

Ball's integral inequality: the complete proof on one page

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October 21, 2026: every step of the proof, without its remarks. The same proof, with remarks, is in the companion note. The cosine product behind (F2) is Euler's (1773), extending Viète's formula.

Theorem (K. Ball, 1986). For every real $s \geq 2$, $\frac{1}{\pi} \int_{\mathbb{R}} \left| \frac{\sin t}{t} \right|^s dt \leq \frac{1}{\pi} \int_{\mathbb{R}} e^{-st^2/(2\pi)} dt = \sqrt{\frac{2}{s}}$, with equality if and only if $s = 2$.

Setup. Let $f(t) = (\sin t/t)^2$ ($f(0) = 1$), $g(t) = e^{-t^2/\pi}$, $p = s/2 \geq 1$. Then $0 \leq f \leq 1$ (as $|\sin t| \leq |t|$), $0 < g \leq 1$, $\int_{\mathbb{R}} g^p = \pi \sqrt{2/s}$; so the claim is $\int f^p \leq \int g^p$, with equality iff $p = 1$.

(F1) $\int_{\mathbb{R}} f = \pi = \int_{\mathbb{R}} g$ (Plancherel: $\frac{\sin t}{t} = \int_{-1}^1 \frac{1}{2} e^{-itx} dx$, so $\int f = 2\pi \int_{-1}^1 \frac{1}{4} dx$; Gaussian integral).

(F2) $f \leq g$ on $(-\pi, \pi)$. As f, g are even and $f(0) = g(0)$, let $0 < t < \pi$. For $0 \leq x < \frac{\pi}{2}$, $(\log \cos x)' = -\tan x \leq -x$; integrating from 0, $0 < \cos x \leq e^{-x^2/2}$. As all $t/2^k \in (0, \frac{\pi}{2})$, the double-angle formula gives $\frac{\sin t}{2^n \sin(t/2^n)} = \prod_{k=1}^n \cos \frac{t}{2^k} \leq \exp(-\frac{t^2}{2} \sum_{k=1}^n 4^{-k})$. Let $n \rightarrow \infty$: as $2^n \sin(t/2^n) \rightarrow t$, $\frac{\sin t}{t} \leq e^{-t^2/6}$. As $\sin t > 0$, squaring gives $f \leq e^{-t^2/3} \leq e^{-t^2/\pi} = g$, by $\pi \geq 3$.

(F3) $f(t) \leq t^{-2}$ for $t \neq 0$, since $\sin^2 t \leq 1$.

Step 1 (weighted layer cake). For $p > 1$ and $0 \leq y \leq 1$, $(p-1) \int_0^1 c^{p-2} y \mathbf{1}[y > c] dc = (p-1) y \int_0^y c^{p-2} dc = y^p$ (convergent as $p-2 > -1$). Put $y = h(t)$, $h \in \{f, g\}$, and integrate over t (Tonelli; the integrand is ≥ 0):

$$\int_{\mathbb{R}} h^p dt = (p-1) \int_0^1 c^{p-2} M_h(c) dc, \quad \text{where } M_h(c) := \int_{\{h>c\}} h(t) dt.$$

So the theorem follows from the **Key Lemma**: $M_f(c) \leq M_g(c)$ for $0 < c < 1$, strictly for $c < \pi^{-2}$. Indeed, all these integrals are finite (as $h^p \leq h$), so for $p > 1$ it gives $\int g^p - \int f^p = (p-1) \int_0^1 c^{p-2} (M_g - M_f) dc > 0$, as the integrand is ≥ 0 , and > 0 for $c < \pi^{-2}$; and $p = 1$ is (F1).

Step 2 (levels $\pi^{-2} \leq c < 1$). Let $E = \{f > c\}$. By (F3), $f(t) \leq t^{-2} \leq \pi^{-2} \leq c$ for $|t| \geq \pi$, so $E \subset (-\pi, \pi)$, where $f \leq g$ by (F2). So $E \subset \{g > c\}$ and $M_f(c) = \int_E f \leq \int_E g \leq \int_{\{g>c\}} g = M_g(c)$.

Step 3 (levels $0 < c < \pi^{-2}$). By (F1), $M_h(c) = \pi - \int_{\{h \leq c\}} h$, so we must show $\int_{\{g \leq c\}} g < \int_{\{f \leq c\}} f$. Let $a = c^{-1/2} > \pi$ and $\tau = \sqrt{\pi \log(1/c)}$. As $c < \pi^{-2}$, $\tau^2 > 2\pi \log \pi \geq 2\pi - 2 \geq 4$, by $\log x \geq 1 - \frac{1}{x}$ and then $\pi \geq 3$.

Sinc side. By (F3), $f \leq c$ for $|t| \geq a$. Split $[a, \infty)$ into $I_i = [a + (i-1)\pi, a + i\pi)$, $i \geq 1$. As $\sin^2$ has period π and mean $\frac{1}{2}$, $\int_{I_i} \sin^2 t dt = \frac{\pi}{2}$. As $t^{-2} \geq (a + i\pi)^{-2}$ on I_i and $y \mapsto (a + y\pi)^{-2}$ decreases,

$$\int_{\{f \leq c\}} f \geq 2 \int_a^\infty \frac{\sin^2 t}{t^2} dt \geq \pi \sum_{i \geq 1} \frac{1}{(a + i\pi)^2} \geq \pi \int_1^\infty \frac{dy}{(a + y\pi)^2} = \frac{1}{a + \pi}.$$

Gaussian side. $g(t) \leq c$ if and only if $|t| \geq \tau$, and $t/\tau \geq 1$ for $t \geq \tau$, so

$$\int_{\{g \leq c\}} g = 2 \int_\tau^\infty e^{-t^2/\pi} dt \leq \frac{2}{\tau} \int_\tau^\infty t e^{-t^2/\pi} dt = \frac{\pi}{\tau} e^{-\tau^2/\pi} = \frac{\pi c}{\tau}.$$

Comparison. Using $\tau > 2$ (from $\tau^2 > 4$), then $\pi\sqrt{c} < 1$, then $a > \pi$:

$$\int_{\{g \leq c\}} g \leq \frac{\pi c}{\tau} < \frac{\pi c}{2} < \frac{\sqrt{c}}{2} = \frac{1}{2a} < \frac{1}{a + \pi} \leq \int_{\{f \leq c\}} f.$$

Steps 2 and 3 prove the Key Lemma, and Step 1 then proves the theorem. ■

Inputs: (F1), $|\sin t| \leq |t|$, $\frac{\sin x}{x} \rightarrow 1$, the double-angle formula, $\tan x \geq x$, $\log x \geq 1 - \frac{1}{x}$, Tonelli, and one numerical fact, $\pi \geq 3$, which the theorem itself implies (Laplace's method as $s \rightarrow \infty$, as BC noted). No Euler sine product or $\zeta(2)$, no case analysis in s , no transport map, nothing on the side lobes beyond (F1), $\sin^2 t \leq 1$ and the period and mean of $\sin^2$, no computer. **Meaning:** for $X \sim f/\pi$ and $Y \sim g/\pi = N(0, \pi/2)$, the Key Lemma says $f(X) \leq_{\text{st}} g(Y)$; Ball (1986) compared moments of $f(X)$ and $g(Y)$.