

Ball's integral inequality via weighted level sets

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Theorem (K. Ball, 1986). *For every real $s \geq 2$,*

$$\frac{1}{\pi} \int_{\mathbb{R}} \left| \frac{\sin t}{t} \right|^s dt \leq \frac{1}{\pi} \int_{\mathbb{R}} e^{-st^2/(2\pi)} dt = \sqrt{\frac{2}{s}},$$

with equality if and only if $s = 2$.

Notation. Let $\varphi(t) = |\sin t/t|$ (with $\varphi(0) = 1$) and $\gamma(t) = e^{-t^2/(2\pi)}$, so $0 \leq \varphi, \gamma \leq 1$. On $\mathbb{R}$ let $d\mu = \varphi^2 dt$ and $d\nu = \gamma^2 dt$. Classically $\mu(\mathbb{R}) = \int \sin^2 t/t^2 dt = \pi$ and $\nu(\mathbb{R}) = \int e^{-t^2/\pi} dt = \pi$.

Proof. Step 1 (reduction). Let $s > 2$. For $0 \leq x \leq 1$ we have $x^{s-2} = \int_0^1 (s-2)u^{s-3} \mathbf{1}[x > u] du$. Put $x = \varphi(t)$, integrate against $d\mu$, and use Tonelli (the set $\{(t, u) : \varphi(t) > u\}$ is open):

$$\int \varphi^s dt = \int \varphi^{s-2} d\mu = \int_0^1 (s-2)u^{s-3} \mu\{\varphi > u\} du.$$

Likewise $\int \gamma^s dt = \int_0^1 (s-2)u^{s-3} \nu\{\gamma > u\} du$. Hence it suffices to show

$$\mu\{\varphi > u\} \leq \nu\{\gamma > u\} \quad (0 < u < 1), \quad \text{strictly for } 0 < u < 1/\pi. \quad (\star)$$

Step 2 ($1/\pi \leq u < 1$). If $|t| \geq \pi$ then $\varphi(t) \leq 1/|t| \leq u$, so $\{\varphi > u\} \subset (-\pi, \pi)$. For $|t| < \pi$ every factor of Euler's product $\frac{\sin t}{t} = \prod_{k \geq 1} (1 - \frac{t^2}{k^2 \pi^2})$ lies in $(0, 1]$. By $1 - x \leq e^{-x}$ and $\sum k^{-2} = \pi^2/6$ we get $\varphi(t) \leq e^{-t^2/6} \leq \gamma(t)$, since $\pi \geq 3$. Thus $\{\varphi > u\} \subset \{\gamma > u\}$ and $\varphi^2 \leq \gamma^2$ on this set, whence $\mu\{\varphi > u\} \leq \int_{\{\varphi > u\}} \gamma^2 \leq \nu\{\gamma > u\}$.

Step 3 ($0 < u < 1/\pi$). Since $\mu(\mathbb{R}) = \nu(\mathbb{R}) = \pi$, the strict form of $(\star)$ at u is equivalent to $\mu\{\varphi \leq u\} > \nu\{\gamma \leq u\}$.

- *Sinc side.* Put $a = 1/u > \pi$; then $\varphi \leq u$ on $\{|t| \geq a\}$. With $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ and one integration by parts,

$$\int_a^\infty \frac{\sin^2 t}{t^2} dt = \frac{1}{2a} - \frac{1}{2} \int_a^\infty \frac{\cos 2t}{t^2} dt, \quad \left| \int_a^\infty \frac{\cos 2t}{t^2} dt \right| = \left| -\frac{\sin 2a}{2a^2} + \int_a^\infty \frac{\sin 2t}{t^3} dt \right| \leq \frac{1}{a^2}.$$

Hence $\mu\{\varphi \leq u\} \geq 2 \int_a^\infty \sin^2 t/t^2 dt \geq u - u^2 > u/2$, as $u < 1/2$.

- *Gaussian side.* $\gamma \leq u$ exactly on $\{|t| \geq \tau\}$, where $\tau = \sqrt{2\pi \log(1/u)}$, and $e^{-\tau^2/\pi} = u^2$. So

$$\nu\{\gamma \leq u\} = 2 \int_\tau^\infty e^{-t^2/\pi} dt \leq 2 \int_\tau^\infty \frac{t}{\tau} e^{-t^2/\pi} dt = \frac{\pi}{\tau} u^2.$$

As $\tau^2 > 2\pi \log \pi \geq 2\pi - 2 \geq 4$ (by $\log x \geq 1 - 1/x$ and $\pi \geq 3$), we have $\tau > 2$, and $\pi u < 1$ gives $\nu\{\gamma \leq u\} < \frac{\pi}{2} u^2 < \frac{u}{2}$.

Together, $\nu\{\gamma \leq u\} < u/2 < \mu\{\varphi \leq u\}$, which proves $(\star)$.

Conclusion. For $s > 2$ both integrals are finite ($\int \varphi^s \leq \int \varphi^2 = \pi$), and $\pi\sqrt{2/s} - \int \varphi^s dt = \int_0^1 (s-2)u^{s-3}(\nu\{\gamma > u\} - \mu\{\varphi > u\}) du > 0$, because the integrand is nonnegative and positive on $(0, 1/\pi)$. For $s = 2$ both sides equal π . $\square$

Numerical input: only $\pi \geq 3$ (Steps 2 and 3), which the theorem itself forces as $s \rightarrow \infty$. The ratio of the two bounds of Step 3 is largest at the end $u = 1/\pi$ of its range, where they are $u - u^2 = 0.217$ and $1/(\pi\tau) = 0.119$.

Remarks

1. **Probabilistic form.** Let X have density $\pi^{-1} \sin^2 t/t^2$ and Y have density $\pi^{-1} e^{-t^2/\pi}$. Then $(\star)$ says that $\varphi(X)$ is stochastically dominated by $\gamma(Y)$. Equivalently, $-\log(\sin^2 X/X^2)$ stochastically dominates the Gamma($\frac{1}{2}, 1$) law, and for $r \geq 0$ the Laplace transforms of these two laws are $F(2+2r)$, where $F(p) = \pi^{-1} \int |\sin t/t|^p dt$, and $(1+r)^{-1/2}$.
2. **Consequences.** Write $(x-c)_+ = x(1-c/x)_+$, where $(1-c/x)_+$ is increasing in x . Then $(\star)$ implies the convex-order majorization $\int (\varphi^2 - c)_+ \leq \int (\gamma^2 - c)_+$ for all $c > 0$, which is the intermediate statement of the stop-loss and Lorenz-curve proofs (table below). For $0 < s < 2$ and $0 < x \leq 1$, $x^{s-2} = 1 + \int_0^1 (2-s)u^{s-3} \mathbf{1}[x < u] du$, so $\int \varphi^s = \pi + \int_0^1 (2-s)u^{s-3} \mu\{\varphi < u\} du$ (as $\varphi > 0$ off a null set), and likewise for γ, ν . By $(\star)$ and $\mu(\mathbb{R}) = \nu(\mathbb{R})$, $\mu\{\varphi < u\} \geq \nu\{\gamma < u\}$ (take left limits), which gives the reverse inequality $\int \varphi^s \geq \int \gamma^s$ for $0 < s < 2$ (for $s \leq 1$ the left side is $+\infty$; $\int \gamma^s = \pi\sqrt{2/s}$ is finite).
3. **Relation to Melbourne–Roberto.** Their key estimate $T' \leq 1$ (§3.1), for the monotone transport T from $\sin^2 t/t^2$ to $e^{-t^2/\pi}$, is the pointwise coupling $\varphi \leq \gamma \circ T$. As T pushes μ to ν , it gives $\mu\{\varphi > u\} \leq \mu\{\gamma \circ T > u\} = \nu\{\gamma > u\}$, so $(\star)$ is implicit in their work, for the same two measures (they write $t = \pi x$). Step 1 is formally the excluded case $s_0 = \lambda_0 = 0$ (equal masses, no crossing) of their weighted Nazarov–Podkorytov lemma (§2.5, Cor. 2.11), where it is just the layer-cake formula.
4. **The weights are Ball’s.** For $2 < s < 4$, Ball (1986, proof of Lemma 3(b)) expands $(\sin^2 t/t^2)^{1+\sigma}$, $s = 2+2\sigma$, in powers of $1 - \sin^2 t/t^2$ and reduces the inequality to $\alpha_n < \beta_n$ for all $n \geq 1$, where $\beta_n = \frac{1}{\pi} \int \varphi^2 (1 - \varphi^2)^n dt$ and $\alpha_n = \frac{1}{\pi} \int \gamma^2 (1 - \gamma^2)^n dt$ (his Table 1 starts $\beta_1 = 0.33$, $\alpha_1 = 0.29$). These are moments under the same two measures $\mu/\pi, \nu/\pi$, and since $(1-y^2)^n$ decreases in y , $(\star)$ gives $\beta_n \geq \alpha_n$ for every n at once. What is new is only the direct verification of $(\star)$. It is formally weaker than the coupling, but that is not what saves the work, since the bounds above, which also hold at $u = 1/\pi$, give $T' \leq 1$ (as a referee observed): one integration by parts replaces the lobe-wise tail estimates, and the layer-cake formula replaces the transport lemma. The weighted distribution functions do not cross, and the proof compares the two measures at equal *heights* u instead of equal quantiles. High levels then reduce to the main lobe. At low levels the masses are of different orders ($\sim \frac{4}{\pi}u$ against $\sim \sqrt{\pi/2}u^2/\sqrt{\log(1/u)}$), so the only side-lobe information needed is $|\sin t| \leq 1$ and that $\sin^2$ averages to $\frac{1}{2}$; no hump heights are needed.
5. **Lebesgue measure fails.** The unweighted distribution functions $|\{\varphi > u\}|$ and $|\{\gamma > u\}|$ cross once, at $u \approx 0.2087$, just below the second-hump height 0.2172. Locating this crossing is the delicate part of the Nazarov–Podkorytov proof. The weighted ones do not cross at all.
6. **Independent version and audits.** An independent write-up found the same proof, bounding the sinc tail one period of $\sin^2$ at a time ($2 \int_a^\infty \sin^2 t/t^2 dt \geq 1/(a+\pi)$). Three independent checks found an earlier version VALID (Step 3 then used $\pi > e$; $\log x \geq 1 - 1/x$ is from that write-up), and an independent checker found this version VALID and suggested its presentational changes. One of the checks carried the argument over to the discrete Ball inequality of Melbourne–Roberto.

Comparison with known proofs

Proof	Length	Main tools	Numerical input
Ball 1986	3–4 pp.	ranges of s and t , series expansions	many explicit estimates
Nazarov–Podkorytov 2000	2–3 pp.	single crossing of distribution functions, hump-wise derivative bounds	a few constants
Melbourne–Roberto 2023	≈ 2 pp. (arXiv v3, pp. 13–15)	monotone transport with $T' \leq 1$; product formula; Mills ratio; lobe-wise tail	second-hump height $< 1/\sqrt{20}$
Stop-loss / Lorenz (independent write-ups)	$\frac{1}{2}$ –1 p. compressed	convex-order majorization checked level- or set-wise; Jordan or sampling identity; Mills	$\frac{5}{2} < e < 3 < \pi$
This note	$\approx \frac{3}{4}$ p.	weighted layer cake; product formula; one integration by parts; Mills	$\pi \geq 3$

References. K. Ball, Cube slicing in $\mathbb{R}^n$, Proc. AMS 97 (1986) 465–473. F. Nazarov, A. Podkorytov, Ball, Haagerup, and distribution functions, Oper. Theory Adv. Appl. 113 (2000) 247–267. J. Melbourne, C. Roberto, Transport-majorization to analytic and geometric inequalities, J. Funct. Anal. 284 (2023) 109717.