

**THE LOCAL NEIGHBORLINESS THRESHOLD OF THE SYMMETRIC
MOMENT CURVE:
AN INTEGRAL CRITERION, SHARP ASYMPTOTICS, AND THE
BARVINOK–LEE–NOVIK CONJECTURE
FOR $k \leq 440$ AND FOR ALL SUFFICIENTLY LARGE k**

PAATA IVANISVILI

This proof was generated on October 27, 2026, by Grok Bot using Cursor's agents, guided by several prompts from Paata Ivanisvili

ABSTRACT. Let ϕ_k be the largest ϕ such that any k points of the symmetric moment curve $U_k(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \dots, \cos(2k-1)t, \sin(2k-1)t)$ lying on an open arc of length ϕ span a face of $\text{conv } U_k$. Barvinok, Lee and Novik proved $\pi/2 < \phi_k < \pi$ and $\phi_k \rightarrow \pi/2$, reduced ϕ_k to configurations collapsed to the two endpoints of the arc, and conjectured that the balanced collapse is always the worst one. We turn their characterization into an explicit two-integral criterion. We use it to prove: $\phi_k < \frac{\pi}{2} + \frac{\log(8\pi(k+1/2))}{4(k-1)}$ for even $k \geq 4$; $\phi_k < \frac{\pi}{2} + \frac{\log(64\sqrt{\pi}(k-1)\sqrt{k-1/2})}{2k-8}$ for odd $k \geq 13$; and $\liminf_k (4k(\phi_k - \pi/2) - \log k) \geq 2 \log 2$. We also prove the effective two-sided bound $\frac{\pi}{2} + \frac{\log k}{4k} < \phi_k < \frac{\pi}{2} + \frac{\log k + 4.1}{4k}$ for every $k \geq 2$; on a finite range of k it rests on explicit closed-form inequalities certified in ball arithmetic. Refining the analysis we obtain the exact second-order term, $\lim_k (4k(\phi_k - \frac{\pi}{2}) - \log k) = \log(8\pi)$, which is the asymptotics of the balanced split. We prove the Barvinok–Lee–Novik conjecture for all $k \leq 440$ (both parities; computer-assisted, ball arithmetic), together with the monotonicity of ϕ_k in that range. We also prove it, analytically but with an ineffective threshold, for all sufficiently large k . The splits far from balanced are excluded by the sharp asymptotics. For near-balanced splits we show that the criterion is strictly concave in the exponent: its second derivative is at most $-4\delta/N + O(N^{-2})$ with $N\delta \asymp \log k$. This is proved with Stein identities for the weights $\sin^a u \sin^b(\ell - u)$ and a moving-frame treatment of the kink $|\sin(u - s)|$. The same estimates, used to first order in the other exponent, show that ϕ_k is strictly decreasing for all sufficiently large k . This answers BLN's monotonicity question asymptotically. The conjecture thus remains open only on a finite but unspecified range $440 < k < K_0$. We also isolate an “endpoint monotonicity” conjecture which implies $\phi_{k+1} < \phi_k$.

1. INTRODUCTION

Let $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ and $B_k = \text{conv } U_k(\mathbb{S}) \subset \mathbb{R}^{2k}$, the Barvinok–Novik orbitope [1, 3]. A *raked trigonometric polynomial* of degree $\leq 2k-1$ is $f(t) = c + \sum_{j=1}^k (a_j \cos(2j-1)t + b_j \sin(2j-1)t)$, i.e. $f = c + \langle C, U_k \rangle$. Faces of B_k through $U(t_1), \dots, U(t_n)$ correspond to nonnegative raked polynomials vanishing at the t_i .

Definition 1 ([2]). ϕ_k is the largest number such that for every open arc $\Gamma \subset \mathbb{S}$ of length ϕ_k and all distinct $t_1, \dots, t_n \in \Gamma$, $n \leq k$, the set $\text{conv}\{U(t_1), \dots, U(t_n)\}$ is a face of B_k .

Known. Barvinok–Novik [1]: $\phi_k > 0$. Lee: $\phi_k > \sqrt{6}k^{-3/2}$. Barvinok–Lee–Novik [2]: $\pi/2 < \phi_k < \pi$ and $\phi_k \rightarrow \pi/2$; $\phi_2 = 2\pi/3$, $\phi_3 = \pi - \arccos \frac{3-\sqrt{5}}{2}$, and $\phi_4 \approx 1.8706585$ in closed form. They also give a characterization (Theorem 13 below). Edges of B_k were determined in [1, 4]. The facial structure of B_3 is unknown.

Date: October 27, 2026.

Conjecture 2 ([2]). *In Theorem 13(2) one can take $m_a = m_b = k$ if k is even and $\{m_a, m_b\} = \{k-1, k+1\}$ if k is odd. For even k this is equivalent to $\phi_k = 2\alpha_k$, where α_k is the smallest positive root of $\cos \alpha + 1 + \sum_{j=1}^{k-1} (-1)^j \frac{(2j-1)!!}{(2j)!!} \tan^{2j} \alpha = 0$.*

BLN also write that their computations suggest ϕ_k is decreasing, but they could not prove it.

Results. For a *split* (p, q) , $p, q \geq 1$, $p + q = k$, and $L \in (0, \pi)$, let $f_{p,q,L}$ be the unique raked polynomial of degree $\leq 2k-1$ with constant term 1, a root of multiplicity $2p$ at 0 and of multiplicity $2q$ at L [2, Cor. 3.5]. Let $L(p, q) \in (0, \pi]$ be the first L at which $f_{p,q,L}$ acquires an additional root ($L(p, q) = \pi$ if none). Put $a = 2p-1$, $b = 2q-1$.

Theorem 3 (integral criterion). *For $L \in (0, \pi)$ let $A(u) = \sin^a u \sin^b(L-u)$ on $(0, L)$, let $s \in (0, L)$ be the argument of $\int_0^L A(u) e^{iu} du$, and put*

$$I_{\text{in}} = \int_0^s A(u) \sin(s-u) du = \frac{1}{2} \int_0^L A(u) |\sin(u-s)| du, \quad I_{\text{out}} = \int_L^\pi \sin^a u \sin^b(u-L) \sin(u-s) du.$$

Then $f_{p,q,L} \geq 0$ on $\mathbb{S}$ iff $I_{\text{out}} \geq I_{\text{in}}$, and $f_{p,q,L}$ is ≥ 0 and has a root outside $\{0, L\}$ iff $I_{\text{out}} = I_{\text{in}}$. Consequently, $f_{p,q,L}$ has a root outside $\{0, L\}$ iff $I_{\text{out}} \leq I_{\text{in}}$. Moreover $\Delta_{p,q}(L) := I_{\text{out}} - I_{\text{in}}$ is positive on $(0, L(p, q))$, vanishes at $L(p, q)$ if $L(p, q) < \pi$, and is negative on $(L(p, q), \pi)$. Finally $\phi_k = \min_{p+q=k} L(p, q)$ and $L(p, q) = L(q, p)$.

Theorem 4 (upper bounds). (i) *For even $k \geq 4$: $\phi_k \leq L(\frac{k}{2}, \frac{k}{2}) < \frac{\pi}{2} + \frac{\log(8\pi(k + \frac{1}{2}))}{4(k-1)}$.*

(ii) *For odd $k \geq 13$: $\phi_k < \frac{\pi}{2} + \frac{\log(64\sqrt{\pi}(k-1)\sqrt{k-1/2})}{2k-8} = \frac{\pi}{2} + \frac{3\log k + O(1)}{4k}$.*

Theorem 5 (lower bounds). (i) $\liminf_{k \rightarrow \infty} (4k(\phi_k - \frac{\pi}{2}) - \log k) \geq 2 \log 2$.

(ii) $\phi_k > \frac{\pi}{2} + \frac{\log k}{4k}$ for every $k \geq 2$.

The proof of (ii) is analytic for $k > 2 \cdot 10^4$. For $301 \leq k \leq 2 \cdot 10^4$ it consists of certifying in ball arithmetic, for each of the $\approx 10^8$ splits, the sign of an explicit closed form in Gamma functions (Proposition 25). For $k \leq 300$ it follows from the enclosures of Theorem 8.

Theorem 6 (effective upper bound). $\phi_k < \frac{\pi}{2} + \frac{\log k + 4.1}{4k}$ for every $k \geq 2$.

For even $k \geq 302$ this follows from Theorem 4(i), and for $k \leq 300$ from Theorem 8. For odd $k \geq 301$ it comes from an explicit criterion for the split $(\frac{k-1}{2}, \frac{k+1}{2})$ (Proposition 19). That criterion is certified in ball arithmetic for $k \leq 200001$ and verified by hand beyond. Together with Theorem 5(ii),

$$\frac{\pi}{2} + \frac{\log k}{4k} < \phi_k < \frac{\pi}{2} + \frac{\log k + 4.1}{4k} \quad \text{for every } k \geq 2.$$

Theorem 7 (sharp asymptotics). $\lim_{k \rightarrow \infty} (4k(\phi_k - \frac{\pi}{2}) - \log k) = \log(8\pi)$, i.e. $\phi_k = \frac{\pi}{2} + \frac{\log k + \log(8\pi)}{4k} + o(\frac{1}{k})$.

This is exactly the asymptotics of the balanced split. It is what Conjecture 2 predicts, but it does not imply the conjecture: a non-balanced split could still win at order $o(1/k)$. The proof (§6) is analytic but does not give an explicit rate. The cruder bounds of Theorems 4, 5(ii) and 6 hold for every k in their stated ranges; Theorem 5(i) is an asymptotic statement.

Theorem 8 (computer-assisted, rigorous). *For every $2 \leq k \leq 440$, the balanced split is the unique minimizer in Theorem 13(2). That is, Conjecture 2 holds, in both parities, and $\phi_k = L(\lfloor k/2 \rfloor, \lceil k/2 \rceil)$. Moreover $\phi_{k+1} < \phi_k$ for $2 \leq k < 440$, and $L(p, q+1) < L(p, q)$ for all $p+q <$*

200. Each ϕ_k is enclosed in an interval of half-width $1.01 \cdot 10^{-12}$ about a floating-point value, printed to 12 decimals (files `numerics/rigorous_200.txt`, `numerics/rigorous_201_300.txt`, `numerics/rigorous_301_400.txt` and `numerics/rigorous_401_440.txt`).

BLN proved $k \leq 4$. The proof is in §8: Theorem 3 turns each comparison into the sign of an explicit closed-form expression at a rational point, and these signs are certified in ball arithmetic (Arb).

Theorem 9 (all sufficiently large k). *There is K_0 such that for every $k \geq K_0$ the balanced split is the unique minimizer in Theorem 13(2), up to $p \leftrightarrow q$. That is, Conjecture 2 holds for all $k \geq K_0$.*

Theorem 10 (monotonicity for large k). *There is K_2 such that $\phi_{k+1} < \phi_k$ for all $k \geq K_2$. Moreover, endpoint monotonicity $L(p, q+1) < L(p, q)$ (Conjecture 11 below) holds for all $k = p+q \geq K_2$ with $0.42k \leq p \leq 0.58k$.*

The proofs (§7) are analytic but ineffective: we do not compute K_0 or K_2 . Together with Theorem 8, the conjecture holds for $k \leq 440$ and for $k \geq K_0$. The splits far from balanced are handled by the sharp asymptotics. For near-balanced splits we prove that the criterion $\log I_{\text{in}} - \log I_{\text{out}}$ is strictly concave in the (real) exponent $2p-1$ at $L = L(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil)$. Its second derivative is at most $-4\delta/N + O(N^{-2})$ with $N\delta \asymp \log k$.

In floating point (§9) we also find, for $k \leq 300$, that $p \mapsto L(p, k-p)$ is strictly decreasing on $p \leq k/2$, and that $4k(\phi_k - \pi/2) - \log k$ approaches $\log(8\pi) = 3.2242$, the value of the upper bound in Theorem 4(i). So the upper bound (i) is asymptotically sharp if the conjecture holds. We also propose:

Conjecture 11 (endpoint monotonicity). *$L(p, q+1) < L(p, q)$ for all $p, q \geq 1$.*

Conjecture 11 implies $\phi_{k+1} \leq L(p_*, q_*+1) < L(p_*, q_*) = \phi_k$, where (p_*, q_*) attains ϕ_k . So it answers BLN's monotonicity question. It is rigorously verified for $p+q < 200$ (Theorem 8).

Relation to the source paper. On an arc of length $< \pi$ the curve U_k meets every hyperplane at most $2k$ times, counting multiplicity [2, Thm. 3.1]; that is, it is a convex curve. The configurations in Theorem 3 are the endpoint principal representations, the analogue in even dimension of the upper and lower hulls in the source paper's Theorem 2.1. ϕ_k is the first time the corresponding supporting hyperplane of the convex hull of the *arc* stops supporting the whole closed curve. The criterion follows the source paper's method: factor the derivative of the support function into explicit sines, and read off signs from integrals of positive functions.

2. FACTS FROM [2]

Theorem 12 ([2], Thm. 3.1, Cor. 3.5). *(a) A nonzero raked f of degree $\leq 2k-1$ has at most $4k-2$ roots on $\mathbb{S}$ counting multiplicity. If its constant term is 0, then $f(t+\pi) = -f(t)$, and it has at most $2k-1$ roots in any open semicircle. (b) If distinct t_i lie in an open semicircle and $\sum m_i = 2k$, there is a unique raked f of degree $\leq 2k-1$ with constant term 1 having a root of multiplicity exactly m_i at t_i .*

Theorem 13 ([2], Thm. 5.1, Lemma 5.7). *(1) For arcs of length $< \phi_k$: if $t_1, \dots, t_n \in \Gamma$ are distinct and m_i are positive even integers with $\sum m_i = 2k$, then the polynomial of Theorem 12(b) has no other roots and is ≥ 0 . ϕ_k is the largest length with this property. (2) For an arc of length ϕ_k with endpoints a, b there are positive even $m_a, m_b = 2k$ such that the polynomial with roots of multiplicities m_a, m_b at a, b is ≥ 0 and has a root in $\Gamma + \pi$. (3) Let Γ be an open arc of length $< \pi$ with endpoint a , and fix positive even m_a, m_b with $m_a + m_b = 2k$. For $b \in \Gamma$ let f_b be the polynomial of Theorem 12(b) with roots of multiplicities m_a at a and m_b at b . Let $x, y, z \in \Gamma$, with x between a and y and y between a and z . If $f_y \geq 0$ and f_y has a root in $\Gamma + \pi$, then $f_x > 0$ off $\{a, x\}$, and f_z is negative somewhere.*

3. THE TWO-CLUSTER CRITERION: PROOF OF THEOREM 3

Lemma 14. *Let g be a raked polynomial of degree $\leq 2k - 1$ with zero constant term, i.e. an odd-harmonic trigonometric polynomial. If g has roots of multiplicity $\geq \mu_j$ at points r_j that are distinct mod π , with $\sum \mu_j = 2k - 1$, then $g(t) = C \prod_j \sin^{\mu_j}(t - r_j)$.*

Proof. $\cos((2j-1)t)$ and $\sin((2j-1)t)$ are homogeneous polynomials of degree $2j-1$ in $(\cos t, \sin t)$. Multiplying by $(\cos^2 t + \sin^2 t)^{k-j}$, odd-harmonic polynomials of degree $\leq 2k - 1$ are exactly the restrictions $F(\cos t, \sin t)$ of binary forms F of degree $2k - 1$. A root of order μ at $t = r$ means the linear form $\ell_r(x, y) = x \sin r - y \cos r$ divides F to order μ , and $\ell_r(\cos t, \sin t) = -\sin(t - r)$. Distinct r_j mod π give non-proportional linear forms. Hence $\prod \ell_{r_j}^{\mu_j}$, of degree $2k - 1$, divides F , and F is a scalar multiple of it. $\square$

Proof of Theorem 3. Write $f = f_{p,q,L}$. Then f' is odd-harmonic of degree $\leq 2k - 1$, with roots of multiplicity exactly $2p - 1$ at 0 and $2q - 1$ at L (Theorem 12(b)). Its antipodes are roots as well, since $f'(t + \pi) = -f'(t)$. By Rolle's theorem there is a further root $\sigma \in (0, L)$. Lemma 14 gives $f'(t) = K g(t)$ with $g(t) = \sin^a t \sin^b(t - L) \sin(t - \sigma)$. Since $f(0) = f(L) = 0$, $\int_0^L g = 0$. On $(0, L)$, $g = -A(u) \sin(u - \sigma)$ (b is odd), so $\text{Im}(e^{-i\sigma} \int_0^L A e^{iu} du) = 0$. The point $\int_0^L A e^{iu} du / \int_0^L A$ lies in the closed circular segment bounded by the arc $\{e^{iu} : 0 \leq u \leq L\}$ and its chord. This is a convex set not containing 0, since $L < \pi$. The point is not one of the extreme points 1, e^{iL} , because $A > 0$ on $(0, L)$. The only points of the segment with argument 0 or L are 1 and e^{iL} , so $s \in (0, L)$. The condition above says $\sigma \equiv s \pmod{\pi}$, and since $\sigma, s \in (0, L)$ with $L < \pi$, $\sigma = s$. Put $F(t) = \int_0^t g$. On $(0, s)$, $g > 0$; on (s, L) , $g < 0$; on (L, π) , $g > 0$, since every factor is positive there ($u - s \in (0, \pi)$). Hence $F > 0$ on $(0, L) \cup (L, \pi]$ and $F(\pi) = \int_L^\pi g = I_{\text{out}} > 0$. Since $g(t + \pi) = -g(t)$, we get $F(t + \pi) = F(\pi) - F(t)$, and F has mean $F(\pi)/2 > 0$. By uniqueness, $f = 2F/F(\pi)$. On $[0, \pi]$, $F \geq 0$ with maximum $\max(F(s), F(\pi))$. On $[\pi, 2\pi]$, $F = F(\pi) - F(\cdot - \pi) \geq 0$ iff $F(s) \leq F(\pi)$, with an extra zero exactly when $F(s) = F(\pi)$. If $F(s) > F(\pi)$, then $F(\pi + s) = F(\pi) - F(s) < 0 < F(\pi)$, so F vanishes somewhere in $(\pi, \pi + s)$, which contains neither 0 nor L . Finally $F(s) = I_{\text{in}}$, and $F(\pi) = I_{\text{out}}$.

Sign pattern. Suppose $\Delta(y) = 0$ for some $y \in (0, \pi)$. Then $f_{p,q,y} \geq 0$, and its extra root $s + \pi$ lies in $(\pi, \pi + y)$. Given $0 < x < y < z < \pi$, choose $\gamma \in (z, \pi)$ and apply Theorem 13(3) with $a = 0$, $\Gamma = (0, \gamma)$, $m_a = 2p$, $m_b = 2q$; note $(\pi, \pi + y) \subset \Gamma + \pi$. It gives $\Delta(x) > 0$ and $\Delta(z) < 0$. So Δ has at most one zero, and if $\Delta(y) = 0$ then $\Delta > 0$ before y and $\Delta < 0$ after y . By Theorem 13(1), $\Delta > 0$ for small L . Δ is continuous in L , because A , s and the integrals are. This gives the sign pattern.

$\phi_k = \min L(p, q)$. If $L < \phi_k$, the configuration $\{0, L\}$ lies in an open arc of length $< \phi_k$, so by Theorem 13(1) $\Delta_{p,q}(L) > 0$ for every split, and $L(p, q) \geq \phi_k$. By (2), some split has $\Delta_{p,q}(\phi_k) = 0$. *Symmetry.* $t \mapsto f_{p,q,L}(L - t)$ is raked with constant term 1, so it equals $f_{q,p,L}$. $\square$

4. UPPER BOUNDS

Proposition 15 (balanced split). *Let k be even, $p = q = k/2$, and $c = \cos(L/2)$. Then*

$$\Delta(L) = c^{2k-1} B(\tfrac{1}{2}, k) - J_k(c), \quad J_k(c) := \int_c^1 (x^2 - c^2)^{k-1} dx.$$

Proof. Here $a = b = k - 1$, and A is symmetric about $L/2$, so $s = L/2$. Substituting $u = L/2 - w$, we have $\sin u \sin(L - u) = \cos^2 w - c^2$, and $I_{\text{in}} = \int_0^{L/2} (\cos^2 w - c^2)^{k-1} \sin w dw = J_k(c)$. Substituting $u = \frac{L+\pi}{2} + w$, we have $\sin u \sin(u - L) = c^2 - \sin^2 w$ and $\sin(u - s) = \cos w$. Hence $I_{\text{out}} = \int_{-c}^c (c^2 - y^2)^{k-1} dy = c^{2k-1} B(\tfrac{1}{2}, k)$. $\square$

Lemma 16. $J_m(c) \geq (1 - c^2)^m / (2m)$ for $c \in (0, 1)$, $m \geq 1$.

Proof. With $x = 1 - y$: $x^2 - c^2 \geq (1 - c^2) - 2y \geq 0$ for $0 \leq y \leq (1 - c^2)/2 \leq 1 - c$. Integrate $((1 - c^2) - 2y)^{m-1}$ over $[0, (1 - c^2)/2]$. $\square$

Lemma 17 (Wendel [5]). *For $x > 0$: $x/\sqrt{x+1/2} \leq \Gamma(x + \frac{1}{2})/\Gamma(x) \leq \sqrt{x}$.*

Proof of Theorem 4(i). By Theorem 3 it suffices to show $\Delta(\frac{\pi}{2} + \delta) < 0$ for $\delta = \delta_+ := \log(8\pi(k + \frac{1}{2}))/4(k-1)$, which satisfies $\delta_+ \leq 0.4$ for $k \geq 4$. By Proposition 15 and Lemma 16 it suffices that $c^{2k-1}B(\frac{1}{2}, k) < (1 - c^2)^k/(2k)$, i.e.

$$2k \log \tan \alpha > \log(2kB(\frac{1}{2}, k)/c), \quad \alpha = \frac{\pi}{4} + \frac{\delta}{2}, \quad c = \cos \alpha.$$

We have $\log \tan \alpha = 2 \operatorname{artanh} \tan \frac{\delta}{2} \geq \delta$ and $c = (\cos \frac{\delta}{2} - \sin \frac{\delta}{2})/\sqrt{2} \geq (1 - \delta)/\sqrt{2}$. By Lemma 17, $B(\frac{1}{2}, k) = \sqrt{\pi} \Gamma(k)/\Gamma(k + \frac{1}{2}) \leq \sqrt{\pi(k + \frac{1}{2})}/k$. So the right side is at most $\frac{1}{2} \log(8\pi(k + \frac{1}{2})) - \log(1 - \delta) \leq \frac{1}{2} \log(8\pi(k + \frac{1}{2})) + 2\delta$. This is $\leq 2k\delta$ for $\delta \geq \delta_+$. Since $\log \tan \alpha > \delta$ strictly for $\delta > 0$, the required strict inequality holds at $\delta = \delta_+$. $\square$

Odd k : a three-cluster configuration. Let k be odd and $\alpha \in (0, \pi/2)$. Consider roots of multiplicity $k-1$ at $\pm\alpha$ and 2 at 0. All multiplicities are even, they sum to $2k$, and they lie in an arc of length 2α . By Theorem 13(1), if the corresponding f is negative somewhere, then $\phi_k \leq 2\alpha$.

Proposition 18. *Let $c = \cos \alpha$, $w(y) = (y^2 - c^2)^{k-2}$ on $[c, 1]$, and $c_\sigma^2 = \int_c^1 wy^2 / \int_c^1 w$. Then $f \geq 0$ on $\mathbb{S}$ iff*

$$\text{LHS} := \int_{c_\sigma}^1 w(y)(y^2 - c_\sigma^2) dy \leq \text{RHS} := 2 \int_0^c (c^2 - y^2)^{k-2}(c_\sigma^2 - y^2) dy.$$

Proof. By uniqueness f is even. f' is odd-harmonic of degree $\leq 2k-1$, with roots of multiplicity $k-2$ at $\pm\alpha$ and 1 at 0, so it is divisible by $(\sin^2 t - \sin^2 \alpha)^{k-2} \sin t$ in the binary-form sense. The quotient Q is a quadratic form, i.e. $Q = A + B \cos 2t + C \sin 2t$, and it is even, so $C = 0$. Rolle's theorem on $(0, \alpha)$ gives a root σ , so $Q \propto \sin^2 t - \sin^2 \sigma$. With $y = \cos u$: $f(t) = K \int_{\cos t}^1 g$, where $g(y) = (c^2 - y^2)^{k-2}(c_\sigma^2 - y^2)$. The condition $f(\alpha) = 0$ is $\int_c^1 g = 0$, which gives the formula for c_σ (recall $k-2$ is odd). The sign of g is $+$ on $|y| < c$, $-$ on $c < |y| < c_\sigma$, and $+$ on $|y| > c_\sigma$. Also $f(\pi) = 2K \int_0^c g > 0$ equals twice the mean, so $K > 0$. Tracking $\int_x^1 g$ for x from 1 down to -1 , its only possible negative value is at $x = -c_\sigma$, where it equals $2 \int_0^c g - \int_{c_\sigma}^1 g$, and $\int_{c_\sigma}^1 g = \text{LHS}$. By evenness this settles $[-\pi, 0]$ as well. $\square$

Proof of Theorem 4(ii). First, $\text{RHS} \leq c^{2k-3}B(\frac{1}{2}, k-1)$. For LHS: let $Y = y^2$ with $y \sim w/Z$ on $[c, 1]$, $Z = J_{k-1}(c)$. Then $\text{LHS} = Z \mathbb{E}(Y - \mathbb{E}Y)_+ = \frac{Z}{2} \mathbb{E}|Y - \mathbb{E}Y|$. For any density with maximum $f_{\max}$, $\mathbb{E}|Y - \mathbb{E}Y| \geq 1/(4f_{\max})$, since $\mathbb{P}(|Y - \mathbb{E}Y| < r) \leq 2rf_{\max}$. Here $f_Y(v) = w(\sqrt{v})/(2\sqrt{v}Z) \leq (1 - c^2)^{k-2}/(2cZ)$. With Lemma 16 this gives $\text{LHS} \geq cZ^2/(4(1 - c^2)^{k-2}) \geq c(1 - c^2)^k/(16(k-1)^2)$. So f is negative somewhere if $2k \log \tan \alpha > \log(16(k-1)^2B(\frac{1}{2}, k-1)) - 4 \log c$. Put $\alpha = \frac{\pi}{4} + \frac{\delta}{2}$ and use $\log \tan \alpha \geq \delta$, $-4 \log c \leq 2 \log 2 + 8\delta$ for $\delta \leq 1/2$, and $B(\frac{1}{2}, k-1) \leq \sqrt{\pi(k - \frac{1}{2})}/(k-1)$. It suffices that $(2k-8)\delta > \log(64\sqrt{\pi}(k-1)\sqrt{k-1/2})$. The resulting δ is $\leq 1/2$ for $k \geq 13$. Strictness: put $\delta_0 = \log(64\sqrt{\pi}(k-1)\sqrt{k-1/2})/(2k-8)$. At $\delta = \delta_0$ the sufficient inequality still holds strictly, because $\log \tan \alpha > \delta$ for $\delta > 0$. So f is negative somewhere at $\alpha_0 = \frac{\pi}{4} + \frac{\delta_0}{2}$. The coefficients of f solve a linear system (constant term 1, and roots of multiplicity at least $k-1, 2, k-1$ at $-\alpha, 0, \alpha$) whose matrix is continuous in α . It is invertible: a kernel vector would be a nonzero raked polynomial with zero constant term and at least $2k$ roots in the open semicircle $(-\frac{\pi}{2}, \frac{\pi}{2})$, which Theorem 12(a) excludes. So f depends continuously on α , and negativity is an open condition in α . Hence f is negative somewhere at some $\alpha' < \alpha_0$, and $\phi_k \leq 2\alpha' < \frac{\pi}{2} + \delta_0$. $\square$

Odd k : the sharp order of magnitude. For odd k the three-cluster bound loses a factor 3 in front of $\log k$. The near-balanced split $p = \frac{k-1}{2}$, $q = \frac{k+1}{2}$ recovers the factor 1. The tools are those of §5 below, used in the opposite direction.

Proposition 19 (general-split upper criterion). *Let $2 \leq p < q$ with $b - a = 2$, and put $N = a + b$, $L = \frac{\pi}{2} + \delta$, $t = \tan \delta$, $w = N^{-1/3}$. Assume $\frac{L}{2} - \frac{2}{a} - w > 0$, $\frac{L}{2} + \frac{2}{a} + w \leq \frac{\pi}{2}$, and $\kappa := (\sqrt{a} + \sqrt{b})^2 / (1 + \sin \delta) > 4$. Put*

$$K_w = \frac{a}{\sin^2(\frac{L}{2} - \frac{2}{a} - w)} + \frac{b}{\sin^2(\frac{L}{2} - w)}, \quad E_{\text{low}} = \frac{1}{4} \sqrt{2\pi/K_w} (1 - \text{erfc}(w\sqrt{K_w/2})) - \frac{1}{6} (152/\kappa^2)^{3/4}.$$

Let $D_1 = 2\Gamma(p + \frac{1}{2})\Gamma(q + \frac{1}{2})/(\Gamma(p)\Gamma(q))$ as in Lemma 24(c). If $E_{\text{low}} > 0$ and $(\delta + t)D_1 + b \log \cos \delta - \frac{bt^2p}{2(q-1)} > \log 2 - \log E_{\text{low}}$, then $\Delta_{p,q}(L) < 0$, and hence $\phi_k \leq L(p, q) < L$.

Proof. We use the notation $M_{x,y}(\ell) = \int_0^\ell \sin^x u \sin^y(\ell - u) du$ and Lemmas 22 and 24 of §5 below; their proofs do not use this subsection. *Outer integral.* $I_{\text{out}} \leq M_{a,b}(\pi - L)$ since $\sin \leq 1$, and by log-concavity (Lemma 24(a),(c)) $\log M_{a,b}(\frac{\pi}{2} - \delta) \leq \log M_{a,b}(\frac{\pi}{2}) - \delta D_1$.

Inner mass. Dropping $u \in (\frac{\pi}{2}, L)$, we get $M_{a,b}(L) \geq \cos^b \delta M_{a,b}(\frac{\pi}{2}) \mathbb{E}_0(1 + tT)^b$, where $u \sim \sin^a u \cos^b u$ on $(0, \frac{\pi}{2})$ and $T = \tan u$. By Jensen and $\log(1 + x) \geq x - x^2/2$, $\mathbb{E}_0(1 + tT)^b \geq \exp(tb\mathbb{E}_0 T - \frac{bt^2}{2}\mathbb{E}_0 T^2)$, with $b\mathbb{E}_0 T = bB(p + \frac{1}{2}, q - \frac{1}{2})/B(p, q) = D_1$ (as in the proof of Lemma 24(c)) and $\mathbb{E}_0 T^2 = B(p + 1, q - 1)/B(p, q) = p/(q - 1)$.

Mean absolute deviation. $I_{\text{in}} = \frac{1}{2} M_{a,b}(L) \mathbb{E}|\sin(u - s)|$ for $u \sim A/M$ on $(0, L)$, $A = e^{-V}$. Let u_1 be the mode, so $V'(u_1) = 0$. Since $V'(\frac{L}{2}) = (b - a) \cot \frac{L}{2} > 0$ and $V'(\frac{L}{2} - \eta) \leq -2a\eta + (b - a)$ (because $\cot(x - \eta) - \cot(x + \eta) \geq 2\eta$ and $\cot(\frac{L}{2} + \eta) \leq 1$), we have $u_1 \in (\frac{L}{2} - \frac{2}{a}, \frac{L}{2})$. On $|u - u_1| \leq w$ we have $V'' \leq K_w$ by the two assumptions on the window. Hence $A(u) \geq A(u_1)e^{-K_w(u - u_1)^2/2}$ there, and the maximal density satisfies $A(u_1)/M \leq 1/\int_{-w}^w e^{-K_w x^2/2} dx$. Since $\mathbb{P}(|u - s| < r) \leq 2rA(u_1)/M$, $\mathbb{E}|u - s| \geq M/(4A(u_1))$. Integrating $(u - u_1)^{2j-1}V'(u)$ by parts, with $V'' \geq \kappa$ (proof of Lemma 22), gives $\mathbb{E}(u - u_1)^2 \leq 1/\kappa$ and $\mathbb{E}(u - u_1)^4 \leq 3/\kappa^2$. Hence $\mathbb{E} \cos(u - u_1) \geq 1 - \frac{1}{2\kappa} \geq \frac{1}{2}$ and $|\mathbb{E} \sin(u - u_1)| \leq \mathbb{E}|u - u_1| \leq \kappa^{-1/2} < \frac{1}{2}$, since $\kappa > 4$. The angle s maximizes $\sigma \mapsto \mathbb{E} \cos(u - \sigma)$, so $\mathbb{E} \cos(u - s) \geq \mathbb{E} \cos(u - u_1)$. If $|s - u_1| \geq \frac{\pi}{2}$, then $\mathbb{E} \cos(u - s) = \cos(s - u_1)\mathbb{E} \cos(u - u_1) + \sin(s - u_1)\mathbb{E} \sin(u - u_1) \leq |\mathbb{E} \sin(u - u_1)| < \frac{1}{2} \leq \mathbb{E} \cos(u - u_1)$, a contradiction. So $|s - u_1| < \frac{\pi}{2}$. From $\mathbb{E} \sin(u - s) = 0$ we get $\tan(s - u_1) = \mathbb{E} \sin(u - u_1)/\mathbb{E} \cos(u - u_1)$, hence $|s - u_1| \leq |\tan(s - u_1)| \leq \kappa^{-1/2}/(1 - \frac{1}{2\kappa}) \leq 2\kappa^{-1/2}$. Thus $\mathbb{E}(u - s)^4 \leq 8(3 + 16)/\kappa^2$. Finally $\mathbb{E}|\sin(u - s)| \geq \mathbb{E}|u - s| - \frac{1}{6}\mathbb{E}|u - s|^3 \geq E_{\text{low}}$. Combining, $\log I_{\text{in}} - \log I_{\text{out}} \geq (\delta + t)D_1 + b \log \cos \delta - \frac{bt^2p}{2(q-1)} - \log 2 + \log E_{\text{low}} > 0$. $\square$

Proof of Theorem 6. Put $C = 4.1$ and $\delta_k = (\log k + C)/(4k)$.

$2 \leq k \leq 300$. The enclosures of Theorem 8 give $\phi_k < \frac{\pi}{2} + \delta_k$, compared in ball arithmetic. The smallest margin is $7.2 \cdot 10^{-4}$, at $k = 300$ (numerics/bln_odd_upper_arb.py, output numerics/odd_upper_arb.txt).

Even $k \geq 302$. By Theorem 4(i), $4k(\phi_k - \frac{\pi}{2}) - \log k < g(k) := \log(8\pi) + \log(1 + \frac{1}{2k}) + \frac{\log(8\pi(k+1/2))}{k-1}$. Both k -dependent terms decrease in k ; for the last one,

$$\frac{d}{dx} \frac{\log(8\pi(x + \frac{1}{2}))}{x - 1} = \frac{(x - 1)/(x + \frac{1}{2}) - \log(8\pi(x + \frac{1}{2}))}{(x - 1)^2} < 0.$$

Hence $g(k) \leq g(302) < 3.26 < C$.

Odd $301 \leq k \leq 200001$. Apply Proposition 19 with $p = \frac{k-1}{2}$, $q = \frac{k+1}{2}$ (so $a = k - 2$, $b = k$) and $\delta = \delta_k$. For each of these 99851 values of k , its four hypotheses and its final inequality

were certified in Arb ball arithmetic (precision 96 bits). The smallest certified margin in the final inequality is 0.0257, at $k = 301$ (same script).

Odd $k \geq 200003$. We verify the hypotheses and the inequality of Proposition 19 by hand. Let $N = 2k - 2$ and $w = N^{-1/3}$, and put $\varepsilon_1 = \frac{2}{a} + w$. Then $\delta := \delta_k \leq 2.04 \cdot 10^{-5}$ and $\varepsilon_1 \leq 0.0135821$, and both decrease in k . So $\frac{L}{2} - \frac{2}{a} - w = \frac{\pi}{4} + \frac{\delta}{2} - \varepsilon_1 > 0$ and $\frac{L}{2} + \frac{2}{a} + w < \frac{\pi}{2}$; and $\kappa \geq 4a/(1 + \delta) > 4$, since $b > a$. Both arguments of $\sin$ in K_w lie in $[\frac{\pi}{4} - \varepsilon_1, \frac{\pi}{2}]$. Moreover $\sin(\frac{\pi}{4} - \varepsilon) = (\cos \varepsilon - \sin \varepsilon)/\sqrt{2} \geq s(\varepsilon)/\sqrt{2}$, where $s(\varepsilon) = 1 - \varepsilon - \varepsilon^2/2$. Hence $K_w \leq 2N/s(\varepsilon_1)^2$, and $\frac{1}{4}\sqrt{2\pi/K_w} \geq \frac{1}{4}\sqrt{\pi/N} s(\varepsilon_1)$. Since $\sin^2 \leq 1$, $K_w \geq a + b = N$, so $\operatorname{erfc}(w\sqrt{K_w/2}) \leq e_1 := \operatorname{erfc}(N^{1/6}/\sqrt{2}) < 10^{-16}$. From $\kappa \geq 4a/(1 + \delta)$ we get $\frac{1}{6}(152/\kappa^2)^{3/4} \leq 0.902 a^{-3/2} \leq \frac{1}{4}\sqrt{\pi/N} \cdot \frac{2.9}{a}$, using $N = 2a + 2$. Therefore

$$E_{\text{low}} \geq \frac{1}{4}\sqrt{\pi/N}(1 - \eta), \quad \eta := \varepsilon_1 + \frac{1}{2}\varepsilon_1^2 + e_1 + \frac{2.9}{a} \leq 0.01369.$$

In particular $E_{\text{low}} > 0$, and since $N < 2k$, $\log 2 - \log E_{\text{low}} < \frac{1}{2} \log k + \frac{1}{2} \log \frac{128}{\pi} - \log(1 - \eta)$. On the other side, Lemma 17 gives $D_1 \geq 2pq/\sqrt{(p + \frac{1}{2})(q + \frac{1}{2})} = (k^2 - 1)/\sqrt{k(k + 2)} \geq k - 1$. Since $\delta \leq t \leq 2\delta$, $\log \cos \delta \geq -\delta^2$ and $q - 1 = p$, the left side of the inequality in Proposition 19 is at least $2\delta(k - 1) - k\delta^2 - \frac{k}{2}t^2 \geq \frac{1}{2}(\log k + C) - 2\delta - 3k\delta^2$. The quantity $3k\delta^2 = 3(\log k + C)^2/(16k)$ decreases in k and is $\leq 2.5 \cdot 10^{-4}$. So the margin in the inequality of Proposition 19 is at least

$$\frac{C}{2} - \frac{1}{2} \log \frac{128}{\pi} + \log(1 - \eta) - 2\delta - 3k\delta^2 \geq 2.05 - 1.85366 - 0.01379 - 0.00005 - 0.00025 > 0.18.$$

In all cases Proposition 19 or the preceding cases give $\phi_k < \frac{\pi}{2} + \delta_k$. $\square$

Remark 20. Letting $C \downarrow \log(128/\pi)$ in the last step shows that Proposition 19 alone gives $\limsup_{k \text{ odd}} (4k(\phi_k - \frac{\pi}{2}) - \log k) \leq \log(128/\pi) = 3.707$. Theorem 7 gives the sharp value $\log(8\pi) = 3.224$ for both parities. The explicit lower bounds on the margin used in the last step are compared with the exact criterion at $k = 200003$, $10^6 + 1$, $10^8 + 1$ and $10^{12} + 1$ in the same script.

5. LOWER BOUND: PROOF OF THEOREM 5

Throughout, $p \leq q$ (by symmetry), $a = 2p - 1$, $b = 2q - 1$, $L = \frac{\pi}{2} + \delta$ with $0 < \delta \leq 0.2$, $t = \tan \delta$, and $M_{x,y}(\ell) = \int_0^\ell \sin^x u \sin^y(\ell - u) du$.

Lemma 21 (variance). *Let $\mu = e^{-V} du$ be a probability measure on an interval, $V \in C^2$, $V'' \geq \kappa > 0$, and $(u - m)e^{-V} \rightarrow 0$ at the endpoints, where $m = \int u d\mu$. Then $\operatorname{Var}_\mu(u) \leq 1/\kappa$. If moreover the interval has length $< \pi$ and $s = \arg \int e^{iu} d\mu$, then $\int \sin^2(u - s) d\mu \leq \operatorname{Var}_\mu(u)$.*

Proof. Integrating by parts, $\int (u - m)V'(u) d\mu = 1$. Also $\int (u - m)V'(m) d\mu = 0$, and $(u - m)(V'(u) - V'(m)) \geq \kappa(u - m)^2$. Hence $\kappa \operatorname{Var} \leq 1$. Next, $\sigma \mapsto \int \cos(u - \sigma) d\mu = |\int e^{iu} d\mu| \cos(\sigma - s)$ is maximal at $\sigma = s$. Hence $\int \sin^2(u - s) \leq 2 \int (1 - \cos(u - s)) \leq 2 \int (1 - \cos(u - m)) \leq \int (u - m)^2$. $\square$

Lemma 22. $I_{\text{in}} \leq \frac{1}{2} M_{a,b}(L) \kappa_\delta^{-1/2}$ with $\kappa_\delta = (\sqrt{a} + \sqrt{b})^2/(1 + \sin \delta)$.

Proof. Apply Lemma 21 to $A/M_{a,b}(L) = e^{-V}$, with $V'' = a/\sin^2 u + b/\sin^2(L - u)$. By Cauchy-Schwarz, V'' is at least $(\sqrt{a} + \sqrt{b})^2/(\sin^2 u + \sin^2(L - u))$, and $\sin^2 u + \sin^2(L - u) = 1 - \cos L \cos(2u - L) \leq 1 + \sin \delta$. Then $I_{\text{in}} = \frac{1}{2} M \int |\sin(u - s)| \leq \frac{1}{2} M (\int \sin^2(u - s))^{1/2}$. $\square$

Lemma 23. $I_{\text{out}} \geq M_{a+1,b+1}(\pi - L)$.

Proof. For $u \in [L, \pi]$ we have $u - s \in [u - L, u] \subset [0, \pi]$, so by concavity of $\sin$, $\sin(u - s) \geq \min(\sin u, \sin(u - L)) \geq \sin u \sin(u - L)$. Substitute $v = u - L$ and use $\sin(L + v) = \sin(\pi - L - v)$. $\square$

Lemma 24. (a) $\ell \mapsto M_{x,y}(\ell)$ is log-concave on $(0, \pi)$ for $x, y \geq 0$. (b) $M_{a,b}(\pi/2) = \frac{1}{2}B(p, q)$, $M_{a+1,b+1}(\pi/2) = \frac{1}{2}B(p + \frac{1}{2}, q + \frac{1}{2})$. (c) $D_1 := (\log M_{a,b})'(\pi/2) = 2 \frac{\Gamma(p+1/2)\Gamma(q+1/2)}{\Gamma(p)\Gamma(q)} \leq 2\sqrt{pq}$. (d) Let $u \sim \sin^{a+1} u \cos^{b+1} u du$ on $(0, \pi/2)$ and $T = \tan u$. Then $(b+1)\mathbb{E}T = D_2 := 2 \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+1/2)\Gamma(q+1/2)}$, and $\sin^2 u \sim \text{Beta}(p + \frac{1}{2}, q + \frac{1}{2})$.

Proof. (a) $M_{x,y} = h_x * h_y$ with $h_x = \sin^x \cdot 1_{(0,\pi)}$ log-concave, and a convolution of log-concave functions is log-concave (Prékopa's theorem [6]). (b),(d) are Beta integrals: $\int_0^{\pi/2} \sin^x u \cos^y u du = \frac{1}{2}B(\frac{x+1}{2}, \frac{y+1}{2})$. (c) $M'_{a,b}(\ell) = b \int \sin^a u \sin^{b-1}(\ell - u) \cos(\ell - u)$, which at $\ell = \pi/2$ gives $bB(p + \frac{1}{2}, q - \frac{1}{2})/B(p, q)$; simplify, and use Lemma 17. $\square$

Proposition 25 (explicit safety criterion). Let $\tau = (4/5)^{p-1/2}(1/5)^{q+1/2}/B(p + \frac{1}{2}, q + \frac{1}{2})$ and suppose $\tau < 1$, $t < 1/2$. Define

$$\Phi(p, q, \delta) = \log \frac{B(p, q)}{B(p+\frac{1}{2}, q+\frac{1}{2})} + \delta D_1 + \frac{tD_2}{(1-2t)(1-\tau)} - (b+1) \log \cos \delta - \log(1-\tau) - \log 2 - \frac{1}{2} \log \kappa_\delta.$$

If $\Phi(p, q, \delta) < 0$, then $\Delta_{p,q}(\frac{\pi}{2} + \delta) > 0$, and hence $L(p, q) > \frac{\pi}{2} + \delta$.

Proof. By Lemmas 22 and 23 it suffices that $\log M_{a,b}(\frac{\pi}{2} + \delta) - \log M_{a+1,b+1}(\frac{\pi}{2} - \delta) < \log 2 + \frac{1}{2} \log \kappa_\delta$. By Lemma 24(a), $\log M_{a,b}(\frac{\pi}{2} + \delta) \leq \log M_{a,b}(\frac{\pi}{2}) + \delta D_1$. Next, $M_{a+1,b+1}(\frac{\pi}{2} - \delta) = \int_0^{\pi/2-\delta} \sin^{a+1} u \cos^{b+1}(u + \delta) du$, and $\cos(u + \delta) = \cos \delta \cos u (1 - tT)$. So $M_{a+1,b+1}(\frac{\pi}{2} - \delta) = \cos^{b+1} \delta M_{a+1,b+1}(\frac{\pi}{2}) \mathbb{E}(1 - tT)_+^{b+1}$. On $\{T \leq 2\}$, $\log(1 - tT) \geq -tT/(1 - 2t)$. By Jensen on the conditional law, $\mathbb{E}(1 - tT)_+^{b+1} \geq P_0 \exp(-(b+1)t\mathbb{E}T/((1 - 2t)P_0))$ with $P_0 = \mathbb{P}(T \leq 2)$. Moreover $1 - P_0 = \mathbb{P}(\sin^2 u > 4/5) \leq \tau$: the $\text{Beta}(p + \frac{1}{2}, q + \frac{1}{2})$ density is decreasing on $[4/5, 1]$ because $p \leq q$. Combine with Lemma 24(b),(d). $\square$

Proof of Theorem 5(i). Fix $c < 2 \log 2$ and put $\delta = (\log k + c)/(4k)$. We show $\max_{p \leq q} \Phi < 0$ for large k . Lemma 17 gives $\log \frac{B(p, q)}{B(p+1/2, q+1/2)} \leq \log k - \frac{1}{2} \log(pq) + \frac{1}{2} \log((1 + \frac{1}{2p})(1 + \frac{1}{2q}))$, $D_2 \leq 2\sqrt{(p + \frac{1}{2})(q + \frac{1}{2})}$, and $D_1 \leq 2\sqrt{pq}$. Also $\tau \leq C\sqrt{k}(0.8)^k$ uniformly in $p \leq q$, using $1/B(\alpha, \beta) \leq 2^{\alpha+\beta-1} \sqrt{(\alpha + \beta)/(2\pi)}$ (log-convexity of Γ and duplication). Further $-\log \cos \delta \leq \delta^2$, and $\frac{t}{\delta(1-2t)(1-\tau)} = 1 + O(\delta + \tau)$. Finally $\delta\sqrt{pq}(\frac{1}{p}) \leq \delta\sqrt{k} \rightarrow 0$, $k\delta^2 \rightarrow 0$, and $\delta k(\delta + \tau) \rightarrow 0$. So, uniformly in p ,

$$\Phi \leq F + o(1), \quad F = -\log y + (\log k + c)y - \log 2 - \log(\sqrt{a} + \sqrt{b}) + \frac{1}{2} \log(1 + \frac{1}{2p}), \quad y = \frac{\sqrt{pq}}{k} \in (0, \frac{1}{2}].$$

Since $(\sqrt{a} + \sqrt{b})^2 = 2k - 2 + 2\sqrt{(2p-1)(2q-1)} \geq 2k(1+2y) - 2 - 2/y$ and $1/y \leq k/\sqrt{k-1}$, we have $\log(\sqrt{a} + \sqrt{b}) \geq \frac{1}{2} \log(2k) + \frac{1}{2} \log(1+2y) - o(1)$. Thus $F \leq h(y) + \frac{1}{2} \log(1 + \frac{1}{2p}) + o(1)$, where $h(y) = -\log y + (\log k + c)y - \frac{1}{2} \log k - \frac{3}{2} \log 2 - \frac{1}{2} \log(1+2y)$ is convex. We have $h(\frac{1}{2}) = \frac{c}{2} - \log 2 < 0$, $h(y_{\min}) = -\frac{3}{2} \log 2 + o(1)$ with $y_{\min} = \sqrt{k-1}/k$, and $h(k^{-1/4}) = -\frac{1}{4} \log k - \frac{3}{2} \log 2 + o(1)$. On $[k^{-1/4}, \frac{1}{2}]$, $p \geq ky^2 \geq \sqrt{k}$, so the extra term is $o(1)$, and by convexity $h \leq \max(h(k^{-1/4}), h(\frac{1}{2})) < 0$. On $[y_{\min}, k^{-1/4}]$, the extra term is at most $\frac{1}{2} \log \frac{3}{2} < 0.203$ and $h \leq -\frac{3}{2} \log 2 + o(1) < -1.03$. Hence $\Phi < 0$ for all splits when k is large. Proposition 25 and Theorem 3 give $\phi_k > \frac{\pi}{2} + \delta$ for large k . $\square$

Proof of Theorem 5(ii). Take $c = 0$. For $k > 2 \cdot 10^4$ the error terms in the previous proof are explicitly small. Here $\delta \leq 1.3 \cdot 10^{-4}$, $(1 + 3\delta)\delta\sqrt{k}/2 \leq 0.009$ (the $1/p$ part of the D_2 correction, including the factor $\frac{t}{(1-2t)(1-\tau)} \leq (1 + 3\delta)\delta$), $3k\delta^2 \leq 10^{-3}$ (the $O(\delta)$ part), $2k\delta^2 = (\log k)^2/(8k) \leq 7 \cdot 10^{-4}$ (the $\log \cos \delta$ term), the $\log(\sqrt{a} + \sqrt{b})$ correction is $\leq (1 + k/\sqrt{k-1})/k < 0.008$, and $\tau < 10^{-900}$. The remaining terms are also small: $\frac{1}{2} \log(1 + \sin \delta) \leq \delta/2$; $\frac{1}{2} \log(1 + \frac{1}{2q}) \leq \frac{1}{2k}$ (as $q \geq k/2$); the $1/q$ and $1/(pq)$ parts of the D_2 correction, at most $\frac{t}{(1-2t)(1-\tau)} \cdot 2\sqrt{pq}(\frac{1}{4q} + \frac{1}{16pq}) \leq \delta$;

and $-\log(1-\tau) \leq 2\tau$. Together they contribute $< 3 \cdot 10^{-4}$. All these bounds decrease in k , and their total is < 0.02 . On the other side, $h(\frac{1}{2}) = -\log 2$, $h(k^{-1/4}) \leq -2.7$, and on the small- p branch $h \leq h(y_{\min}) \leq -0.976$, so even after adding $\frac{1}{2} \log \frac{3}{2}$ it stays ≤ -0.77 . On the large- p branch, $p \geq \sqrt{k}$ and the extra term $\frac{1}{2} \log(1 + \frac{1}{2p})$ is $\leq \frac{1}{4\sqrt{k}} < 0.002$. So in every case the error total is < 0.025 , against a margin of at least 0.69. Hence $\Phi < 0$.

For $301 \leq k \leq 2 \cdot 10^4$ we certified $\Phi(p, k - p, \frac{\log k}{4k}) < 0$, together with the hypotheses $\tau < 1$ and $t < \frac{1}{2}$ of Proposition 25, in Arb ball arithmetic (precision 96 bits) for each of the 99977500 pairs (k, p) with $p \leq k/2$. All Gamma arguments are multiples of $\frac{1}{2}$, so $\log \Gamma(n/2)$ is tabulated once. The largest upper endpoint of the certified enclosures of Φ is -0.6127 , at $k = 301$, $p = 1$ (`numerics/bln.lower_arb.py`, output `numerics/lower_arb.txt`; about 80 seconds on 8 cores). For $2 \leq k \leq 300$, the enclosures of Theorem 8 give $\phi_k - \frac{\pi}{2} - \frac{\log k}{4k} > 0$; the smallest margin is 0.0026, at $k = 300$ (same script). $\square$

6. SHARP ASYMPTOTICS: PROOF OF THEOREM 7

Lemma 26 (Laplace bounds for strongly log-concave laws). *Let $\mu = e^{-V} du/Z$ be a probability measure on an open interval J , $V \in C^2(J)$, $V'' \geq \kappa > 0$, and $e^{-V} \rightarrow 0$ at the endpoints. Let $V'(u_1) = 0$, $K = V''(u_1)$, $y = u - u_1$. Suppose $|V'' - K| \leq \eta K$ on $[u_1 - w, u_1 + w] \subset J$ with $\eta \leq \frac{1}{2}$, and put $\lambda = (1 - \eta)Kw^2/2 \geq 1$. Then, with an absolute constant C :*

$$\left| \mathbb{E}|y| - \sqrt{2/(\pi K)} \right| \leq C(\eta + e^{-\lambda})K^{-1/2}, \quad |\mathbb{E}y| \leq C(\eta + e^{-\lambda})K^{-1/2}, \quad \mathbb{E}y^2 \leq \kappa^{-1}, \quad \mathbb{E}y^4 \leq 3\kappa^{-2}.$$

Proof. Let $\varphi(y) = V(u_1 + y) - V(u_1)$ and $K_{\pm} = (1 \pm \eta)K$. For $|y| \leq w$, Taylor's formula gives $K_-y^2/2 \leq \varphi \leq K_+y^2/2$. For $|y| > w$, convexity and $\varphi'(\pm w) \geq \pm K_-w$ give $\varphi(y) \geq K_-w(|y| - w/2)$. Hence the parts of $\int e^{-\varphi}$ and $\int |y|e^{-\varphi}$ over $|y| > w$ are at most $\frac{2}{K_-w}e^{-\lambda}$ and $\frac{2e^{-\lambda}}{K_-w}(w + \frac{1}{K_-w})$. Over $|y| \leq w$ they lie between the corresponding Gaussian integrals with K_+ and K_- , namely $\sqrt{2\pi/K_{\pm}}(1 - O(e^{-\lambda}))$ and $\frac{2}{K_{\pm}}(1 - O(e^{-\lambda}))$. Taking ratios gives the first bound. For the second, compare $\int_0^w ye^{-\varphi}$ and $\int_{-w}^0 |y|e^{-\varphi}$; both lie in $[\frac{1}{K_+}(1 - e^{-\lambda}), \frac{1}{K_-}]$. The last two bounds come from integrating $y^{2j-1}V'(u_1 + y) \geq \kappa y^{2j}$ by parts ($j = 1, 2$). $\square$

Fix $\epsilon = \frac{1}{5}$ and call a split *central* if $\epsilon k \leq p \leq q$ and *extreme* if $p < \epsilon k$ (with $p \leq q$). Throughout, $o(1)$ is uniform over central splits as $k \rightarrow \infty$, and $0 \leq \delta \leq 2 \log k/k$, $L = \frac{\pi}{2} + \delta$, $L' = \pi - L$. Write $u_1(\ell)$ for the mode of $A_{\ell} = \sin^a u \sin^b(\ell - u)$ on $(0, \ell)$, and $r_{ab} = 2\sqrt{ab}/(a + b)$.

Lemma 27 (central splits). (a) $\tan^2 u_1(\frac{\pi}{2}) = a/b$, $|u'_1(\ell)| \leq 1$, and $K(\ell) := V''_{\ell}(u_1(\ell)) = 2N(1 + O(|\ell - \frac{\pi}{2}|))$. Moreover $|V'''_{\ell}| = O(N)$ within distance 0.1 of $u_1(\ell)$. (b) $I_{\text{in}} = \frac{1}{2}M_{a,b}(L)(\pi N)^{-1/2}(1 + o(1))$. (c) $I_{\text{out}} = M_{a,b}(L')r_{ab}(1 + o(1))$. (d) $\Psi := \log M_{a,b}(L) - \log M_{a,b}(L') = 2\delta D_1 + O(N\delta^2) + o(1)$, and $D_1 = 2\sqrt{pq}(1 + O(1/k))$.

Proof. (a) u_1 solves $F(u, \ell) := a \cot u - b \cot(\ell - u) = 0$, and $u'_1 = -\partial_{\ell}F/\partial_u F = \frac{b \csc^2(\ell - u)}{a \csc^2 u + b \csc^2(\ell - u)} \in (0, 1)$. At $\ell = \frac{\pi}{2}$, $\sin^2 u_1 = a/N$, so $K = a \cdot \frac{N}{a} + b \cdot \frac{N}{b} = 2N$. For central splits $\tan^2 u_1(\frac{\pi}{2}) \in [\frac{1}{8}, 1]$, so u_1 and $\ell - u_1$ stay at distance ≥ 0.25 from 0 for $|\ell - \frac{\pi}{2}| \leq 0.05$. On that range $\csc^2$ and $\csc^2 \cdot \cot$ are bounded, which gives the remaining claims. (b) Apply Lemma 26 to A_L/M , with $\kappa \geq N/2$ (Lemma 22), $w = N^{-1/3}$, $\eta = O(N^{-1/3})$ and $\lambda \asymp N^{1/3}$. We get $\mathbb{E}|y| = (\pi N)^{-1/2}(1 + o(1))$ and $\mathbb{E}y = o(N^{-1/2})$. From $\mathbb{E} \sin(u - s) = 0$, $\tan(s - u_1) = \mathbb{E} \sin y / \mathbb{E} \cos y$, and $|\mathbb{E} \sin y - \mathbb{E}y| \leq \mathbb{E}|y|^3 = O(N^{-3/2})$. To pass to $s - u_1$ itself, note that $\mathbb{E} \cos y \geq 1 - \frac{1}{2}\mathbb{E}y^2 \geq 1 - \frac{1}{N}$. Also s maximizes $\sigma \mapsto \mathbb{E} \cos(u - \sigma)$, so $\mathbb{E} \cos(u - s) \geq \mathbb{E} \cos y$. If $|s - u_1| \geq \frac{\pi}{2}$, then $\mathbb{E} \cos(u - s) \leq |\mathbb{E} \sin y| \leq \mathbb{E}|y| < \mathbb{E} \cos y$, a contradiction. Hence $|s - u_1| < \frac{\pi}{2}$, and $s - u_1 = \arctan(\mathbb{E} \sin y / \mathbb{E} \cos y) = o(N^{-1/2})$. Hence $\mathbb{E}|\sin(u - s)| = \mathbb{E}|y| + O(|s - u_1| + \mathbb{E}|y|^3 + |s - u_1|^3) = (\pi N)^{-1/2}(1 + o(1))$, and $I_{\text{in}} = \frac{1}{2}M \mathbb{E}|\sin(u - s)|$.

(c) The substitution $v = \pi - u$ maps the outer density to $A_{L'}$ and $\sin(u-s)$ to $\sin(v+s)$. By (a) and Lemma 26, $\mathbb{E}|v - u_1(L')| = O(N^{-1/2})$. Also $v_1 + s = u_1(L') + u_1(L) + o(1) = 2u_1(\frac{\pi}{2}) + O(\delta) + o(1)$, and $\sin 2u_1(\frac{\pi}{2}) = \frac{2\sqrt{a/b}}{1+a/b} = r_{ab}$, and $r_{ab} \geq 0.79$ for central splits once k is large (the limit value at $p = \epsilon k$ is 0.8). Hence $\mathbb{E} \sin(v+s) = r_{ab}(1+o(1))$. (d) Upper bound: $\log M(\frac{\pi}{2} + \delta) \leq \log M(\frac{\pi}{2}) + \delta D_1$ by concavity. Also $M(\frac{\pi}{2} - \delta) = \cos^b \delta M(\frac{\pi}{2}) \mathbb{E}_0(1 - tT)_+^b$, and on $\{T \leq 1/(2t)\}$, $(1 - tT)^b \geq e^{-btT - bt^2T^2}$. By Jensen, with $b\mathbb{E}_0T = D_1$, $\mathbb{E}_0T^2 = \frac{p}{q-1} \leq 2$, and $\mathbb{P}(T > \frac{1}{2t}) \leq \mathbb{E}_0T^b(2t)^b$ super-polynomially small, we get $\log M(\frac{\pi}{2} - \delta) \geq \log M(\frac{\pi}{2}) - tD_1 - O(N\delta^2)$. Lower bound: by concavity $\log M(\frac{\pi}{2} - \delta) \leq \log M(\frac{\pi}{2}) - \delta D_1$; and as in Proposition 19, $\log M(\frac{\pi}{2} + \delta) \geq \log M(\frac{\pi}{2}) + tD_1 - O(N\delta^2)$. Since $t - \delta = O(\delta^3)$, (d) follows; the bound on D_1 is Lemma 17. $\square$

Proof of Theorem 7. By Theorem 3, a split is safe at L iff $\log I_{\text{out}} > \log I_{\text{in}}$. For central splits, by Lemma 27, this is iff $\Psi < \log(2r_{ab}\sqrt{\pi N}) + o(1)$. Put $\delta = (\log k + c)/(4k)$ and $r = 2\sqrt{pq}/k \in [0.8, 1]$. Then $r_{ab} = r(1 + O(1/k))$, $N = 2k - 2$, and $N\delta^2 = o(1)$. So the condition reads

$$(*) \quad \frac{r}{2}(\log k + c) < \frac{1}{2} \log k + \frac{1}{2} \log(8\pi) + \log r + o(1).$$

Upper bound. The balanced split ($p = \lfloor k/2 \rfloor$) has $r = 1 - O(k^{-2})$. By (*) it is unsafe for $c > \log 8\pi$ and k large. Hence $\phi_k \leq L(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil) \leq \frac{\pi}{2} + \frac{\log k + \log 8\pi + o(1)}{4k}$. *Lower bound, central splits.* Let $c = \log 8\pi - \epsilon'$. Using $\log r \geq -(1-r)/r \geq -\frac{5}{4}(1-r)$, the right side of (*) minus the left side is at least $(1-r)(\frac{1}{2} \log k + \frac{1}{2} \log 8\pi - \frac{5}{4}) + \frac{r}{2}\epsilon' - o(1) \geq 0.4\epsilon' - o(1) > 0$. So every central split is safe for large k . *Lower bound, extreme splits.* The proof of Theorem 5(i) applies verbatim with $c = \log 8\pi$ in the range $y = \sqrt{pq}/k \leq \sqrt{p/k} < \sqrt{\epsilon}$. There $h(\sqrt{\epsilon}) = (\sqrt{\epsilon} - \frac{1}{2}) \log k + O(1) \rightarrow -\infty$ and $h(y_{\min}) \rightarrow -\frac{3}{2} \log 2$, so by convexity $\Phi < 0$ for large k , uniformly over extreme splits. Hence $\phi_k = \min_p L(p, k-p) \geq \frac{\pi}{2} + \frac{\log k + \log 8\pi - \epsilon'}{4k}$ for large k , for every $\epsilon' > 0$. $\square$

7. ALL SUFFICIENTLY LARGE k : PROOF OF THEOREM 9

Write $L_{\text{bal}} = L(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil)$. By symmetry we only consider $p \leq q$.

Lemma 28 (far-from-balanced splits). *There is K_A such that for $k \geq K_A$ every split with $p \leq q$ and $p \leq 0.41k$ satisfies $L(p, k-p) > L_{\text{bal}}$.*

Proof. By the upper-bound part of the proof of Theorem 7, $L_{\text{bal}} < \frac{\pi}{2} + \frac{\log k + \log 8\pi + 1/40}{4k}$ for large k . Let $c_+ = \log 8\pi + \frac{1}{20}$ and $\delta_+ = (\log k + c_+)/ (4k)$. For a central split ($p \geq k/5$) with $p \leq 0.41k$ we have $r = 2\sqrt{pq}/k \in [0.8, 0.984]$. At $c = c_+$ the right side of (*) minus the left side equals $\frac{1-r}{2}(\log k + \log 8\pi) - \frac{r}{40} + \log r \geq 0.008 \log k - \frac{1}{4}$. This tends to ∞ , so the split is safe at δ_+ for large k . For extreme splits ($p < k/5$) we repeat the proof of Theorem 5(i) with $c = c_+$, in the range $y = \sqrt{pq}/k < 0.4$. The restriction $c < 2 \log 2$ in that proof is used only at $y = \frac{1}{2}$, which lies outside this range. Here $h(0.4) = (0.4 - \frac{1}{2}) \log k + O(1) \rightarrow -\infty$ and $h(y_{\min}) \rightarrow -\frac{3}{2} \log 2$. Since h is convex, $\Phi \leq \max h + \frac{1}{2} \log \frac{3}{2} + o(1) < 0$, and Proposition 25 shows these splits are safe at δ_+ . In both cases $L(p, q) > \frac{\pi}{2} + \delta_+ > L_{\text{bal}}$. $\square$

Setting for near-balanced splits. For the rest of this section $a, b > 0$ are real, $N = a + b$, $x = a/N \in [0.4, 0.6]$, and $\ell \in \Lambda := [\frac{\pi}{2} - \frac{1}{10}, \frac{\pi}{2} + \frac{1}{10}]$. (For the splits of k we have $N = 2k - 2$.) Let $\mu = \mu_{a,\ell}$ be the probability measure on $(0, \ell)$ with density $A_\ell/M_{a,b}(\ell)$, where $A_\ell(u) = \sin^a u \sin^b(\ell - u) = e^{-V(u)}$. Write E , Var , Cov for its moments, u_1 for its mode ($V'(u_1) = 0$), and $\psi_\ell(u) = \log \sin u - \log \sin(\ell - u)$, so that $\partial_a V = -\psi_\ell$ when $b = N - a$ with N fixed, and $\partial_b V = -\log \sin(\ell - u)$ when a is fixed. Put $\Theta(u) = 1 + |\log \sin u| + |\log \sin(\ell - u)| + |\cot u| + |\cot(\ell - u)|$, $B = [u_1 - \frac{1}{4}, u_1 + \frac{1}{4}]$ and $B_1 = [u_1 - \frac{1}{14}, u_1 + \frac{1}{14}]$. Constants C, C_m, c_0 and the implied constants in $O(\cdot)$ do not depend on a, ℓ, N . All statements hold for $N \geq N_1$, where N_1 may depend on the number of moments used.

Lemma 29 (geometry). (i) $u_1, \ell - u_1 \in [0.58, 0.99]$. (ii) $V'' \geq N$ on $(0, \ell)$. (iii) On B : $\sin u, \sin(\ell - u) \geq 0.32$, $V'' \leq 10N$, $|V^{(j)}| \leq CN$ and $|\psi_\ell^{(j)}| \leq C$ for $j \leq 5$. (iv) On $(0, \ell)$: $|\psi_\ell^{(j)}| \leq C\Theta^{j+1}$ and $|V^{(j)}| \leq CN\Theta^j$ for $1 \leq j \leq 5$.

Proof. u_1 solves $a \cot u = b \cot(\ell - u)$, and $0 < \partial_\ell u_1 < 1$ (proof of Lemma 27(a)). At $\ell = \frac{\pi}{2}$, $\tan^2 u_1 = a/b \in [\frac{2}{3}, \frac{3}{2}]$, so u_1 and $\frac{\pi}{2} - u_1$ lie in $[0.684, 0.887]$; this gives (i). (ii) holds because $\csc^2 \geq 1$. On B both u and $\ell - u$ lie in $[0.33, 1.24]$, which gives (iii). (iv) holds because the derivatives of $\log \sin$ are polynomials in $\cot$. $\square$

Lemma 30 (tails and moments). (i) $Z := \int_0^\ell e^{-(V-V(u_1))} du \geq c_0 N^{-1/2}$, so the density of μ is at most $C\sqrt{N}$. (ii) $E|u - u_1|^m \leq C_m N^{-m/2}$. (iii) $E[\Theta^m; u \notin B_1] \leq C_m e^{-N/700}$.

Proof. (i) On B , $V - V(u_1) \leq 5N(u - u_1)^2$. (ii) By Lemma 29(ii), $V(u) - V(u_1) \geq \frac{N}{2}(u - u_1)^2$, so $E|u - u_1|^m \leq Z^{-1} \int |y|^m e^{-Ny^2/2} dy$. (iii) Off B_1 we have $V - V(u_1) \geq N/392$. With $\theta = 1/600$, $e^{-(V-V(u_1))} \leq e^{-(1-\theta)N/392} e^{\theta V(u_1)} A_\ell^\theta$, and $e^{\theta V(u_1)} \leq (\sin 0.58)^{-\theta N} \leq e^{0.61\theta N}$. Hence the left side is at most $Z^{-1} e^{-0.00153N} \int_0^\ell \Theta^m A_\ell^\theta du$, and the last integral is bounded once $\theta a, \theta b \geq m + 2$. $\square$

Lemma 31 (Stein identities). (i) If $g \in C^1(0, \ell)$ and $|g| + |g'| \leq C\Theta^m$, then $E g'(u) = E[g(u)V'(u)]$. (ii) Let $|c - u_1| \leq C/N$ and $y = u - c$. Then $Ey = O(N^{-1})$, $Ey^2 = V''(c)^{-1} + O(N^{-2})$, $Ey^3 = O(N^{-2})$, and $E|y|^m = O(N^{-m/2})$.

Proof. (i) Integrate by parts on $(\epsilon, \ell - \epsilon)$; the boundary terms vanish because e^{-V} vanishes to order $\geq 0.4N$ at 0 and ℓ . (ii) The moment bound follows from Lemma 30(ii). By Lemma 29, $|V'(c)| = |V'(c) - V'(u_1)| \leq C$, and on B_1 $V'(u) = V'(c) + V''(c)y + \frac{1}{2}V'''(c)y^2 + R$ with $|R| \leq CN|y|^3$. The contributions of $u \notin B_1$ to the identities below are $O(e^{-N/800})$ by Lemma 30(iii), since $|V'| \leq N\Theta$. Applying (i) with $g = 1$, $g = y^2$ and $g = y$: $0 = V'(c) + V''(c)Ey + O(NEy^2)$, so $Ey = O(N^{-1})$. Next $2Ey = V'(c)Ey^2 + V''(c)Ey^3 + O(NEy^4)$, so $Ey^3 = O(N^{-2})$. Finally $1 = V'(c)Ey + V''(c)Ey^2 + \frac{1}{2}V'''(c)Ey^3 + O(NEy^4)$, so $Ey^2 = V''(c)^{-1} + O(N^{-2})$. $\square$

Lemma 32 (smooth functionals). Let $F, G \in C^3(0, \ell)$, possibly depending on a and ℓ , with $|F^{(j)}|, |G^{(j)}| \leq C$ on B for $j \leq 3$ and $|F|, |G| \leq C\Theta^m$ on $(0, \ell)$. Then $EF = F(u_1) + O(N^{-1})$, $\text{Cov}(F, G) = F'(u_1)G'(u_1)/V''(u_1) + O(N^{-2})$ (in particular $\text{Cov}(F, G) = O(N^{-1})$ and $\text{Var } F = F'(u_1)^2/V''(u_1) + O(N^{-2})$), and $\kappa(F, G, G) := E[(F - EF)(G - EG)^2] = O(N^{-2})$.

Proof. Let $y = u - u_1$ and $Y = y - Ey$. By Lemma 31 with $c = u_1$: $Ey = O(N^{-1})$, $EY^3 = Ey^3 - 3EyEy^2 + 2(Ey)^3 = O(N^{-2})$, and $\text{Var } y = V''(u_1)^{-1} + O(N^{-2})$. On B_1 write $F = F(u_1) + F'(u_1)y + R_F$ with $|R_F| \leq Cy^2$, and $R_F = \frac{1}{2}F''(u_1)y^2 + R'_F$ with $|R'_F| \leq C|y|^3$; similarly for G . Contributions from $u \notin B_1$ are exponentially small by Lemma 30(iii). The first claim follows. Next, $\text{Cov}(F, G) = F'G' \text{Var } y + F' \text{Cov}(y, R_G) + G' \text{Cov}(y, R_F) + \text{Cov}(R_F, R_G)$. Here $\text{Cov}(y, R_F) = \frac{1}{2}F'' \text{Cov}(y, y^2) + \text{Cov}(y, R'_F) = O(N^{-2})$, since $\text{Cov}(y, y^2) = Ey^3 - EyEy^2$. The same holds for G , and $|\text{Cov}(R_F, R_G)| \leq (ER_F^2 ER_G^2)^{1/2} = O(N^{-2})$. Finally $F - EF = F'(u_1)Y + Q_F$ with $|Q_F| \leq C(y^2 + N^{-1})$ on B_1 , and similarly for G . Expanding, $\kappa = F'G'EY^3 + E O(Y^2|Q| + |Y||Q|^2 + |Q|^3)$, where $|Q| = |Q_F| + |Q_G|$, and this is $O(N^{-2})$. $\square$

Lemma 33 (odd functions). Let $|c - u_1| \leq C/N$. Suppose that for $|y| \leq \frac{1}{15}$ we have $g(c + y) = -g(c - y)$ and $|g(c + y)| \leq |y|^m$, and that $|g| \leq C\Theta^{m'}$ on $(0, \ell)$. Then $Eg(u) = O(N^{-(m+1)/2})$.

Proof. Let $\Xi(y) = V(c + y) - V(c)$, so $Eg = \int g(c + y)e^{-\Xi(y)} dy / \int e^{-\Xi(y)} dy$. For $|y| \leq \frac{1}{15}$ we have $c + y \in B_1$. There $|\Xi'(0)| = |V'(c)| \leq C$, $N \leq \Xi'' \leq 10N$ and $|\Xi'''| \leq CN$, so $-C|y| + \frac{N}{2}y^2 \leq \Xi(\pm y) \leq C|y| + 5Ny^2$ and $|\Xi(y) - \Xi(-y)| \leq C(|y| + N|y|^3)$. Hence $\int e^{-\Xi} \geq c_0 N^{-1/2}$.

By oddness,

$$\begin{aligned} \left| \int_{-1/15}^{1/15} g(c+y)e^{-\Xi(y)} dy \right| &= \left| \int_0^{1/15} g(c+y)(e^{-\Xi(y)} - e^{-\Xi(-y)}) dy \right| \\ &\leq C \int_0^\infty y^m (y + Ny^3) e^{Cy - Ny^2/2} dy = O(N^{-\frac{m+2}{2}}). \end{aligned}$$

The rest of the domain contributes $O(e^{-N/800}) \int e^{-\Xi}$: on B_1 because $\Xi(y) \geq -C|y| + \frac{N}{2}y^2$ with $|y| > \frac{1}{15}$, and off B_1 by Lemma 30(iii). $\square$

Lemma 34 (the angular mean). *Let $\ell = L$ and let $s = s(a) \in (0, L)$ be defined by $E \sin(u-s) = 0$, as in Theorem 3. Then $|s - u_1| = O(N^{-1})$, $s'(a) = \psi'_L(s)/V''(s) + O(N^{-2})$ (in particular $s' = O(N^{-1})$), and $s''(a) = O(N^{-2})$. Here derivatives are taken in a with $b = N - a$ and L fixed. Also $\partial_b s = O(N^{-1})$ when a and L are fixed.*

Proof. With $y_1 = u - u_1$, $s - u_1$ is an argument of Ee^{iy_1} , since $s = \arg Ee^{iu}$. Moreover $E \cos y_1 \geq \frac{1}{2}$, so $\operatorname{Re} Ee^{iy_1} > 0$, and $|s - u_1| < L < \frac{3\pi}{2}$. Hence $s - u_1$ is the principal argument, $\tan(s - u_1) = E \sin y_1 / E \cos y_1$, and $|s - u_1| \leq C|E \sin y_1| = O(N^{-1})$ because $E \sin y_1 = Ey_1 + O(E|y_1|^3)$. The a -derivative of the log-density of μ is $\psi - E\psi$, with $\psi = \psi_L$. Differentiating $E \sin(u - s(a)) = 0$ gives $s' = P/Q$ with $P = \operatorname{Cov}(\sin(u - s), \psi)$ and $Q = E \cos(u - s) = 1 + O(N^{-1})$. Put $y = u - s$ and Taylor-expand ψ at s on B_1 :

$$P = (\psi(s) - E\psi)E \sin y + \psi'(s)E[y \sin y] + \frac{1}{2}\psi''(s)E[y^2 \sin y] + O(Ey^4) = \psi'(s)Ey^2 + O(N^{-2}).$$

This uses $E \sin y = 0$, $E[y \sin y] = Ey^2 + O(Ey^4)$ and $E[y^2 \sin y] = Ey^3 + O(E|y|^5) = O(N^{-2})$ (Lemma 31(ii) with $c = s$). Since $Ey^2 = V''(s)^{-1} + O(N^{-2})$, the formula for s' follows. Next, $P' = \kappa(\sin(u - s), \psi, \psi) - s' \operatorname{Cov}(\cos(u - s), \psi) = O(N^{-2})$ and $Q' = \operatorname{Cov}(\cos(u - s), \psi) + s' E \sin(u - s) = O(N^{-1})$, both by Lemma 32. So $s'' = (P'Q - PQ')/Q^2 = O(N^{-2})$. Finally, differentiating $E \sin(u - s) = 0$ in b gives $\partial_b s = \operatorname{Cov}(\sin(u - s), \log \sin(L - u))/Q = O(N^{-1})$. $\square$

The three terms. Fix $L = \frac{\pi}{2} + \delta \in \Lambda$ with $\delta \geq 0$, and $L' = \pi - L$. Let $\mu = \mu_{a,L}$ (variable u) and $\mu' = \mu_{a,L'}$ (variable v), with expectations E and E' . Theorem 3, together with the substitution $v = \pi - u$ in I_{out} , gives $I_{\text{in}} = \frac{1}{2}M_{a,b}(L) E|\sin(u - s)|$ and $I_{\text{out}} = M_{a,b}(L') E' \sin(v + s)$. Hence

$$G(a) := \log I_{\text{in}} - \log I_{\text{out}} = T_1 + T_2 - T_3 - \log 2,$$

$$T_1 = \log \frac{M_{a,b}(L)}{M_{a,b}(L')}, \quad T_2 = \log E|\sin(u - s)|, \quad T_3 = \log E' \sin(v + s),$$

with $b = N - a$ and $s = s(a)$. Primes denote derivatives in a .

Proposition 35 (mass term). $T_1'' \leq -4\delta/N + C(\delta^3 N^{-1} + N^{-2})$.

Proof. $\partial_a^2 \log M_{a,N-a}(\ell) = \operatorname{Var}_{\mu_{a,\ell}}(\psi_\ell)$. By Lemma 32 this equals $q(x, \ell)/N + O(N^{-2})$, where $q = \psi'_\ell(u_1)^2/K$ and $K = V''(u_1)/N = x \csc^2 u_1 + (1 - x) \csc^2(\ell - u_1)$. Put $\gamma = \cot u_1$. The mode equation $x \cot u_1 = (1 - x) \cot(\ell - u_1)$ gives $\psi'_\ell(u_1) = \gamma/(1 - x)$, $K = 1 + x\gamma^2/(1 - x)$ and $q = \gamma^2/((1 - x)(1 - x + x\gamma^2))$. Hence $\partial q/\partial \gamma = 2\gamma/(1 - x + x\gamma^2)^2$, $\partial_\ell u_1 = (1 - x) \csc^2(\ell - u_1)/K$ and $\partial_\ell \gamma = -\csc^2 u_1 \partial_\ell u_1$. At $\ell = \frac{\pi}{2}$ we have $\gamma^2 = (1 - x)/x$, $K = 2$ and $\partial_\ell u_1 = \frac{1}{2}$, so

$$\partial_\ell q(x, \frac{\pi}{2}) = -\frac{\gamma}{2(1 - x)^2} \cdot \frac{1}{2x} = -\frac{1}{4(x(1 - x))^{3/2}} \leq -2.$$

Since q is real-analytic on $[0.4, 0.6] \times \Lambda$, $q(x, L) - q(x, L') = 2\delta \partial_\ell q(x, \frac{\pi}{2}) + O(\delta^3)$, and $T_1'' = (q(x, L) - q(x, L'))/N + O(N^{-2})$. $\square$

Proposition 36 (outer term). $|T_3''| \leq CN^{-2}$.

Proof. Let $F_a(v) = \sin(v + s(a))$ and $\chi = \psi_L$. On the set B of μ' we have $v + s \in [0.9, 2.24]$ by Lemmas 29(i) and 34, so $F_a \geq 0.78$ there and $E'F_a \geq \frac{1}{2}$. The a -derivative of the log-density of μ' is $\chi - E'\chi$. Therefore

$$\begin{aligned} (E'F_a)' &= \text{Cov}'(F_a, \chi) + s'E' \cos(v + s), \\ (E'F_a)'' &= \kappa'(F_a, \chi, \chi) + 2s' \text{Cov}'(\cos(v + s), \chi) + s''E' \cos(v + s) - s'^2 E'F_a. \end{aligned}$$

By Lemmas 32 and 34, $(E'F_a)' = O(N^{-1})$ and $(E'F_a)'' = O(N^{-2})$, so $T_3'' = (E'F_a)''/E'F_a - ((E'F_a)'/E'F_a)^2 = O(N^{-2})$. $\square$

Proposition 37 (inner term). $|T_2''| \leq CN^{-2}$.

Proof. We work in the frame that moves with s . Let $q_a(y) = A_L(s(a) + y)/M_{a,b}(L)$ for $y \in (-s, L - s)$, extended by 0; this is the density of $y = u - s$. Then $h(a) := E|\sin(u - s)| = \int |\sin y| q_a(y) dy$, and since $\partial_a V = -\psi$ and $\partial_a \log M_{a,b}(L) = E\psi$,

$$\begin{aligned} S_a(y) &:= \partial_a \log q_a(y) = \psi(s + y) - E\psi - s'V'(s + y), \\ \partial_a S_a(y) &= 2s'\psi'(s + y) - \text{Var} \psi - s''V'(s + y) - s'^2 V''(s + y). \end{aligned}$$

A_L vanishes to order $\geq 0.4N$ at 0 and L . Hence $q_a(y)$ and its first two a -derivatives (which equal $q_a S_a$ and $q_a(S_a^2 + \partial_a S_a)$ inside the support) extend continuously by 0. They are bounded uniformly on $\{|y| \leq L\} \times [a_0 - \eta, a_0 + \eta]$ for small $\eta > 0$, so we may differentiate under the integral sign. This gives $E_q[S_a] = 0$ and $E_q[S_a^2 + \partial_a S_a] = 0$ (differentiate $\int q_a = 1$ twice), together with

$$(\log h)'' = \text{Var}_w(S_a) + E_w[\partial_a S_a], \quad E_w[f] := E_q[|\sin y| f]/h.$$

Size of S_a . By Lemma 32, $E\psi = \psi(u_1) + O(N^{-1})$. By Lemma 34, $|s - u_1| = O(N^{-1})$, $|V'(s)| \leq C$ and $s' = \psi'(s)/V''(s) + O(N^{-2})$. Hence $S_a(0) = O(N^{-1})$, $\partial_y S_a(0) = \psi'(s) - s'V''(s) = O(N^{-1})$, and $\partial_y^2 S_a = \psi'' - s'V''' = O(1)$ on $B_1 - s$. So $|S_a(y)| \leq C(N^{-1} + y^2)$ for $s + y \in B_1$, while $|S_a| \leq CN\Theta(s + y)^2$ everywhere. *Lower bound for h .* The density of u is at most $C\sqrt{N}$ (Lemma 30(i)). With $t = c_0 N^{-1/2}$ small enough we get $\mathbb{P}(|y| \leq t) \leq \frac{1}{2}$. Since $|y| < L < 1.7$ we have $|\sin y| \geq \frac{1}{2} \min(|y|, 1)$, hence $h \geq \frac{1}{2} E \min(|y|, 1) \geq \frac{t}{4} \geq cN^{-1/2}$. *Variance part.* $\text{Var}_w(S_a) \leq E_q[|\sin y| S_a^2]/h \leq C E_q[|y|(N^{-1} + y^2)^2]/h + O(e^{-N/800}) = O(N^{-2})$. *Drift part.* Let $D(y) = \partial_a S_a(y) - \partial_a S_a(0)$. Since $E_w - E_q$ annihilates constants, $E_w[\partial_a S_a] = E_q[\partial_a S_a] + (E_w - E_q)[D]$, and $E_q[\partial_a S_a] = -E_q[S_a^2] = O(N^{-2})$. On $B_1 - s$, using $s' = O(N^{-1})$, $s'' = O(N^{-2})$ and Lemma 29(iii), we have $D(y) = \beta y + D_2(y)$ with $\beta = 2s'\psi''(s) - s''V''(s) - s'^2 V'''(s) = O(N^{-1})$ and $|D_2(y)| \leq Cy^2/N$. Everywhere $|D| \leq C\Theta(s + y)^4$. Moreover $E_q y = Eu - s = O(N^{-1})$, and $E_w y = E[|\sin(u - s)|(u - s)]/h = O(N^{-3/2})/h = O(N^{-1})$ by Lemma 33 with $m = 2$. Also $E_w y^2 + E_q y^2 = O(N^{-1})$. Hence $(E_w - E_q)[D] = \beta(E_w y - E_q y) + O(N^{-2}) = O(N^{-2})$. So $(\log h)'' = O(N^{-2})$, and $T_2 = \log h$. $\square$

Proof of Theorem 9. Let $L = L_{\text{bal}}$ and $\delta = L - \frac{\pi}{2}$. Then $\delta \geq \phi_k - \frac{\pi}{2} > \frac{\log k}{4k}$ for large k by Theorem 5(i). The upper-bound part of the proof of Theorem 7 gives $\delta \leq \frac{\log k + \log 8\pi + 1}{4k} \leq \frac{2 \log k}{k}$ for large k , so $L \in \Lambda$. By Lemma 28 it remains to treat splits with $0.41k < p \leq q$. For these $a = 2p - 1 \in [0.4N, 0.6N]$ when $k \geq 10$. By Propositions 35–37, for all real $a \in [0.4N, 0.6N]$,

$$G''(a) \leq -\frac{4\delta}{N} + C\left(\frac{\delta^3}{N} + \frac{1}{N^2}\right) \leq -\frac{\log k}{kN} + \frac{C'}{N^2} < 0 \quad (k \geq K_B).$$

Moreover $G(a) = G(N - a)$. Indeed, $u \mapsto L - u$ and $s \mapsto L - s$ exchange a and b in I_{in} , and $u \mapsto \pi + L - u$ does so in I_{out} . The balanced exponents $a_{\text{bal}} = 2\lfloor \frac{k}{2} \rfloor - 1$ and $N - a_{\text{bal}}$ satisfy $G = 0$ at L . If k is even, then $a_{\text{bal}} = \frac{N}{2}$ is the maximum point of the strictly concave symmetric function G , so $G(a) < 0$ for every other a in the window. If k is odd, then $G(\frac{N}{2} \pm 1) = 0$ and strict concavity gives $G < 0$ outside $[\frac{N}{2} - 1, \frac{N}{2} + 1]$; every other admissible $a = 2p - 1$ satisfies

$|a - \frac{N}{2}| \geq 3$. In both cases every non-balanced split in the window has $I_{\text{out}} > I_{\text{in}}$ at L , i.e. $\Delta_{p,q}(L) > 0$, and hence $L(p, q) > L_{\text{bal}}$ by Theorem 3. Take $K_0 = \max(K_A, K_B, 10)$. $\square$

Monotonicity.

Proposition 38 (derivatives in b). *Let $L = \frac{\pi}{2} + \delta \in \Lambda$ with $\delta \geq 0$, and let T_1, T_2, T_3 be as above, now regarded as functions of b with a and L fixed. Then $\partial_b T_1 \geq 0.22\delta - C\delta/N$ and $|\partial_b T_2| + |\partial_b T_3| \leq C/N$.*

Proof. T_1 . We have $\partial_b \log M_{a,b}(\ell) = m_\ell := E_\ell \log \sin(\ell - u)$, and $\partial_b T_1 = m_L - m_{L'}$. The log-density of $\mu_{a,b,\ell}$ has ℓ -derivative $b \cot(\ell - u) - E b \cot(\ell - u)$, and A_ℓ vanishes to order b at $u = \ell$. Hence $\partial_\ell m_\ell = E \cot(\ell - u) + b \text{Cov}(\log \sin(\ell - u), \cot(\ell - u))$. By Lemma 32, and since $\partial_\ell u_1 = b \csc^2(\ell - u_1)/V''(u_1)$ (proof of Proposition 35),

$$\partial_\ell m_\ell = \cot(\ell - u_1) - \frac{b \cot(\ell - u_1) \csc^2(\ell - u_1)}{V''(u_1)} + O(N^{-1}) = \cot(\ell - u_1) (1 - \partial_\ell u_1) + O(N^{-1}).$$

By Lemma 29(i), $\cot(\ell - u_1) \geq \cot 0.99 \geq 0.65$. Also $1 - \partial_\ell u_1 = x \csc^2 u_1/K$, and this is at least $0.4 \csc^2(0.99)/\csc^2(0.58) \geq 0.17$. So $\partial_\ell m_\ell \geq 0.11 - C/N$ on Λ , and $m_L - m_{L'} \geq 2\delta(0.11 - C/N)$. T_3 . The b -derivative of the log-density of μ' is $\log \sin(L' - v) - E' \log \sin(L' - v)$. So $\partial_b E' \sin(v + s) = \text{Cov}'(\sin(v + s), \log \sin(L' - v)) + \partial_b s E' \cos(v + s) = O(N^{-1})$ by Lemmas 32 and 34, and $E' \sin(v + s) \geq \frac{1}{2}$. T_2 . As in Proposition 37, $\partial_b \log h = E_w[S^b]$ with $S^b(y) = \log \sin(L - s - y) - E \log \sin(L - u) - \partial_b s V'(s + y)$. By Lemmas 32 and 34, $S^b(0) = O(N^{-1})$, $\partial_y S^b(0) = O(1)$ and $|\partial_y^2 S^b| \leq C$ on $B_1 - s$. Hence $E_w[S^b] = S^b(0) + \partial_y S^b(0) E_w y + O(E_w y^2) + O(e^{-N/800}) = O(N^{-1})$, because $E_w y = O(N^{-1})$ (Lemma 33) and $E_w y^2 = O(N^{-1})$. $\square$

Proof of Theorem 10. Let (p, q) be a split of k with $0.42k \leq p \leq 0.58k$, and put $a = 2p - 1$, $b_0 = 2q - 1$, $L = L(p, q)$, $\delta = L - \frac{\pi}{2}$. Then $\delta \geq \phi_k - \frac{\pi}{2} > \frac{\log k}{4k}$ for large k (Theorem 5(i)). Also $\delta \leq \frac{\log k}{k}$ for large k . Indeed, apply Lemma 27 to $(\min(p, q), \max(p, q))$, using $L(p, q) = L(q, p)$. At $\delta = \frac{\log k}{k}$ it gives $\Psi = 2r \log k + o(\log k)$, which exceeds $\log(2r_{ab}\sqrt{\pi N}) + o(1)$, so the split is unsafe there. For $b \in [b_0, b_0 + 2]$ we have $a/(a + b) \in [0.4, 0.6]$ once k is large. Let $g(b) = \log I_{\text{in}} - \log I_{\text{out}}$, computed at L with exponents (a, b) . Then $g(b_0) = 0$, and by Proposition 38,

$$g'(b) = \partial_b T_1 + \partial_b T_2 - \partial_b T_3 \geq 0.22\delta - \frac{C}{N} > \frac{0.05 \log k}{k} - \frac{C}{2k - 2} > 0 \quad (k \geq K'_2).$$

So $g(b_0 + 2) > 0$, i.e. $\Delta_{p,q+1}(L) < 0$, and $L(p, q + 1) < L$ by Theorem 3. This proves endpoint monotonicity for $k \geq K'_2$. Now take $K_2 = \max(K_0, K'_2)$. For $k \geq K_2$ we have $\phi_k = L(p_*, q_*)$ with $(p_*, q_*) = (\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil)$ (Theorem 9). Since $(p_*, q_* + 1)$ is a split of $k + 1$, $\phi_{k+1} \leq L(p_*, q_* + 1) < L(p_*, q_*) = \phi_k$. $\square$

Numerical check of the three terms (`numerics/bln_decomp.py`, `numerics/decomp.txt`). At $a = \frac{N}{2}$ and $L = L_{\text{bal}}$, as N runs from 40 to 1280: $N^2 T_1''$ runs from -11.7 to -19.3 , against $-4N\delta$ from -12.1 to -19.4 ; $N^2 T_2''$ runs from 0.06 to 0.005 ; $N^2 T_3'' \rightarrow -4$; and $N^2 G''$ runs from -7.9 to -15.3 . These are consistent with the propositions above. For Proposition 38 (`numerics/bln_decomp_b.py`, `numerics/decomp_b.txt`), at the balanced split for $k = 20, \dots, 640$: $N \partial_b T_1$ agrees with $2N\delta \cdot \frac{1}{2} \sqrt{x/(1-x)}$ to three decimals (from 2.98 to 4.84), $N \partial_b T_2 \rightarrow -\frac{1}{2}$, and $N \partial_b T_3 \rightarrow 0$. These numbers concern only near-balanced splits. They say nothing about the size of K_0 or K_2 , which also depend on the $o(1)$ terms in Lemma 28, and we did not make the constants explicit.

8. RIGOROUS VERIFICATION FOR $k \leq 440$: PROOF OF THEOREM 8

Fix a split with $a = 2p - 1$, $b = 2q - 1$, and a rational $L \in (0, \pi)$. Expanding by the binomial theorem,

$$P(u) := \sin^a u \sin^b(u - L) = (2i)^{-a-b} \sum_{j=0}^a \sum_{l=0}^b \binom{a}{j} \binom{b}{l} (-1)^{a+b-j-l} e^{-i(2l-b)L} e^{i(2j-a+2l-b)u}.$$

Put $F_1(t) = \int_0^t P \sin$ and $F_2(t) = \int_0^t P \cos$. These are finite sums of $\int_0^t e^{inu} du = (e^{int} - 1)/(in)$ with n odd. In the notation of the proof of Theorem 3, $F = \cos s F_1 - \sin s F_2$. Since $P < 0$ on $(0, L)$, $s = \text{atan2}(-F_1(L), -F_2(L))$, and

$$\Delta_{p,q}(L) = F(\pi) - F(s) = \cos s (F_1(\pi) - F_1(s)) - \sin s (F_2(\pi) - F_2(s)).$$

Every quantity here is evaluated in Arb ball arithmetic (python-flint, precision $200 + 6k$ bits, which absorbs the 2^{a+b} cancellation), giving a rigorous enclosure of $\Delta_{p,q}(L)$. By the sign pattern of Theorem 3, a certified $\Delta_{p,q}(L) > 0$ proves $L(p, q) > L$, and a certified $\Delta_{p,q}(L) < 0$ proves $L(p, q) < L$.

For each k we take a rational L^* (denominator $\leq 10^{12}$) halfway between the floating-point values of L_{bal} and $\min_{p < \lfloor k/2 \rfloor} L(p, k-p)$. We certify $\Delta_{\text{bal}}(L^*) < 0$ and $\Delta_{p,k-p}(L^*) > 0$ for every other $p \leq k/2$, which gives $L_{\text{bal}} < L^* < L(p, k-p)$. The other comparisons in Theorem 8 use separating rational points in the same way. The enclosures of ϕ_k come from certified signs at rational points within $5 \cdot 10^{-15}$ of $L_f \pm 10^{-12}$. The floating-point values only choose the test points; correctness rests on the certified signs. Code: `numerics/bln_rigorous.py`; the run for $k \leq 200$ (all three claims) takes about 50 minutes on one core (output `numerics/rigorous.200.txt`), and the run `bln_rigorous.py 201 300` (the conjecture for $201 \leq k \leq 300$ and $\phi_{k+1} < \phi_k$ for $200 \leq k < 300$) about 40 minutes (`numerics/rigorous.201-300.txt`). The range $301 \leq k \leq 400$ was run in four chunks, `bln_rigorous.py` with arguments 301 337, 338 363, 364 383 and 384 400, in parallel. Each chunk also certifies $\phi_{k+1} < \phi_k$ across its lower seam. This took about 85 minutes of wall time, at roughly 80 seconds per k near $k = 350$ (`numerics/rigorous.301-400.txt`). Near the balanced split the quantities compared are of size about $2^{-(a+b)/2}$, so the rational test points are converted to balls at 4000 bits; with fewer bits the first k of a run above $k \approx 370$ was not certified. The range $401 \leq k \leq 440$ (and $\phi_{k+1} < \phi_k$ for $400 \leq k < 440$) was run as `bln_rigorous.py 401 440` on one core, at about 12 minutes per k (`numerics/rigorous.401-440.txt`).

Two further certificates. Theorems 5(ii) and 6 use Theorem 8 for $k \leq 300$. Above that, on a finite range, they use the explicit closed forms of Propositions 25 and 19. These are certified in Arb by `numerics/bln_lower_arb.py` (all 99977500 pairs (k, p) with $301 \leq k \leq 2 \cdot 10^4$; output `numerics/lower_arb.txt`) and by `numerics/bln_odd_upper_arb.py` (odd $301 \leq k \leq 200001$; output `numerics/odd_upper_arb.txt`). Both scripts read the enclosures of Theorem 8 from the output files above. `bln_rigorous.py` certifies $\phi_k \in (L_f - 1.01 \cdot 10^{-12}, L_f + 1.01 \cdot 10^{-12})$ and prints L_f rounded to 12 decimals, so the scripts use the printed value $\pm 2 \cdot 10^{-12}$.

9. NUMERICAL EVIDENCE

Computation of $L(p, q)$ (`numerics/bln_fast.py`). The quantities s , I_{in} and I_{out} of Theorem 3 are integrals of positive functions. We evaluate them in log scale with 1200-point Gauss–Legendre rules split at the peaks, and solve $\log I_{\text{out}} = \log I_{\text{in}}$ by Brent’s method. Doubling the number of nodes changes L by at most $6 \cdot 10^{-14}$. The method reproduces $\phi_2 = 2\pi/3$, $\phi_3 = 1.962719002$ and $\phi_4 = 1.870658532$ to all digits.

Results (`numerics/sweep_300.txt`). For every $2 \leq k \leq 300$: the minimizing split is the balanced one (Conjecture 2, both parities); $p \mapsto L(p, k-p)$ is strictly decreasing on $1 \leq p \leq k/2$, with the

smallest gap $1.3 \cdot 10^{-7}$; and ϕ_k is strictly decreasing in k . Conjecture 11 is rigorously verified for $p + q < 200$ (Theorem 8).

k	ϕ_k (numerical)	$4k(\phi_k - \frac{\pi}{2}) - \log k$	lower (Prop. 25)	upper (Prop. 15)
10	1.7115206092	3.3264	+0.457	3.383
20	1.6493102586	3.2854	+0.923	3.312
100	1.5904114962	3.2409	+1.242	3.246
300	1.5782417442	3.2307	+1.322	3.232
10^4	—	—	+1.382	3.2245

The last two columns are in the normalization $4k\delta - \log k$. The first gives the largest c such that $\Phi < 0$ for all splits at $\delta = (\log k + c)/(4k)$. The second gives the smallest δ satisfying the sufficient inequality $2k \log \tan \alpha > \log(2kB(\frac{1}{2}, k)/c)$ in the proof of Theorem 4(i), for even k . The limit $\log(8\pi) = 3.22417$ is matched by the heuristic Laplace expansion of the criterion. For a split with $\rho = p/k$ and $r = 2\sqrt{\rho(1-\rho)}$, one has $I_{\text{in}} \approx M(L)/(2\sqrt{\pi N})$ ($N = 2k - 2$), $I_{\text{out}} \approx rM(\pi - L)$, and $\log M(\frac{\pi}{2} + \delta) - \log M(\frac{\pi}{2} - \delta) \approx 4\sqrt{pq}\delta$. This predicts $L(p, q) \approx \frac{\pi}{2} + \frac{\log k + \log 8\pi + 2\log r}{4kr}$, which is minimized at $r = 1$ on the admissible range $r \geq 2/\sqrt{k}$. The predicted gaps $L(p, k-p) - L(\frac{k}{2}, \frac{k}{2}) \approx (\frac{\log k + \log 8\pi}{2} - 1) \frac{(\rho - 1/2)^2}{k}$ match the observed ones (for example $1.1 \cdot 10^{-7}$ predicted against $1.3 \cdot 10^{-7}$ observed at $k = 300$). The proof of Theorem 9 does not use this expansion. Instead it controls the second derivative of the criterion in the exponent (§7).

REFERENCES

- [1] A. Barvinok, I. Novik, *A centrally symmetric version of the cyclic polytope*, Discrete Comput. Geom. 39 (2008) 76–99.
- [2] A. Barvinok, S. J. Lee, I. Novik, *Neighborliness of the symmetric moment curve*, Mathematika 59 (2013) 223–249; arXiv:1104.5168.
- [3] R. Sanyal, F. Sottile, B. Sturmfels, *Orbitopes*, Mathematika 57 (2011) 275–314.
- [4] C. Vinzant, *Edges of the Barvinok–Novik orbitope*, Discrete Comput. Geom. 46 (2011) 479–487.
- [5] J. G. Wendel, *Note on the gamma function*, Amer. Math. Monthly 55 (1948) 563–564.
- [6] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) 34 (1973) 335–343.
- [7] J. de Dios Pont, P. Ivanisvili, J. Madrid, *A new proof of the description of the convex hull of space curves with totally positive torsion*, Michigan Math. J. 76 (2026) 1043–1089.