2000 = 6.8504 \cdot 10^{-5}$.
 - (3) $|P\nabla_\zeta V(w(q), q, q)| \leq G_x = 1.95$ (993 boxes) and $|\nabla_u V(w(q), q, q)| \leq G_u = 2.0$ (751 boxes), Taylor forms.
 - (4) $M_{xx} \leq 191.130$, $M_{xu} \leq 28.8434$, $C_2 \leq 0.968167$: the closed formulas above (centred quantities written as sums of differences to avoid cancellation) evaluated on 117 boxes of half-width 0.005 in q times 2196 boxes of half-width 0.0015 covering $|\zeta - U| \leq R_X$.
 - (5) Inequality (A.6). All three deficits are non-decreasing in m and m is decreasing in s , so on an interval $[s_1, s_2]$ it suffices that $\gamma S_{\min}(m(s_2) - \frac{1}{2000+s_1-\bar{d}}) > \sum \text{Def}(m(s_1))$. This is checked on 4 844 intervals $[s, 1.002s]$ covering $[d, 2 \cdot 10^4]$; the largest ratio of the right side to the left side is 0.644, at $s = 1.253$. For $s \geq 2 \cdot 10^4$, using $m \leq \frac{1}{s}$, the vertex bounds $\max_\rho(mG\rho - C\rho^2) \leq \frac{m^2 G^2}{4C}$ (the coefficients C evaluated at $m(2 \cdot 10^4)$, where they are smallest) and $m \geq \frac{1}{s} - \frac{a}{3s^2}$, inequality (A.6) follows from

$$\gamma S_{\min} \frac{s(2000 - \bar{d} - \frac{a}{3}) - \frac{a}{3}(2000 - \bar{d})}{2000 + s - \bar{d}} > \frac{C_\omega^2}{4S_{\min}c_4} + \frac{G_x^2}{4C_x} + \frac{G_u^2}{4C_u},$$

whose left side increases in s ; at $s = 2 \cdot 10^4$ it equals 459.08, and the right side is 14.83.

B.3 Constants

chosen	closed form (value)	certified
shift $A = 2000$	$b_{\min} = c - \frac{2}{3}\kappa = 0.406582$	$q_4 = 0.3480573, M_5 \leq 9.3740$
$c_4 = 0.2, r_q = 0.13605$	$r_u = 0.0126844, \delta_u = 0.123769$	vertex bound 0.0118696
$\sigma_V = 10^{-3}$	$\underline{d} = 1.253191, \bar{d} = 2.784392$	$\omega_2 = -3.692642, M_3 \leq 169.88$
$C_\omega = 4.13, r_x = 0.017$	$S_{\min} = 1.856001, R_X = 0.0777$	$M_{xx} \leq 191.130$
$R_w = 0.0607, \Lambda = 0.1$	$a = 31.62783, \theta = 0.746143$	$M_{xu} \leq 28.8434$
$G_x = 1.95, G_u = 2.0$	$b' = 0.406514, c_x = 0.454797$	$C_2 \leq 0.968167$

B.4 Reproduction

The scripts are in `verification/`. `python3 run_all.py` runs (C1)–(C4) in about 65 seconds and stores the logs in `verification/results/`; `python3 numerics.py` and `python3 threshold.py` reproduce Section 3. Requirements: Python ≥ 3.10 , `python-flint`, `mpmath`, `sympy`, `scipy`.

References

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Disclaimer

This is a raw draft produced by an AI (Grok Bot), prompted by Paata Ivanisvili on September 23, 2026. A similar (equivalent) result was obtained for an arbitrary number n of partitions; for simplicity only the case $n = 3$ is presented here.